$$\leq F^{-1}\left(\sum_{j=1}^{n-1} j^{q} \sum_{i=1}^{s} a_i(r_i(j))^{m_i} \leq F^{-1}(M) < \infty, \quad n \in N,\right.$$

i.e. some kind of summability of this solution with any weight. A similar reasoning can be made for the system

$$u_k(n) = u_k(1) + g_k(n) \sum_{j=1}^{n} j^{p} \sum_{i=1}^{j-1} i^{q} \sum_{t=1}^{s} a_t \, |u_t(i)|^{m_t}, \quad n \in N, \; k \in S.$$

In this case the system has many solutions each of which depends of the initial values $[u_1(1), \ldots, u_s(1)]$. For fixed initial values, we obtain uniqueness of the solution. Changing in the first example $f_k(n)$ by $u_k(1)$ for all $n \in N$, $k \in S$ and assuming all respective conditions are fulfilled we obtain the boundedness or the summability of this solution.

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Institute of Mathematics
Technical University
Piotrowo 3a
60 — 965 Poznań
Poland

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ON PSEUDO SYMMETRIC MANIFOLDS

Dedicated to Professor Kentaro Yano on his 74th birthday

BY
M. C. CHAKI

Introduction. In this paper the author introduces a type of nonflat Riemannian manifold (M^n, g) $(n \geq 2)$ whose curvature tensor R satisfies the condition

$$(\nabla_X R)(Y, Z) W = 2A(X) R(Y, Z) W + A(Y) R(X, Z) W + A(Z) R(Y, X) W + A(W) R(Y, Z) X + g(R(Y, Z) W, X) \rho, \tag{1}$$

where A is a non-zero 1-form,

$$g(X, \rho) = A(X) \text{ for every vector field } X, \tag{2}$$

and ∇ denotes the operator of covariant differentiation with respect to the meric g. Such a manifold shall be called a *pseudo symmetric manifold* and the 1-form A shall be called its associated 1-form. The name pseudo symmetric is chosen because if the 1-form A in (1) becomes zero, then the manifold reduces to a symmetric manifold of Cartan. An n-dimensional pseudo symmetric manifold shall be denoted by $(PS)_n$.

In this paper it is shown that in a $(PS)_2$ the associated 1-form is closed. The question whether a $(PS)_n$ can be of constant curvature is answered. $(P, S)_n$ with Codazzi type of Ricci tensor and conformally flat $(PS)_n$ are studied. In the last section it is proved that, under a certain condition, a simply connected conformally flat $(PS)_n$ $(n \geq 3)$ can be isometrically immersed in a Euclidean space E^{n+1} as a hypersurface.

1. Preliminaries. Let S and r denote the Ricci tensor of type (0, 2) and the scalar curvature respectively and L denote the symmetric endomorphism of the tangent space at each point, corresponding to the Ricci tensor, i.e.,

$$g(LX, Y) = S(X, Y) \text{ for any vector fields } X, Y. \tag{1.1}$$

Let B be the 1-form defined by

$$B(X) = A(LX). \tag{1.2}$$

From (1) we get

$$(\nabla_X S)(Y, Z) = 2A(X) S(Y, Z) + A(Y) S(X, Z) + A(Z) S(Y, X) + A(R(X, Y)Z) + A(R(X, Z) Y). \tag{1.3}$$

Contracting (1.3) we have

$$dr(X) = 2A(X) r + 4B(X). \tag{1.4}$$

From (1.4) we obtain

$$dr(X) A(Y) - dr(Y) A(X) = 4[A(Y) B(X) - A(X) B(Y)]. \tag{1.5}$$

Further, we get

(1.6) $$r\,dA(X, Y)+4[A(Y)\,B(X)-A(X)\,B(Y)]+2dB(X, Y)=0.$$

We shall use these formulas later.

2. Two-dimensional pseudo symmetric manifolds. In this section we consider a $(PS)_2$. It is known [1] that every (M^2, g) is a recurrent manifold whose 1-form of recurrence Q is given by

(2.1) $$Q(X)=X\,.\,(\log r).$$

Since $dQ(X, Y)=XQ(Y)-YQ(X)-Q([X, Y])$, it follows from (2.1) that $dQ(X, Y)=0$ which means that the 1-form Q is closed. Again since every (M^2, g) is an Einstein manifold we have

(2.2) $$S(X, Y)=(r/2)\,g(X, Y).$$

Therefore from (1.4) we get

(2.3) $$X\cdot r=2A(X)\,r+2rg(X, \rho)=4A(X)\,r \quad \text{by (2.2).}$$

(2.4) $$r[Q(X)-4A(X)]=0 \qquad \text{(by (2.1)).}$$

Now, in an (M^2, g), $r=0$ implies that the manifold is flat which is inadmissible by definition. Hence $r\neq 0$ and therefore from (2.4) we get

(2.5) $$A(X)=(1/4)\,Q(X).$$

Since Q is closed it follows from (2.5) that A is also so. We can therefore state the following theorem:

Theorem 1. *In a $(PS)_2$ the associated 1-form is closed.*

It is to be noted that in a $(PS)_2$ the scalar curvature r cannot be constant, because if it is constant, then from (2.3) it will follow that either $r=0$ or $A(X)=0$, none of which is admissible.

3. $(PS)_n$ $(n>2)$ of constant scalar curvature. If a $(PS)_n$ $(n>2)$ is of constant scalar curvature, then from (1.4) we get

(3.1) $$B(X)=-(r/2)\,A(X).$$

Again if (3.1) holds, then from (1.4) it follows that r is constant. Hence a $(PS)_n$ $(n>2)$ is of constant scalar curvature if and only if (3.1) holds.

Let us now suppose that (3.1) holds. Then from (1.1) we get, in virtue of (2).

$$S(\rho, \rho)=A(L\rho)=-(r/2)\;A(\rho)$$

Hence

(3.2) $$-S(\rho, \rho)/g(\rho, \rho)=r/2.$$

From (3.2) it follows that the Ricci curvature in the direction of ρ is $r/2$ [2]. This leads to the following theorem:

Theorem 2. *If a $(PS)_n$ $(n>2)$ is of constant scalar curvature, then the Ricci curvature in the direction of ρ defined by (2), is $r/2$.*

4. Einstein $(PS)_n$ $(n>2)$. It is known [3] that in an Einstein (M^n, g) $(n>2)$ the scalar curvature r is constant. Hence in an Einstein $(PS)_n$ $(n>2)$ we have $dr(X)=0$. Therefore from (1.4) we get

(4.1) $$A(X)\,r+2S(X, \rho)=0.$$

Since in an Einstein manifold (M^n, g) we have $S(X, Y)=r/n\;g(X, Y)$, (4.1) can be written as

$$A(X)\,r+(2r/n)\,A(X)=0, \text{ or,}$$

(4.2) $$[(n+2)/n]\,r(A(X))=0.$$

Since $A(X)$ cannot be zero, it follows from (4.2) that $r=0$. Hence we can state:

Theorem 3. *An Einstein $(PS)_k$ $(n>2)$ is of zero scalar curvature.*

If possible, let a $(PS)_n$ $(n>2)$ be of constant curvature. Then we have

(4.3) $$R(X, Y)\,Z=k[g(Y, Z)\,X-g(Z, X)\,Y].$$

Since every manifold of constant curvature is an Einstein manifold it follows from Theorem 3 that in a $(PS)_n$ $(n>2)$ of constant curvature, the scalar curvature $r=0$. But $r=0$ implies $k=0$. So from (4.3) it follows that $R(X, Y)\,Z=0$ which is inadmissible by definition. This leads to the following corollary of the above theorem:

Corollary 1. *A $(PS)_n$ $(n>2)$ cannot be of constant curvature.*

Since a 3-dimensional Einstein manifold is of constant curvature [3], it follows from the above corollary that a $(PS)_3$ cannot be an Einstein manifold. Hence we have another corollary which can be stated as follows:

Corollary 2. *Einstein $(PS)_3$ does not exist.*

5. $(PS)_n$ $(n>2)$ with Codazzi type of Ricci tensor.

In this section we consider a $(PS)_n$ $(n>2)$ in which the Ricci tensor is a Codazzi tensor [4] i.e.,

(5.1) $$(\nabla_X S)\,(Y, Z)=(\nabla_Z S)\,(Y, X).$$

Using (1.3) we get

(5.2) $$(\nabla_X S)\,(Y, Z)+(\nabla_Y S)\,(Z, X)+(\nabla_Z S)\,(X, Y)=4[A(X)\,S(Y, Z)+A(Y)\,S(Z, X)+A(Z)\,S(X, Y)].$$

In virtue of (5.1) the above equarion takes the form

(5.3) $$3(\nabla_X S)\,(Y, Z)=4[A(X)\,S(Y, Z)+A(Y)\,S(Z, X)+A(Z)\,S(X, Y)].$$

Contracting (5.3) we get

$$3\,dr(X)=4[A(X)\,r+2B(X)]=2\,dr(X) \quad \text{(by (1.4)).}$$

Hence

(5.4) $$dr(X)=0.$$

It is known [5] that in a Riemannian manifold (M^n, g) $(n>3)$

(5.5) $$(\text{div } C)\,(X, Y)\,Z=[(n-3)/(n-2)]\,[(\nabla_X S)\,(Y, Z)-(\nabla_Z S)\,(Y, X)+[1/[2(n-1)]]\,\{g(X, Y)\,dr(Z)-g(Y, Z)\,dr(X)\}],$$

where C denotes the conformal curvature tensor and div denotes divergence with respect to ∇. In virtue of (5.1) and (5.4) it follows from (5.5) that $(\text{div } C)\,(X, Y)\,Z=0$, which means that the tensor C is conservative [6]. Hence we can state the following theorem:

Theorem 4. *If in a* $(PS)_n$ $(n>3)$ *the Ricci tensor is a Codazzi tensor, then its conformal curvature tensor is conservative.*

We now consider a $(PS)_3$ in which the Ricci tensor is a Codazzi tensor. Let l be the tensor of type $(1,3)$ defined by

(5.6) $$l(X, Y)\,Z=(\nabla_X S)(Y, Z)-(\nabla_Z S)(Y, X)+[1/[2/(n-1)]]\,[g(Y, X)\,dr(Z)-g(Y, Z)\,dr(X)].$$

If the Ricci tensor is a Codazzi tensor, then in virtue of (5.1) and (5.4) it follows from (5.6) that $l(X, Y)\,Z=0$. But it is known [5] that an (M^3, g) is conformally flat if and only if $l(X, Y)\,Z=0$. We can therefore state the following theorem:

Theorem 5. *If in a* $(PS)_3$ *the Ricci tensor is a Codazzi tensor, then the manifold is conformally flat.*

6. Conformally flat $(PS)_n$ $(n\geq 3)$. It is known [5] that in a conformally flat (M^n, g) $(n\geq 3)$

$$(\nabla_X S)(Y, Z)-(\nabla_Z S)(Y, X)=[1/(2(n-1))]\,[dr(X)\,g(Y, Z)-dr(Z)\,g(X, Y)].$$

Hence in a conformally flat $(PS)_n$ $(n\geq 3)$, we have

(6.1) $$(\nabla_X S)(Y, Z)-(\nabla_Z S)(Y, X)=[1/(n-1)]\,[rA(X)\,g(Y, Z)-rA(Z)\,\rho(X, Y)+2B(X)\,g(Y, Z)-2B(Z)\,g(X, Y)] \quad (\text{by } (1.4)]).$$

Again from (1.3) we get

(6.2) $$(\nabla_X S)(Y, Z)-(\nabla_Z S)(Y, X)=3A(R(X, Z)\,Y)+A(X)\,S(Y, Z)-A(Z)\,S(X, Y).$$

From (6.1) and (6.2) we have

(6.3) $$3(n-1)'R(X, Y, Z, \rho)=-(n-1)\,A(X)\,S(Y, Z)+(n-1)\,A(Y)\,S(X, Z)+rA(X)\,g(Y, Z)-rA(Y)\,g(X, Z)+2B(X)\,g(Y, Z)-2B(Y)\,\rho(X, Z).$$

where $'R(X, Y, Z, W)=g(R(X, Y)\,Z, W)$.

Again, in a conformally flat (M^n, g) $(n\geq 3)$ we have

(6.4) $$'R(X, Y, Z, W)=[1/(2-n)][S(Y, Z)g(X, W)-S(X, Z)g(Y, W)+g(Y,Z)S(X, W)-g(X,Z)S(Y, W)]+[r/((n-1)(n-2))]\,[g(X,Z)g(Y, W)-g(Y,Z)g(X, W)].$$

From (6.4) we get

(6.5) $$3(n-1)'R(X, Y, Z, \rho)=[3(n-1)/(n-2)]\,[S(Y,Z)\,A(X)-S(X,Z)A(Y)+g(Y,Z)\,B(X)-g(X,Z)\,B(Y)]+[3r/(n-2)]\,[g(X,Z)\,A(Y)-g(Y,Z)\,A(X)].$$

From (6.3) and (6.5) we have

(6.6) $$(n-1)\,A(X)\,S(Y,Z)-(n-1)\,A(Y)\,S(X,Z)-\gamma A(X)\,g(Y,Z)+\gamma A(Y)g(X,Z)+B(X)g(Y,Z)-B(Y)g(X,Z)=0.$$

It is to be noted that in a conformally flat $(PS)_n$ the scalar curvature r cannot be zero, because if $r=0$ then it will follow from (1.4) that $B(X)=0$. As a result it will follow from (6.6) that $S(Y, Z)=0$ and consequently (6.4) will give $'R(X, Y, Z, W)=0$ which is inadmissible by definition.

Putting $Z=P$ in (6.6) we get

(6.7) $$A(X)\,B(Y)-A(Y)\,B(X)=0.$$

Therefore from (1.6) we have

(6.8) $$r\,dA(X, Y)+2dB(X, Y)=0.$$

Since $r\neq 0$ it follows from (6.8) that $dB(X, Y)$ is not, in general, equal to zero. So the 1-form B is not, in general, closed. If, however, the associated 1-form A is closed, then from (6.8) it follows that B is also closed.

From (6.7) we have

(6.9) $$B(X)=tA(X),$$

where t is a scalar.

Let us now suppose that t is a constant. Then using (6.9) it follows from (6.6) that

(6.10) $$S(Y, Z)=[(r-t)/(n-1)]\,g(Y, Z)+[(nt-r)/(n-1)]\,T(Y)T(Z),$$

where

(6.11) $$T(X)=\frac{A(X)}{\sqrt{A(\rho)}}.$$

Therefore (6.4) can be expressed as

(6.12) $$'R(X, Y, Z, W)=\sigma^2[g(Y, Z)\,g(X, W)-g(X, Z)\,g(Y, W)]+(\sigma\bar{\sigma}-\sigma^2)\,[g(X, W)\,T(Y)\,T(Z)-g(Y, W)\,T(X)\,T(Z)+g(Y, Z)\,T(X)\,T(W)-g(X, Z)\,T(Y)T(W)],$$

where

(6.13) $$\sigma=\sqrt{\frac{r-2t}{(n-1)(n-2)}} \quad \text{and} \quad \bar{\sigma}=t\sqrt{\frac{n-2}{(r-2t)(n-1)}}.$$

We now put

(6.14) $$D(Y, Z)=\sigma g(Y, Z)+(\bar{\sigma}-\sigma)\,T(Y)\,T(Z).$$

Then (6.12) can be written as

$$'R(X, Y, Z, W)=D(Y, Z)\,D(X, W)-D(X, Z)\,D(Y, W).$$

From this we get

(6.15) $$R(X, Y)\,Z=D(Y, Z)\,GX-D(X, Z)\,GY,$$

where

(6.16) $$g(GX, Y)=D(X, Y).$$

Contracting (6.12) we get

(6.17) $$S(Y, Z)=[\sigma^2(n-2)+\sigma\bar{\sigma}]\,g(Y, Z)+(n-2)\,\sigma(\bar{\sigma}-\sigma)\,T(Y)\,T(Z).$$

In virtue of (6.17) we can express (6.14) as

(6.18) $$(n-2)\,D(Y, Z)=(1/\sigma)\,S(Y, Z)-\bar{\sigma}g(Y, Z).$$

From (6.18) it follows that

(6.19) $$(\nabla_X D)(Y, Z)-(\nabla_Z D)(Y, X)=[T(Y)/\sigma]\,(\bar{\sigma}-\sigma)\,[(Z.\sigma)T(X)-(X.\sigma)\,T(Z)]$$

Now, from (6.13) we have $\sigma\bar{\sigma}=t/(n-1)$.

Therefore

(6.20) $(X\,.\,\sigma)\,\bar\sigma+(X\,.\,\bar\sigma)\,\sigma=0$ (since t is a constant).

Contracting (6.17) we get

$$r=(n-1)\,(n-2)\,\sigma^2+2(n-1)\,\sigma\bar\sigma.$$

Therefore

(6.21) $X\,.\,r-2(n-1)\,(X\,.\,\sigma)\,[(n-2)\,\sigma+\bar\sigma]=2(n-1)\,\sigma(X\,.\,\bar\sigma).$

In virtue of (1.4) and (6.20) it follows from (6.21) that

(6.22) $$(X\,.\,\sigma)=\frac{(r+2t)}{(n-1)(n-2)\sigma\sqrt{A(\rho)}}\,T(X).$$

Using (6.22) from (6.19) we get

(6.23) $(\nabla_X D)\,(Y,Z)-(\nabla_Z D)\,(Y,X)=0.$

Since $(\nabla_Z D)\,(X,Y)=g[(\nabla_Z G)\,(X),\,Y]$, the equation (6.23) implies that

(6.24) $(\nabla_Z G)\,(X)=(\nabla_X G)\,(Z).$

In view of the fact that the tensor field G of type (1,1) satisfies the equations (6.15) and (6.24), we can state the following theorem in virtue of the fundamental theorem for hypersurfaces [7].

Theorem 6. *If in a simply connected conformally flat $(PS)_n(n\geqslant 3)$ the scalar t given by (6.9) is a constant, then it can be isometrically immersed in a Euclidean space E^{n+1} as a hypersurface.*

It may be mentioned that the author has shown elsewehere that there exists an example of a manifold which is Pseudo symmetric but not locally symmetric [see [8] and [9]].

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27 Sashi Bhusan de Street
Calcutta — 700012
India

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PAIRS OF METRICS ON MANIFOLDS

BY

THEMIS KOUFOGIORGOS and CHRISTOS BAIKOUSSIS

1. **Introduction.** Let M be a connected n-manifold and suppose there is a pair of Riemannian metrics G and G' on M such that the sectional curvature of M with respect to G is never zero. On M there are two Riemannian connections ∇ and $\bar\nabla$ defined by G and G' respectively. If $(\nabla_X L)\,Y=(\nabla_Y L)\,X$ and $(\bar\nabla_X L^{-1})\,X=(\bar\nabla_Y L^{-1})\,X$ for all $X,\,Y\in TM$, where L is the linear transformation $G^{-1}\circ G'$, then the metrics G and G' are proportional. This generalizes a result of [1] which is valid for $n=2$. Realy our assumptions $(\nabla_X L)\,Y=(\nabla_Y L)\,X$ and $(\bar\nabla_X L^{-1})\,Y=(\bar\nabla_Y L^{-1})\,X$ are equivalent with the relations (2.1) of [1]. Also we note that our result generalizes a characterization of the n-sphere amongst hypersurfaces in Euclidean $(n+1)$-space.

2. **Preliminaries.** The general situation that we examine is an n-dimensional connected manifold M together with a pair of metrics on it, namely a pair of symmetric 2-covariant tensor fields on M, which are positive definite. We denote them by G (or $<,>$) and G' (or $<,>_{G'}$) and by ∇, $\bar\nabla$ the Riemannian connections of G and G', respectively. From this pair of tensor fields we form a (1,1) tensor field, or linear transformation field on M; the field may be thought of as $G^{-1}\circ G'$. In fact, at each point $p\in M$, both G and G' are symmetric bilinear forms on the tangent space T_pM at p. Hence, these bilinear forms give nonsingular linear maps from the tangent space T_pM of M, at each point p, to the cotangent space T_p^*M in the usual way. So, if we compose the inverse of the linear map G with the linear map G', we get a linear transformation on T_pM, which we denote by $G^{-1}\circ G'(p)$. We also denote the resulting linear transformation field on M by $L=G^{-1}\circ G'$.

3. **Main result.** We will prove the following

Theorem. *Let M be a connected n-manifold with two metrics G, G' such that the sectional curvature of M with respect to G is never zero. If $(\nabla_X L)\,Y=(\nabla_Y L)\,X$ and $(\bar\nabla_X L^{-1})\,Y=(\bar\nabla_Y L^{-1})\,X$ for all $X,\,Y\in TM$, then $G=cG'$, where $c\in R$ is a constant.*

Before proceeding, we shall give the geometrical meaning of this theorem, in the case where M is a convex hypersurface in Euclidean space E^{n+1} and G, G' are the first and the second fundamental forms, respectively. The relations

(*) $$(\nabla_X L)\,Y=(\nabla_Y L)\,X$$