Proof. The first inequality in (4.5) is trivial.
Using Lemma 3.1, we have

$$J(u^*)=J(u)-<\mu(u),\ u^*-u>+\iint_{\Sigma_0}\left(\frac{\partial z(u^*)}{\partial n_x}-\frac{\partial z(u)}{\partial r_x}\right)^2 d\Sigma$$

$$>J(u)-<\mu(u),\ u^*-u>$$

$$=J(u)-<\mu(u),\ u^*-w(u)>-<\mu(u),\ w(u)-u>$$

$$\geqslant J(u)-<\mu(u),\ w(u)-u>,$$

which is just the second inequality in (4.5).
Theorem 4.3 bellow shows that algorithm A is convergent.

Theorem 4.3. *Let $u_1\in\mathcal{U}_{ad}$, $u_2=\tilde{u}(u_1)$, ..., $u_{m+1}=\tilde{u}(u_m)$, ..., be the sequence constructed in algorithm* (A). *If $\{u_m\}$ is finite, then the last element is an optimal control for problem* (Q). *If $\{u_m\}$ is infinite, then any weak limit of $\{u_m\}$ is an optimal control for problem* (Q). *Furthermore, we have*

$$(4.6)\qquad J(u_{m+1})-J(u^*)\leqslant(J(u_m)-J(u^*))\ \mathrm{Max}\left\{1-\frac{J(u_m)-J(u^*)}{4M\|w(u_n)-u_n\|^2},\ \frac{1}{2}\right\}$$

where u^* is any optimal control for problem (Q).

If M_1 is the radius of $\mathcal{U}_{ad}$, then (4.6) *implies that*

$$(4.7)\qquad J(u_{m+1})-J(u^*)\leqslant(J(u_m)-J(u^*))\ \mathrm{Max}\left\{1-\frac{J(u_m)-J(u^*)}{4MM_1^2},\ \frac{1}{2}\right\}.$$

Proof. The theorem is evident if $\{u_m\}$ is finite. Now assume that $\{u_m\}$ is infinite. Then (4.6) follows by (4.4), (4.2) and (4.5). Relation (4.7) is a trivial consequence of (4.6). From (4.7), we deduce that $J(u_m)-J(u^*)$ is strictly decreasing and tends to zero. So by Lemma 2.3, we deduce that any weak limit of $\{u_m\}$ is an optimal control for problem (P).

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ON THE DISCRETE INEQUALITIES OF GRONWALL-BELLMAN TYPE

BY

J. POPENDA

1. **Introduction.** Gronwall-Bellman inequalities play an important role in the study of various problems of integro-differential and summary-difference equations (see [8], [9]). Many types of the discrete analogies of integral and differential inequalities known in the continuous case have been studied recently. Among them, there have been considered discrete inequalities of one independent variable [4, 6], two variables [1, 10, 11] also involving higher order partial differences [5], many variables [3], in the distributive lattice [7]. Finite systems of difference inequalities have been studied in [2] using comparison theorems. In the present note we shall consider finite systems of discrete inequalities of one variable, however not using comparison methods strictly.

2. **Definitions and Notations.** We denote by N the set $\{n_0, n_0+1, n_0+2, \ldots\}$ where n_0 is any fixed nonnegative integer, by S the set $\{1, 2, \ldots, s\}$, and by R, R_0, R_+ the set of real numbers, nonnegative real numbers and positive real numbers, respectively.

For the function $y: N\to R$ the difference operator Δ is defined as usually $\Delta y(n)=y(n+1)-y(n)$, $\Delta^2 y(n)=\Delta(\Delta y(n))$. In the following section we shall use the following functions with their prescribed properties:

(i) $g_k, u_k: N\to R_0$, $f_k: N\to R$, $k\in S$,

(ii) $B: N\times R^s\to R$, B is a continuous function of s-variables for every fixed first variable n, $n\in N$

(ii)$_1$ $0\leqslant B(n, y_1, \ldots y_s)\leqslant B(n, x_1, \ldots, x_s)$ for $n\in N$, $0\leqslant y_k\leqslant x_k$, $k\in S$,

(ii)$_2$ $B(n, ay_1, \ldots, ay_s)\leqslant A(a)\ B(n, y_1, \ldots, y_s)$ for arbitrary $a\geqslant 1$,

where the function A is continuous, nondecreasing and positive on $[1, \infty)$. Instead to write $B(n, y_1, \ldots, y_s)$ we shall use $B(n, [y_i]s)$.

(iii) $G: N\times R_0\to R_0$. G is continuous for every fixed $n\in N$,

$0\leqslant G(n, y)\leqslant G(n, x)$ for $n\in N$, $0\leqslant y\leqslant x$,

$G(n, ay)\leqslant E(a)\, G(n, y)$ for nonnegative numbers a,

where the function E is continuous, nonnegative and nondecreasing on R_0.

(iv) $h_k: N\to R_0$, $k\in S$.

Furthermore it is supposed

$$\prod_{j=n_0}^{n_0-1} y(j)=1 \quad \text{and} \quad \sum_{j=n_0}^{n_0-1} y(j)=0.$$

3. Main Results. Theorem 1. *Let the conditions (i), (ii), (iii) be satisfied. If furthermore*

$$(1)\qquad u_k(n)\leqslant f_k(n)+g_k(n)\sum_{j=n_0}^{n} G\Big(j, \sum_{i=n_0}^{j-1} B(i, [u_t(i)]_S)\Big), \quad n\in N,\ k\in S$$

then

$$(2)\qquad u_k(n)\leqslant f_k(n)+g_k(n)\sum_{j=n_0}^{n} G\Big(j, F^{-1}\Big(\sum_{i=n_0}^{j-1} B(i, [r_t(i)]_S)\Big)\Big),$$

for $n\in N_F$, $k\in S$, *where*

$$(3)\qquad r_k(n)=\max\Big[f_k(n),\ g_k(n)\sum_{j=n_0}^{n} G(j, 1)\Big], \quad n\in N,$$

F^{-1} *is the inverse function of* F, *defined by*

$$F(x)=\int_0^x \frac{ds}{A(1+E(s))} \quad \textit{for } x\geqslant 0,$$

$$N_F=\Big\{n\in N:\ \sum_{j=n_0}^{n-1} B(j, [r_t(j)]_S)\in \operatorname{dom} F^{-1}\Big\}.$$

Proof. Denoting

$$(4)\qquad b(n)=\sum_{j=n_0}^{n-1} B(j, [u_t(j)]_S), \quad n\in N,$$

inequality (1) can be rewritten as follows

$$(5)\qquad u_k(n)\leqslant f_k(n)+g_k(n)\sum_{j=n_0}^{n} G(j, b(j)), \quad n\in N,\ k\in S.$$

Hence by (i), (ii), (iii) we obtain from (4) and (5)

$$\begin{aligned}\Delta b(n)&=B(n, [u_t(n)]_S)\leqslant\\ &\leqslant B\Big(n, \Big[f_t(n)+g_t(n)\sum_{j=n_0}^{n} G(j, b(j))\Big]_S\Big)\leqslant\\ &\leqslant B\Big(n, \Big[f_t(n)+g_t(n)\sum_{j=n_0}^{n} G(j, b(n))\Big]_S\Big)\leqslant\\ &\leqslant B\Big(n, \Big[f_t(n)+E(b(n))\, g_t(n)\sum_{j=n_0}^{n} G(j, 1)\Big]_S\Big), \quad n\in N.\end{aligned}$$

and by $(\mathrm{ii})_1$, $(\mathrm{ii})_2$ and definition (3) we get

$$\begin{aligned}\Delta b\, n&\leqslant B(n, (1+E(b(n)))\, [r_t(n)]_S)\leqslant\\ &\leqslant A(1+E(b(n)))\, B(n, [r_t(n)]_S), \quad n\in N.\end{aligned}$$

Dividing the last inequality by the positive term $A(1+E(b(n)))$, and using monotonicity property of integral operator we have

$$\int_{b(n)}^{b(n+1)} \frac{ds}{A(1+E(s))}\leqslant \frac{\Delta b(n)}{A(1+E(b(n)))}\leqslant B(n, [r_t(n)]_S), \quad n\in N.$$

Summing the above inequality from n_0 to $z-1$, we obtain, using the definition of the function F, the estimation

$$F(b(z))-F(b(n_0))\leqslant \sum_{j=n_0}^{z-1} B(j, [r_t(j)]_S), \quad z\in N.$$

As a result of $F(b(n_0))=F(0)=0$, the following bound

$$(6)\qquad b(n)\leqslant F^{-1}\Big(\sum_{j=n_0}^{n-1} B(j, [r_t(j)]_S)\Big), \quad n\in N$$

holds. The desired estimation (2) follows on using (6), (5) and monotonicity of the function G. The set N_F is obvious.

Theorem 2. *Let the conditions (i)–(iv) be satisfied. If furthermore*

$$(7)\qquad \begin{aligned}u_k(n)&\leqslant f_k(n)+\sum_{j=n_0}^{n-1}\sum_{t=1}^{s} h_t(j)\, u_t(j)+\\ &+g_k(n)\sum_{j=n_0}^{n} G\Big(j, \sum_{i=n_0}^{j-1} B(i, [u_t(i)]_S)\Big), \quad n\in N,\ k\in S,\end{aligned}$$

then

$$(8)\qquad \begin{aligned}u_k(n)&\leqslant Z_{f_k}(n)+g_k(n)\sum_{j=n_0}^{n} G\Big(j, F^{-1}\Big(\sum_{i=n_0}^{j-1} B(i, [r_t(i)]_S)\Big)\Big)+\\ &+\Big[\prod_{j=n_0}^{n-1} h(j)\Big]\sum_{j=n_0}^{n-1} z_g(j)\sum_{i=n_0}^{j} G\Big(i, F^{-1}\Big(\sum_{l=n_0}^{i-1} B(l, [r_t(l)]_S)\Big)\Big),\end{aligned}$$

for $n\in N_F$, $k\in S$, *where*

$$(9)\qquad \begin{aligned}r_k(n)&=\max\Big[Z_{f_k}(n),\ g_k(n)\sum_{j=n_0}^{n} G(j, 1)+\\ &+\Big[\prod_{j=n_0}^{n-1} h(j)\Big]\sum_{j=n_0}^{n-1} z_g(j)\sum_{i=n_0}^{j} G(i, 1)\Big], \quad k\in S,\end{aligned}$$

$$h(n)=1+\sum_{t=1}^{s} h_t(n), \quad z_p(n)=\Big[\prod_{j=n_0}^{n}\frac{1}{h(j)}\Big]\sum_{t=1}^{s} h_t(n)\, p_t(n),$$

$$Z_{p_k}(n)=p_k(n)+\Big[\prod_{j=n_0}^{n-1} h(j)\Big]\sum_{j=n_0}^{n-1} z_p(j),$$

F, F^{-1}, N_F are defined in Theorem 1 with suitable change of r_k.

Proof. We define

$$a(n)=\sum_{j=n_0}^{n-1}\sum_{t=1}^{s} h_t(j)\,u_t(j), \quad n\in N. \tag{10}$$

Hence

$$a(n)\leqslant\left[\prod_{j=n_0}^{n-1} h(j)\right]\sum_{j=n_0}^{n-1} z_f(j)+\left[\prod_{j=n_0}^{n-1} h(j)\right]\sum_{j=n_0}^{n-1} z_g(j)\sum_{i=n_0}^{j} G\left(i, \sum_{l=n_0}^{i-1} B(l, [u_t(l)]_s)\right).$$

Taking into account the above estimation, we get from (7)

$$\begin{aligned} u_k(n)\leqslant Z_{f_k}(n)+g_k(n)\sum_{j=n_0}^{n} G\left(j, \sum_{i=n_0}^{j-1} B(i, [u_t(i)]_s)\right)+\\ +\left[\prod_{j=n_0}^{n-1} h(j)\right]\sum_{j=n_0}^{n-1} z_g(j)\sum_{i=n_0}^{j} G\left(i, \sum_{l=n_0}^{i-1} B(l, [u_t(l)]_s)\right).\end{aligned} \tag{11}$$

Calculating the difference of the function b defined by (4) we have

$$\begin{aligned}\Delta b(n)\leqslant B\Bigg(n, \Bigg[Z_{f_t}(n)+g_t(n)\sum_{j=n_0}^{n} G(j, b(j))+\\ +\left[\prod_{j=n_0}^{n-1} h(j)\right]\sum_{j=n_0}^{n-1} z_g(j)\sum_{i=n_0}^{j} G(i, b(i))\Bigg]_s\Bigg), \quad n\in N.\end{aligned} \tag{12}$$

With regard to (9) and definition of the function F^{-1}, we obtain from (12).

$$b(n)\leqslant F^{-1}\left(\sum_{j=n_0}^{n-1} B(j, [r_t(j)]_s)\right), \quad n\in N_F.$$

This estimation together with inequality (11) leads us to the desired bound (8). The proof is complete.

Remark. If

$$r_k(n)=\max\left[Z_{f_k}(n), g_k(n)\sum_{j=n_0}^{n} G(j, 1), \left[\prod_{j=n_0}^{n-1} h(j)\right]\sum_{j=n_0}^{n-1} z_g(j)\sum_{i=n_0}^{j} G(i, 1)\right],$$

$$F(x)=\int_0^x \frac{ds}{A(1+2E(s))}$$

and the rest of the assumptions and definitions of Theorem 2 hold, then estimation (8) remains true.

Example. Let us consider the system of recurrence equations of the form

$$u_k(n)=f_k(n)+g_k(n)\sum_{j=1}^{n} j^p\sum_{i=1}^{j-1} i^q\sum_{t=1}^{s} a_t\,|u_t(i)|^{m_t}, \quad \begin{matrix} n=1, 2, \dots,\\ k=1, 2, \dots, s,\end{matrix} \tag{13}$$

where $m_t\in R_+$, $t=1, \dots, s$.

By the solution of (13) we mean the sequence of s-vectors $\{[u_1(n), u_2(n), \dots, u_s(n)]\}_{n=1, 2, \dots}$ which fulfils (13) for all $n\in N:=\{1, 2, \dots\}$.
The system (13) has one solution existing on N because of $[u_1(1), u_2(1), \dots, u_s(1)]$ is uniquely determined.
We shall study the system (13) where $f, g: N\to R_0$, $a_t\in R_0$, $t=1, \dots, s$, $p, q\in R$.
If

$$\max_{1\leqslant t\leqslant s} m_t:=m\leqslant 1, \quad \sum_{j=1}^{\infty} j^q\sum_{t=1}^{s} a_t(r_t(j))^{m_t}=M<\infty, \qquad \text{or} \tag{14}$$

$$\max_{1\leqslant t\leqslant s} m_t:=m>1, \quad \sum_{j=1}^{\infty} j^q\sum_{t=1}^{s} a_t(r_t(j))^{m_t}=M<(m-1)^{-1}, \tag{15}$$

where

$$r_k(n)=\max\left[f_k(n), g_k(n)\sum_{j=1}^{n} j^p\right], \quad n\in N, \ k=1, \dots, s, \ \max_{1\leqslant t\leqslant s}\sup_{n\in N} r_t(n)<\infty, \tag{16}$$

then this solution is bounded (i.e. all its coordinates are bounded). This follows from Theorem 1. It suffices to observe that in the considered system

$$B(n, x_1, \dots, x_s)=n^q(a_1|x_1|^{m_1}+\dots+a_s|x_s|^{m_s}),$$

for which conditions (ii) hold. The function A can be taken $A(x)=x^m$. Furthermore $G(n, x)=n^p x$; hence $E(x)=x$.
Calculating the function F we obtain

$$F(x)=[1/(1-m)]\,[(1+x)^{1-m}-1] \quad \text{for the case } m\neq 1,$$

$$F(x)=\ln(1+x) \quad \text{when } m=1.$$

Hence

$$F^{-1}(x)=(1+(1-m)\,x)^{1/(1-m)}-1 \quad \text{or} \quad F^{-1}(x)=e^x-1.$$

respectively. The function F^{-1} is increasing and continuous on the respective interval. It is evident by (14) or (15) that

$$\sum_{j=1}^{n} j^q\sum_{t=1}^{s} a_t(r_t(j))^{m_t}\in \ \operatorname{dom} F^{-1},$$

for all $n\in N$. Therefore for all $n\in N$ we obtain the estimation

$$u_k(n)\leqslant f_k(n)+g_k(n)\sum_{j=1}^{n} j^p F^{-1}(M),$$

which follows from (2) by (14), (15) and the monotonicity of F^{-1}. The boundedness of the solution follows now from (16).
Furthermore, as result of estimation (6) and definition (4), we obtain

$$\sum_{j=1}^{n-1} j^q[a_1(u_1(j))^{m_1}+\dots+a_s(u_s(j))^{m_s}]\leqslant$$

$$\leq F^{-1}\left(\sum_{j=1}^{n-1} j^q \sum_{i=1}^{k} a_i(r_i(j))^{m_i} \leq F^{-1}(M) < \infty, \quad n \in N.\right.$$

i.e. some kind of summability of this solution with any weight. A similar reasoning can be made for the system

$$u_k(n) = u_k(1) + g_k(n) \sum_{j=1}^{n} j^p \sum_{i=1}^{j-1} i^q \sum_{l=1}^{s} a_l \, |u_l(i)|^{m_l}, \quad n \in N, \; k \in S.$$

In this case the system has many solutions each of which depends of the initial values $[u_1(1), \ldots, u_s(1)]$. For fixed initial values, we obtain uniqueness of the solution. Changing in the first example $f_k(n)$ by $u_k(1)$ for all $n \in N$, $k \in S$ and assuming all respective conditions are fulfilled we obtain the boundedness or the summability of this solution.

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ON PSEUDO SYMMETRIC MANIFOLDS

Dedicated to Professor Kentaro Yano on his 74th birthday

BY
M. C. CHAKI

Introduction. In this paper the author introduces a type of nonflat Riemannian manifold (M^n, g) $(n \geq 2)$ whose curvature tensor R satisfies the condition

$$(1) \qquad \begin{aligned}(\nabla_X R)(Y, Z) W = {} & 2A(X) R(Y, Z) W + A(Y) R(X, Z) W + \\ & + A(Z) R(Y, X) W + A(W) R(Y, Z) X + g(R(Y, Z) W, X) \rho,\end{aligned}$$

where A is a non-zero 1-form,

$$(2) \qquad g(X, \rho) = A(X) \text{ for every vector field } X,$$

and ∇ denotes the operator of covariant differentiation with respect to the meric g. Such a manifold shall be called a *pseudo symmetric manifold* and the 1-form A shall be called its associated 1-form. The name pseudo symmetric is chosen because if the 1-form A in (1) becomes zero, then the manifold reduces to a symmetric manifold of Cartan. An n-dimensional pseudo symmetric manifold shall be denoted by $(PS)_n$.

In this paper it is shown that in a $(PS)_2$ the associated 1-form is closed. The question whether a $(PS)_n$ can be of constant curvature is answered. $(P, S)_n$ with Codazzi type of Ricci tensor and conformally flat $(PS)_n$ are studied. In the last section it is proved that, under a certain condition, a simply connected conformally flat $(PS)_n$ $(n \geq 3)$ can be isometrically immersed in a Euclidean space E^{n+1} as a hypersurface.

1. Preliminaries. Let S and r denote the Ricci tensor of type (0, 2) and the scalar curvature respectively and L denote the symmetric endomorphism of the tangent space at each point, corresponding to the Ricci tensor, i.e.,

$$(1.1) \qquad g(LX, Y) = S(X, Y) \text{ for any vector fields } X, Y.$$

Let B be the 1-form defined by

$$(1.2) \qquad B(X) = A(LX).$$

From (1) we get

$$(1.3) \qquad \begin{aligned}(\nabla_X S)(Y, Z) = {} & 2A(X) S(Y, Z) + A(Y) S(X, Z) + \\ & + A(Z) S(Y, X) + A(R(X, Y)Z) + A(R(X, Z) Y).\end{aligned}$$

Contracting (1.3) we have

$$(1.4) \qquad dr(X) = 2A(X) r + 4B(X).$$

From (1.4) we obtain

$$(1.5) \qquad dr(X) A(Y) - dr(Y) A(X) = 4[A(Y) B(X) - A(X) B(Y)].$$