Therefore

$$(X \cdot \sigma)\,\bar{\sigma} + (X \cdot \bar{\sigma})\,\sigma = 0 \quad \text{(since } t \text{ is a constant)}. \tag{6.20}$$

Contracting (6.17) we get

$$r = (n-1)(n-2)\,\sigma^2 + 2(n-1)\,\sigma\bar{\sigma}.$$

Therefore

$$X \cdot r - 2(n-1)(X \cdot \sigma)\,[(n-2)\,\sigma + \bar{\sigma}] = 2(n-1)\,\sigma(X \cdot \bar{\sigma}). \tag{6.21}$$

In virtue of (1.4) and (6.20) it follows from (6.21) that

$$(X \cdot \sigma) = \frac{(r+2t)}{(n-1)(n-2)\sigma\sqrt{A(\rho)}}\,T(X). \tag{6.22}$$

Using (6.22) from (6.19) we get

$$(\nabla_X D)(Y, Z) - (\nabla_Z D)(Y, X) = 0. \tag{6.23}$$

Since $(\nabla_Z D)(X, Y) = g[(\nabla_Z G)(X), Y]$, the equation (6.23) implies that

$$(\nabla_Z G)(X) = (\nabla_X G)(Z). \tag{6.24}$$

In view of the fact that the tensor field G of type (1,1) satisfies the equations (6.15) and (6.24), we can state the following theorem in virtue of the fundamental theorem for hypersurfaces [7].

Theorem 6. *If in a simply connected conformally flat $(PS)_n (n \geq 3)$ the scalar t given by (6.9) is a constant, then it can be isometrically immersed in a Euclidean space E^{n+1} as a hypersurface.*

It may be mentioned that the author has shown elsewehere that there exists an example of a manifold which is Pseudo symmetric but not locally symmetric [see [8] and [9]].

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PAIRS OF METRICS ON MANIFOLDS

BY

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1. **Introduction.** Let M be a connected n-manifold and suppose there
is a pair of Riemannian metrics G and G' on M such that the sectional cur-
vature of M with respect to G is never zero. On M there are two Riemannian
connections ∇ and $\bar{\nabla}$ defined by G and G' respectively. If $(\nabla_X L)\, Y = (\nabla_Y L)\, X$
and $(\bar{\nabla}_X L^{-1})\, X = (\bar{\nabla}_Y L^{-1})\, X$ for all $X, Y \in TM$, where L is the linear transfor-
mation $G^{-1} \circ G'$, then the metrics G and G' are proportional. This generalizes
a result of [1] which is valid for $n = 2$. Realy our asumptions $(\nabla_X L)\, Y = (\nabla_Y L)\, X$
and $(\bar{\nabla}_X L^{-1})\, Y = (\bar{\nabla}_Y L^{-1})\, X$ are equivalent with the relations (2.1) of [1].
Also we note that our result generalizes a characterization of the n-sphere
amongst hypersurfaces in Euclidean $(n+1)$-space.

2. **Preliminaries.** The general situation that we examine is an n-dimen-
sional connected manifold M together with a pair of metrics on it, namely
a pair of symmetric 2-covariant tensor fields on M, which are positive
definite. We denote them by G (or $<, >$) and G' (or $<, >_{G'}$) and by
∇, $\bar{\nabla}$ the Riemannian connections of G and G', respectively. From this
pair of tensor fields we form a (1,1) tensor field, or linear transformation
field on M; the field may be thought of as $G^{-1} \circ G'$. In fact, at each point
$p \in M$, both G and G' are symmetric bilinear forms on the tangent space
$T_p M$ at p. Hence, these bilinear forms give nonsingular linear maps from
the tangent space $T_p M$ of M, at each point p, to the cotangent space $T_p^* M$
in the usual way. So, if we compose the inverse of the linear map G with the
linear map G', we get a linear transformation on $T_p M$, which we denote by
$G^{-1} \circ G'(p)$. We also denote the resulting linear transformation field on
M by $L = G^{-1} \circ G'$.

3. **Main result.** We will prove the following

Theorem. *Let M be a connected n-manifold with two metrics G, G'
such that the sectional curvature of M with respect to G is never zero. If $(\nabla_X L)\, Y =$
$(\nabla_Y L)\, X$ and $(\bar{\nabla}_X L^{-1})\, Y = (\bar{\nabla}_Y L^{-1})\, X$ for all $X, Y \in TM$, then $G = cG'$, where
$c \in R$ is a constant.*

Before proceeding, we shall give the geometrical meaning of this
theorem, in the case where M is a convex hypersurface in Euclidean space
E^{n+1} and G, G' are the first and the second fundamental forms, respectively.
The relations

$$(\nabla_X L)\, Y = (\nabla_Y L)\, X \tag{*}$$

are the Mainardi-Codazzi equations for M, while $(\bar\nabla_X L^{-1})Y=(\bar\nabla_Y L^{-1})X$ are obtained by interchanging the metrics G and G' on (*). Finally, in this case the map L is the Weingarten map and the theorem states that M is an open subset of a hypersphere. For $n=2$ see [1].

It is easy to verify that $<X,Y>_{G'}=<LX,Y>$, $<LX,Y>=$ $=<X,LY>$ and $<(\nabla_X L)Y,Z>=<Y,(\nabla_X L)Z>$.

For the proof of the theorem we need the following two lemmas.

Lemma 1. *If* $(\nabla_X L)Y=(\nabla_Y L)X$ *then* $L(\bar\nabla_X Y-\nabla_X Y)=\frac{1}{2}(\nabla_X L)Y$.

Proof. Since $<X,Y>_{G'}=<LX,Y>$, $Z<X,Y>_{G'}=Z<LX,Y>$ implies

(1) $<L(\bar\nabla_Z X-\nabla_Z X),Y>+<L(\bar\nabla_Z Y-\nabla_Z Y),X>=<(\nabla_X L)Y,Z>$.

This identity remains true when X, Y and Z are replaced by Y, Z and X respectively. So we get

$$<L(\bar\nabla_Z Y-\nabla_Z Y),X>=<L(\bar\nabla_X Y-\nabla_X Y),Z>,$$

hence (1) can be rewritten as

$$<L(\bar\nabla_Y X-\nabla_Y X),Z>+<L(\bar\nabla_X Y-\nabla_X Y),Z>=<(\nabla_X L)Y,Z>.$$

Finally, the lemma is proved noting that $\bar\nabla_Y X-\nabla_Y X=\bar\nabla_X Y-\nabla_X Y$.

Lemma 2. *If* $(\nabla_X L)Y=(\nabla_Y L)X$ *and* $(\bar\nabla_X L^{-1})Y=(\bar\nabla_Y L^{-1})X$ *then* $(\nabla_X L)(L^{-1}Y)=(\nabla_Y L)(L^{-1}X)$.

Proof. From the relation $(\bar\nabla_X L^{-1})Y=(\bar\nabla_Y L^{-1})X$ we have

$$L\bar\nabla_X(L^{-1}Y)-L\bar\nabla_Y(L^{-1}X)=\bar\nabla_X Y-\bar\nabla_Y X.$$

Applying Lemma 1 we get

$$L(\nabla_X L^{-1})Y+\frac{1}{2}(\nabla_X L)(L^{-1}Y)-L(\nabla_Y L^{-1})X-\frac{1}{2}(\nabla_Y L)(L^{-1}X)=0.$$

Because $L(\nabla_X L^{-1})Y=-(\nabla_X L)(L^{-1}Y)$ we immediately obtain the conclusion.

Proof of the theorem. We assume that the theorem is not valid, that is there exists some point $p\in M$ such that $G(p)\neq cG'(p)$. Then we have $G\neq cG'$ in a neighbourhood U of p. It is well known [2, Theorem 2] that there exists an open, dense subset A of M where the eigenvalues $l_1,\dots,l_n$ of $L=G^{-1}\circ G'$ are smooth. We choose, in a neighbourhood W of $U\cap A$, a basis of smooth vector fields $X_1,\dots,X_n$ (eigenvectors) which simultaneously orthonormalizes G and orthogonalizes G'. So in this basis the linear transformation L has a diagonal matrix as well, with diagonal entries $l_1,\dots,l_n$. By Lemma 2 we have $(\nabla_{X_i}L)(L^{-1}X_j)=(\nabla_{X_j}L)(L^{-1}X_i)$ on W. But $L^{-1}X_i=$ $=\frac{1}{l_i}X_i$ and thus

(2) $(l_i-l_j)(\nabla_{X_i}L)X_j=0,\quad \forall\ i,j=1,\dots,n\quad (i\neq j).$

Case 1. Assume that $l_i\neq l_j$ for all $i,j=1,\dots,n$ $(i\neq j)$. Then the conditions (2) are fulfilled if

(3) $(\nabla_{X_i}L)X_j=0$ for all $i,j=1,\dots,n$ $(i\neq j)$.

Let now $\nabla_{X_i}X_j=\sum_{k=1}^{n}f_{ij}^k X_k$. From this and (3) we deduce

(4) $0=(\nabla_{X_i}L)X_j=(X_i l_j)X_j+\sum_k (l_j-l_k)f_{ij}^k X_k$ (no summation for j).

Thus

(5) $\nabla_{X_i}X_j=f_{ij}^j X_j$ for all i and j (no summation for j).

Now, we shall calculate the sectional curvature $R(X_i,X_j,X_i,X_j)$. Using (5) and $<X_i,X_j>=0$ we get $<\nabla_{X_i}\nabla_{X_j}X_j,X_i>=0$. In a similar way we obtain $<\nabla_{X_j}\nabla_{X_i}X_j,X_i>=0$ and $<\nabla_{[X_i,X_j]}X_j,X_i>=0$. Hence

$$R(X_i,X_j,X_i,X_j)=<R(X_i,X_j)X_j,X_i>=$$
$$=<\nabla_{X_i}\nabla_{X_j}X_j-\nabla_{X_j}\nabla_{X_i}X_j-\nabla_{[X_i,X_j]}X_j,X_i>=0\quad (i\neq j),$$

a contradiction, since the sectional curvature of M with respect to G is never zero.

Case 2. Now, we assume that at least two elements of the l_i's are equal. With no loss of generality, we suppose that $l_1=l_2=\dots=l_m$ for some $2\leq m<n$ and $l_i\neq l_j$ for all $i=1,\dots,m$ and $j=m+1,\dots,n$.
Using (4) we get

(6a) $<\nabla_{X_i}X_j,X_k>=0$ for all $i,k=1,\dots,m$ and $j=m+1,\dots,n$.

(6b) $<\nabla_{X_j}X_i,X_l>=0$ for all $i=1,\dots,m$ and $j,l=m+1,\dots,n$.

Differentiating the identity $<X_i,X_j>=0$ and using (6a, b) we obtain

$$<\nabla_{X_j}X_j,X_i>=0,\ <\nabla_{X_i}X_i,X_j>=0,\ 1\leq i\leq m,\ m+1\leq j\leq n$$

which yield $<\nabla_{X_i}X_i,\nabla_{X_j}X_j>=0$. So we have $<\nabla_{X_i}\nabla_{X_j}X_j,X_i>=0$. Also from (6a) for $k=i$, by differentiating with respect to X_j and using (6b) we get $<\nabla_{X_j}\nabla_{X_i}X_j,X_i>=0$. Finally

$$<\nabla_{[X_i,X_j]}X_j,X_i>=<\nabla_{\sum_k h_k X_k}X_j,X_i>=\sum_k h_k<\nabla_{X_k}X_j,X_i>=0.$$

Hence, $R(X_i,X_j,X_i,X_j)=0$, $1\leq i\leq m$, $m+1\leq j\leq n$), a contradiction.

Case 3. If $l_1=l_2=\dots=l_n=l$, then $(\nabla_{X_i}L)X_j=(X_i l)X_j$. Hence $(X_i l)X_j=(X_j l)X_i$ for all $i,j=1,\dots,n$. So $l=c=constant$, since $X_i l=0$ for all i.
Thus the assumption that M contains a point p such that $G(p)\neq cG'(p)$ gives a contradiction and the theorem is proved.

Let M be a Riemannian manifold with Riemannian metric G. If the sectional curvature of M is positive everywhere, then the Ricci tensor S defines another metric $G'=S$ on M.

As an application of the theorem we obtain, for the pair (G,G') of metrics G and $G'=S$, the following

Corollary. *Let M be a connected Riemannian n-manifold whose sectional curvature is positive everywhere. If $(\nabla_X L)Y = (\nabla_Y L)X$ and $(\bar{\nabla}_X L^{-1})Y = (\bar{\nabla}_Y L^{-1})X$ for all $X, Y \in TM$, then M is an Einstein manifold.*

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NEW LIFTING PROPERTIES OF WEAK FIBRATIONS

BY

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In this note we study weak versions of lifting properties introduced by A. Strom in [3], and further investigated in [2], to obtain new characterizations of weak fibrations in the sense of A. Dold (see [1] for definitions, where they are called „h-Faserungen").

Throughout this paper we work in the category of topological spaces and continuous maps.

Definition 1. *A map $p: E \to B$ has the weak right lifting property (WRLP) with respect to a map $i: A \to X$, if for every commutative square*

$$(^*) \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r, "g"'] & B \end{tikzcd}$$

in Top *there exists a map $h: X \to E$ such that $p \circ h = g$ and $h \circ i \underset{p}{\simeq} f$* (i.e. *there exists a homotopy H from $h \circ i$ to f such that $pH(a, t) = pH(a, o)$ for all $(a, t) \in A \times I$, in short a fiberhomotopy from $h \circ i$ to f*).

Let **G** denote the class of weak fibrations, **C** the class of closed cofibrations, **D** the class of weak cofibrations and **W** the class of homotopy equivalences (see [1] for definitions).

Theorem 1. *For a map $p: E \to B$, the following are equivalent:*

(i) $p \in \mathbf{G}$

(ii) *p has the WRLP w.r.t. all $i \in \mathbf{D} \cap \mathbf{W}$,*

(iii) *p has the WRLP w.r.t. all $i \in \mathbf{C} \cap \mathbf{W}$,*

(iv) *p has WRLP w.r.t. all $i \in \mathbf{W}$.*

Proof. (i)$\Rightarrow$(ii) : Consider a commutative square (*) with $p \in \mathbf{G}$ and $i \in \mathbf{D} \cap \mathbf{W}$. By [1] (2.14) and (2.29) $i(A)$ is a strong deformation retract of X, i.e. there exists a retraction $r: X \to A$ and a homotopy $\varphi: X \times I \to X$ from $i \circ r$ to 1_X, rel $i\,(A)$. Then $g \circ \varphi: X \times I \to B$ satisfies $g\varphi(x, 0) = gir\,(x) = pfr\,(x)$.

Since $p \in G$, by [1] (6.1) there exists a homotopy $F: X \times I \to E$ such that $p \circ F = g \circ \varphi$ and $F_0 \underset{p}{\simeq} f \circ r$.

Define $h: X \to E$ by $h(x) = F(x, 1)$, then $ph(x) = pF(x, 1) = g\varphi(x, 1) = g(x)$ and one verifies easily that the map