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MAXIMUM PRINCIPLE FOR CONTROL PROBLEMS GOVERNED BY BILINEAR FINITE DIMENSIONAL INEQUALITIES

BY

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This paper is concerned with necessary optimality conditions (maximum principle) for control problems governed by bilinear finite dimensional inequalities.

A simple example is also presented, in which the optimality conditions established are applied and the optimal pair is found.

1. Necessary optimality conditions. We consider the following control problem :

Problem (P_K) : *Minimize*

$$\int_0^T [g(t, y(t)) + h(u(t))]dt + \varphi_0(y(T)), \quad 0 < T < \infty, \tag{1.1}$$

over the set of all functions $y \in W^{1,2}([0, T] : R^n)$ and $u \in L^2(0, T)$ *subject to* :

$$y'(t) + (F + \partial I_K)(y(t)) + u(t)F_1(y(t)) \ni f(t) \quad a.e. \ t \in]0, T[, \tag{1.2}$$

$$y(0) = y_0 \in K, \tag{1.3}$$

where $K = \{z \in R^n : 0 \leqslant z_i \leqslant a_i, \ i = 1, 2, \ldots, n\}$.

Using the definition of the operator ∂I_K, the relations (1.2) and (1.3) can be equivalently written as :

$$\begin{aligned}
&y_i'(t) + (Fy(t))_i + u(t)(F_1 y(t))_i = f_i(t) \ \text{a.e.} \ \{t \in [0, T] ; \ 0 < y_i(t) < a_i\},\\
&y_i'(t) + (Fy(t))_i + u(t)(F_1 y(t))_i \geqslant f_i(t) \ \text{a.e.} \ \{t \in [0, T] ; \ y_i(t) = 0\},\\
&y_i'(t) + (Fy(t))_i + u(t)(F_1 y(t))_i \leqslant f_i(t) \ \text{a.e.} \ \{t \in [0, T] ; \ y_i(t) = a_i\},\\
&0 \leqslant y_i(t) \leqslant a_i, \ \text{a.e.} \ t \in]0, T[, \ y_i(0) = y_{0,i} \in [0, a_i].
\end{aligned} \tag{1.4}$$

where $i = 1, 2, \ldots, n$.

The elements which have appeared satisfy the following properties :

(i) $F : R^n \to R^n$ is locally Lipschitz and there exist $D_1, D_2 \in R_+$ such that

$$(Fy, y) \geqslant -D_1 \|y\|^2 - D_2 \ \text{for all} \ y \in R^n.$$

(ii) $F_1 : R^n \to R^n$ is a M-Lipschitzian function.

(iii) $$f \in W^{1,2}([0, T]; R^n).$$

(iv) $$\varphi_0 : R^n \to R_+.$$

(v) $h : R^n \to \overline{R}$ is convex, lower semicontinuous and there exist

$b_1 \in R_+^*$ and $b_2 \in R$ such that $h(u) \geq b_1 |u|^2 - b_2$ for all $u \in R^n$.

(vi) the function $g : [0, T] \times R^n \to R_+$ is measurable in t, $g(t, 0) \in L^\infty(0, T)$ and for every $\delta > 0$ there exists $L_\delta > 0$, independent of t, so that:

$$|g(t, y) - g(t, z)| + |\varphi_0(y) - \varphi_0(z)| \leq L_\delta \| y - z \|$$

for all $t \in [0, T]$ and $\|y\| + \|z\| \leq \delta$.

We may prove that if $u \in L^2(0, T)$ then the bilinear finite dimensional variational inequality (1.2) with the initial condition (1.3) has a unique solution $y \in W^{1,2}([0, T]; R^n)$, and the control problem (P_K) has, at least, one optimal pair.

In the following theorem the maximum principle for the control problem (P_K) is presented.

Theorem 1.1. *Let $(y^*, u^*) \in W^{1,2}([0, T]; R^n) \times L^2(0, T)$ be an arbitrary optimal pair for the (P_K) problem.*

Then there exist $p \in BV([0, T]; R^n)$, the dual extremal arc, and an element $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in L^\infty(0, T; R^n)$ so that:

(1.5) $$p_i'(t) - \left[\left(\frac{\partial F}{\partial y}(y^*(t))\right)^* p(t)\right]_i - u^*(t)\left[\left(\frac{\partial F_1}{\partial y}y^*(t)\right)^* p(t)\right]_i = \xi_i(t),$$

(1.6) $$\xi(t) \in \partial g(t, y^*(t)) \text{ a.e. } \{t \in [0, T];\ 0 < y_i^*(t) < a_i\},$$

(1.7) $$p_i(t) = 0 \text{ a.e. } \{t \in [0, T];\ y_i^*(t) = 0,\ (F(0))_i + u^*(t)(F_1(0))_i \neq f_i(t)\},$$

(1.8) $$p_i(t) = 0 \text{ a.e. } \{t \in [0, T];\ y_i^*(t) = a_i,\ (F(a))_i + u^*(t)(F_1(a))_i \neq f_i(t)\},$$

(1.9) $$p(T) + \partial\varphi_0(y^*(T)) \ni 0,$$

(1.10) $$-(p(t), F_1 y^*(t)) \in \partial h(u^*(t)) \text{ a.e. } t \in]0, T[.$$

where $i = 1, 2, \ldots, n$ and $a = (a_1, a_2, \ldots, a_n)$.

Proof. The idea of the proof consists in approximating the control problem (P_K) by a family of smooth problems (P_K^ε) for which the necessary conditions are immediate and passing to limit in the approximating equations (see [1]).

For every $\varepsilon > 0$ we construct the approximating control problems:

Problem (P_K^ε): *Minimize*

$$\int_0^T [g^\varepsilon(t, y(t)) + h(u(t)) + \tfrac{1}{2}|u(t) - u^*(t)|^2] dt + \varphi_0^\varepsilon(y(T))$$

over the set of all functions $y \in W^{1,2}([0, T]; R^n)$ and $u \in L^2(0, T)$ subject to:

$$\begin{cases} y'(t) + F^\varepsilon(y(t)) + \gamma^\varepsilon(y(t)) + u(t)F_1^\varepsilon(y(t)) = f(t), \text{ a.e. } t \in]0, T[\\ y(0) = y_0 \end{cases}$$

where $\gamma^\varepsilon : R^n \to R^n$ is defined by $\gamma^\varepsilon(y) = (\beta_1^\varepsilon(y_1), \ldots, \beta_n^\varepsilon(y_n))$ and $\beta_i^\varepsilon(y_i) = \int_R [\beta_{\varepsilon,i}(y_i - \varepsilon^2\theta) - \beta_{\varepsilon,i}(-\varepsilon^2\theta)]\rho(\theta)d\theta + \beta_{\varepsilon,i}(0)$, $i = 1, 2, \ldots, n$.

Here ρ is a C_0^∞-mollifier function on R and

$$\beta_{\varepsilon,i}(y_i) = \varepsilon^{-1}(y_i - P_K y_i) = \begin{cases} \varepsilon^{-1} y_i & \text{if } y_i \leq 0, \\ 0 & \text{if } 0 < y_i \leq a_i, \\ \varepsilon^{-1}(y_i - a_i) & \text{if } y_i > a_i. \end{cases}$$

By integration we obtain:

$$\beta_i^\varepsilon(y_i) = \frac{y_i - a_i}{\varepsilon}\int_{-\infty}^{\varepsilon^{-2}(y_i - a_i)} \rho(\theta)d\theta + \frac{y_i}{\varepsilon}\int_{\varepsilon^{-2}y_i}^{\infty} \rho(\theta)d\theta +$$

$$+ \varepsilon\left(\int_0^1 \theta\rho(\theta)d\theta - \int_{-\infty}^{\varepsilon^{-2}(y_i - a_i)} \theta\rho(\theta)d\theta - \int_{\varepsilon^{-2}y_i}^{\infty} \theta\rho(\theta)d\theta\right).$$

and by derivation we have:

$$\dot\beta_i^\varepsilon(y_i) = 1/\varepsilon \int_{-\infty}^{\varepsilon^{-2}(y_i - a_i)} \rho(\theta)d\theta + 1/\varepsilon \int_{\varepsilon^{-2}y_i}^{\infty} \rho(\theta)d\theta.$$

We have used the properties of the ρ function.

The function g^ε and φ_0^ε are defined by the relations:

$$g^\varepsilon(t, y) = \int_{R^n} g(t, y - \varepsilon^2\theta)\rho_n(\theta)d\theta,$$

$$\varphi_0^\varepsilon(y) = \int_{R^n} \varphi_0(y - \varepsilon^2\theta)\rho_n(\theta)d\theta,$$

where ρ_n is a C_0^∞-mollifier function on R^n.

Let $(y^\varepsilon, u^\varepsilon)$ be an optimal pair for (P_K^ε) and $p^\varepsilon \in BV([0, T]; R^n)$ the corresponding dual extremal arc, which satisfies the relations:

(1.11) $$\frac{dp^\varepsilon}{dt}(t) - \left(\frac{\partial F^\varepsilon}{\partial y}(y^\varepsilon(t))\right)^* p^\varepsilon(t) - \nabla\gamma^\varepsilon(y^\varepsilon(t))p^\varepsilon(t) - u^\varepsilon(t)\left(\frac{\partial F_1^\varepsilon}{\partial y}(y^\varepsilon(t))\right)^* p^\varepsilon(t) =$$
$$= \nabla g^\varepsilon(t, y^\varepsilon(t)) \text{ a.e. } t \in]0, T[.$$

(1.12) $$p^\varepsilon(T) + \nabla\varphi^\varepsilon(y^\varepsilon(T)) = 0.$$

(1.13) $$-(p^\varepsilon(t), F_1^\varepsilon(y^\varepsilon)t)) \in \partial h(u^\varepsilon(t)) + u^\varepsilon(t) - u^*(t) \text{ a.e. } t \in]0, T[.$$

If $u^\varepsilon \to u^*$ strongly in $L^2(0, T)$, we may prove that for $\varepsilon \to 0$, we have the following convergences:

(1.14) $y^\varepsilon \to y^*$ strongly in $C([0, T]; R^n)$; weakly in $W^{1,2}([0, T]; R^n)$, (1.15) $\gamma^\varepsilon(y^\varepsilon)\to f-y^{*\prime}-Fy^*-u^*F_1y^*$ weakly in $L^2(0, T; R^n)$.

(1.16) $\nabla g^\varepsilon(t, \dot{y}^\varepsilon(t))\to\xi(t)\in\partial g(t, y^*(t))$, weakly in $L^\infty(0, T; R^n)$ a.e. $t\in]0, T[$, (see [1], p. 182).

(1.17) $$\nabla\varphi_0^\varepsilon(y^\varepsilon(T))\to\xi^0\in\partial\varphi_0(y^*(T)).$$

(1.18) $$\frac{\partial F^\varepsilon}{\partial y}(y^\varepsilon(t))\to\xi^1(t)\in\frac{\partial F}{\partial y}(y^*(t)) \text{ weakly in } L^2(0, T; R^{n^2})$$

a.e. $t\in]0, T[$, (see [3])

(1.19) $$\frac{\partial F_1^\varepsilon}{\partial y}(y^\varepsilon(t))\to\xi^2(t)\in\frac{\partial F_1}{\partial y}(y^*(t)), \text{ weakly in } L^2(0, T; R^{n^2}) \text{ a.e. } t\in]0, T[.$$

Taking the scalar product of equation (1.11) with $p^\varepsilon(t)$, integrating on the interval $[t, T]$ and using Gronwall's lemma, we get $\|p^\varepsilon\|_{C([0T];R^n)}\leqslant C$ for every $\varepsilon>0$.

Taking the scalar product of equation (1.11) with $q^\varepsilon=(\operatorname{sgn} p_1^\varepsilon, \ldots, \operatorname{sgn} p_n^\varepsilon)$ and integrating on the interval $[0, T]$ we obtain the inequality:

(1.20) $$\sum_{i=1}^n\int|\beta_i^\varepsilon(y_i^\varepsilon(t))p_i^\varepsilon(t)|dt\leqslant C, \text{ for every } \varepsilon>0.$$

Thus, the sequence $\{\nabla\gamma^\varepsilon(y^\varepsilon)p^\varepsilon\}_{\varepsilon>0}$ is bounded in $L^1(0, T; R^n)$.

Since the sequence $\left\{\frac{dp^\varepsilon}{dt}\right\}_{\varepsilon>0}$ is bounded in $L^1(0, T; R^n)$ and the sequence $\{p^\varepsilon\}$ is bounded in $L^\infty(0, T; R^n)$ we infer, by Helly's theorem, that there exists $p\in BV([0, T]; R^n)$ and a subsequence of $\{\varepsilon\}$, renoted $\{\varepsilon\}$, so that $p^\varepsilon(t)\to p(t)$ for every $t\in[0, T]$.

We infer:

(1.21) $$p^\varepsilon\to p \text{ strongly in } L^r(0, T; R^n),\ 1\leqslant r<\infty.$$

Thus, from (1.21), (1.18) and (1.19) it results:

(1.18′) $$\left(\frac{\partial F^\varepsilon}{\partial y}(y^\varepsilon(t))\right)^* p^\varepsilon(t)\to\left(\frac{\partial F}{\partial y}(y^*(t))\right)^* p(t) \text{ a.e. } t\in]0, T[.$$

(1.19′) $$\left(\frac{\partial F_1^\varepsilon}{\partial y}(y^\varepsilon(t))\right)^* p^\varepsilon(t)\to\left(\frac{\partial F_1}{\partial y}(y^*(t))\right)^* p(t) \text{ a.e. } t\in]0, T[.$$

Relation (1.20) insures the existence of a measure $\mu=(\mu_1, \mu_2, \ldots, \mu_n)\in((L^\infty(0, T))^*)^n$ so that on a generalized subsequence of $\{\varepsilon\}$, renoted $\{\varepsilon\}$, we have:

(1.22) $$\beta_i^\varepsilon(y_i^\varepsilon)p_i^\varepsilon\to\mu_i \text{ weakly in } (L^\infty(0,T))^* \text{ for } i=1, 2, \ldots, n.$$

Then making ε to tend to 0 in equations (1.11), (1.12) and (1.13) it follows that p satisfies equations (1.9), (1.10) and

$$p_i'(t)-\left[\left(\frac{\partial F}{\partial y}(y^*(t))\right)^* p(t)\right]_i-\mu_i-u^*(t)\left[\left(\frac{\partial F_1}{\partial y}(y^*(t))\right)^* p(t)\right]_i\in L^\infty(0, T)$$

a.e. $t\in]0, T[$ $i=1, 2, \ldots, n$.

$$p'(t)-\left(\frac{\partial F}{\partial y}(y^*(t))\right)^* p(t)-\mu-u^*(t)\left(\frac{\partial F_1}{\partial y}(y^*(t))\right)^* p(t)\in\partial g(t, y^*(t)) \text{ a.e. } t\in]0, T[.$$

In this particular case we may show that:

(1.22′) $\beta_i^\varepsilon(y_i^\varepsilon)p_i^\varepsilon\to\mu_i=0$ strongly on $L^1(A_i)$ for $i=1, 2, \ldots, n$, where $A_i=\{t\in[0, T]; 0<y_i^*(t)<a_i\}$.

Using the convergences (1.22′) and (1.14) we obtain: $\mu_i y_i^*=0$ so that:

(1.23) $$\left(p'(t)-\left[\left(\frac{\partial F}{\partial y}(y^*(t))\right)^* p(t)\right]_i-u^*(t)\left[\left(\frac{\partial F_1}{\partial y}(y^*(t))\right)^* p(t)\right]_i-\xi_i(t)\right)y_i^*(t)=0$$

a.e. $t\in]0, T[$, $i=1, 2, \ldots, n$.

where $\xi=(\xi_1, \xi_2, \ldots, \xi_n)\in L^\infty(0, T; R^n)$ satisfies the equation $\xi(t)\in\partial g(t, y^*(t))$ a.e. $t\in]0, T[$. From (1.23) it results (1.5) on the A_i set.

We may demonstrate (see [3]) that:

(1.24) $$p_i^\varepsilon\beta_i^\varepsilon(y_i^\varepsilon)\to 0 \text{ strongly on } L^1(B_i),\ i=1, 2, \ldots, n \text{ where}$$
$$B_i=\{t\in[0, T]; y_i^*(t)=0\}.$$

Now, relations (1.24), (1.15) and (1.21) lead to:

$$p_i(t)(f_i(t)-(F(0))_i-u^*(t)(F_1(0))_i)=0 \quad i=1, 2, \ldots, n$$

which implies (1.7) relation for $f_i(t)\neq(F(0))_i+u^*(t)(F_1(0))_i$.

By a similar argument we obtain relation (1.8).

2. Example. *Problem* (P_1):

maximize $\int_0^T y(t)dt$

over the set of all functions $y\in W^{1,2}(0, T)$ *and* $u\in U$ *subject to*:

(2.1) $$\begin{cases} y'(t)+\partial I_K(y(t))-u(t)y(t)\ni 0 \text{ a.e. } t\in]0, T[,\\ y(0)=y_0\in K,\end{cases}$$

where $U=\{u\in L^2(0, T); 0\leqslant u(t)\leqslant\gamma, \forall t\in[0, T]; \int_0^T u(t)dt=\delta\}$ *with* $0\leqslant\delta\leqslant\gamma T$ *and* $K=\{z\in R; 0\leqslant z\leqslant a\}$, $a>0$.

This problem is a particular case of the problem (P_K), for $n=1$, $F=0$, $F_1=-\mathrm{Id}_R$, the identity function, $g(\cdot, y)=-y$, $f\equiv\varphi_0\equiv 0$ and $h=I_U$, the indicator function of the set U.

Remark 2.1. a) If $y_0=0$ the variational inequality (2.1) admits only the trivial solution $y\equiv 0$ on $[0, T]$.

b) If $y_0=a$ the variational inequality (2.1) admits only the constant solution $y\equiv a$ on $[0, T]$.

Let $(y^*, u^*)\in W^{1,2}(0, T)\times U$ be an optimal pair for the control problem (P_1)

For $y_0\in]0, a[$ we can distinguish the following cases :

Case 2.1.

If $y_0\in]0, ae^{-\delta}]$ then we can see that the equation :

$$\text{(2.1)}' \qquad \begin{cases} y'(t)-u(t)y(t)=0 \text{ on }]0, T[, \\ y(0)=y_0 \end{cases}$$

has the solution $y_u(t)=y_0\exp(\int_0^t u(s)ds)$ which is also solution for the initial state system (2.1).

The adjoint state system for the pair (y^*, u^*) is :

$$\text{(2.2)} \qquad p'(t)+u^*(t)p(t)+1=0 \text{ a.e. } t\in]0, T[,$$

$$\text{(2.3)} \qquad p(T)=0.$$

$$\text{(2.4)} \qquad p(t)y(t)\in \partial h(u^*(t)) \text{ a.e. } t\in]0, T[.$$

From the relation (2.4) we have assured the existence of an element $\rho\in R$ such that :

$$\text{(2.5)} \qquad \begin{aligned} &u^*(t)=\gamma && \text{a.e. } \{t\in]0, T[\, ;\ p(t)y^*(t)>\rho\}. \\ &0<u(t)<\gamma && \text{a.e. } \{t\in]0, T[\, :\ p(t)y^*(t)=\rho\}. \\ &u(t)=0 && \text{a.e. } \{t\in]0, T[\, :\ p(t)y^*(t)<\rho\}. \end{aligned}$$

The relations (2.1)$'$, written for y^*, together with (2.2) imply :

$$(p(t)y^*(t))'=-y^*(t)\leqslant -y_0<0 \text{ a.e. on }]0, T[,$$

which means that py^* is decreasing (so that $py\geqslant 0$ because $p(T)=0$).

By (2.5) there is $t_1\in]0, T[$ so that the control is :

$$\text{(2.6)} \qquad u^*(t)=\begin{cases} \gamma \text{ a.e. } t\in]0, t_1[, \\ 0 \text{ a.e. } t\in]t_1, T[. \end{cases}$$

where $t_1=\delta/\gamma$ is determined from the condition $\int_0^T u^*(t)dt=\delta$; the optimal state is :

$$\text{(2.7)} \qquad y^*(t)=\begin{cases} y_0e^{\gamma t} & t\in \left[0, \dfrac{\delta}{\gamma}\right] \\ y_0e^{\delta} & t\in \left[\dfrac{\delta}{\gamma}, T\right] \end{cases}$$

and the cost functional has the optimal value :

$$\text{(2.8)} \qquad \int_0^T y^*(t)dt=e^{\delta}\left(\frac{y_0}{\gamma}(1-\delta)+y_0T\right)-\frac{y_0}{\gamma}.$$

Case 2.2.

If $y_0\in]ae^{-\delta}, a[$, then for every $u\in U$ there is $t_u\in]0, T]$, $t_u=\inf\{t\in]0, T[, y_0\exp(\int_0^t u(s)ds=a\}$ so that the solution of (2.1) is :

$$\text{(2.9)} \qquad y_u(t)=\begin{cases} y_0\exp(\int_0^t u(s)ds), & 0\leqslant t\leqslant t_u, \\ a & , t_u\leqslant t\leqslant T. \end{cases}$$

We note that $t^*=t_{u^*}<T$. Indeed, if $t^*=T$, then $y^*<a$ on $[0, T[$ and so the state system and the adjoint system remain the same as in the Case 2.1. Therefore, from (2.7) we have :

$$y^*(t^*)=y^*(T)=y_0e^{\delta}.$$

which contradicts the fact that $y^*(t^*)=a<y_0e^{\delta}$.

In the following, we shall prove that :

$$\text{(2.10)} \qquad u^*(t)=\gamma \text{ a.e. } t\in]0, t^*[.$$

Assume, by contradiction, that (2.10) is not true, i.e. there exists a measurable set $B\subset]0, t^*[$ with $\text{mes}(B)>0$ such that :

$$\text{(2.11)} \qquad \int_B u^*(t)dt<\gamma\,\text{mes}(B).$$

We shall prove that (2.11) implies

$$\text{(2.12)} \qquad u^*=0 \text{ a.e. on }]t^*, T[.$$

Assume that this is not the case. Note that on the interval $[t^*, T[$, u^* can be modified without any modification of y^*. Therefore, by taking B with mes(B) small enough, we can construct, starting from u^*, the following control :

$$\tilde{u}(t)=\begin{cases} \gamma & \text{a.e. } t\in B, \\ u^*(t) & \text{a.e. } t\in]0, t^*[\,/\,B, \\ \alpha(t) & \text{a.e. } t\in]t^*, T[, \end{cases}$$

where α is chosen in such a way that $\tilde{u}\in U$. Then it is easy to see that :

$$\int_0^T y_{\tilde{u}}(t)dt>\int_0^T y^*(t)dt.$$

which contradicts the optimality of the pair (y^*, u^*). Therefore, (2.11) implies (2.12).

But (2.12) leads to

$$\int_0^{t^*} u^*(t)\mathrm{d}t = \delta \text{ and so } a = y^*(t^*) = y_0 e^{\delta},$$

which is a contradiction, showing that (2.10) is valid.

Now, we have $y^*(t) = y_0 e^{\delta t}$ on $[0, t^*]$.

From $a = y^*(t^*) = y_0 e^{\delta t^*}$ it follows that $t^* = \gamma^{-1} \ln a/y_0$.

The control u^* is not identically zero on $[t^*, T]$ because

$$\int_{t^*}^{T} u^*(t)\mathrm{d}t = \gamma t^* = \ln a/y_0 < \delta.$$

On this interval we can take any u^* such that : $\int_{t^*}^{T} u^*(t)\mathrm{d}t = \delta - \ln a/y_0$, for example :

$$u^*(t) = \begin{cases} \gamma & \text{a.e. } t \in]t^*, t^* + \gamma^{-1}(\delta - \ln a/y_0)[\\ 0 & \text{a.e. } t \in]t^* + \gamma^{-1}(\delta - \ln a/y_0), T[. \end{cases} \tag{2.13}$$

Therefore, one of the optimal control is :

$$u(t) = \begin{cases} \gamma & \text{a.e. } t \in]0, \delta/\gamma[, \\ 0 & \text{a.e. } t \in]\delta/\gamma, T]. \end{cases} \tag{2.14}$$

The optimal corresponding state is given by :

$$y^*(t) = \begin{cases} y_0 e^{\delta t} & t \in [0, \gamma^{-1} \ln a/y_0], \\ a & t \in [\gamma^{-1} \ln a/y_0, T]. \end{cases} \tag{2.15}$$

and the optimal value of the cost functional is :

$$\int_0^T y^*(t)\mathrm{d}t = \gamma^{-1}(a(1 + T\gamma - \ln a/y_0) - y_0). \tag{2.16}$$

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THE MANIFOLD OF RIEMANNIAN METRICS ON A COMPACT MANIFOLD WITH BOUNDARY

BY

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0. **Introduction.** The manifold $\mathcal{M}$ of Riemannian structures, which a given smooth orientable manifold M, without boundary may carry is con- sidered by D. Ebin in [3] and N. Koiso in [9]. The action of the group $\mathcal{D}$ of diffeomorphisms of the manifold on $\mathcal{M}$ is analysed and through this analysis one begins to know something about the space $\mathcal{M}/\mathcal{D}$ of equivalent Riemannian structures. The posibility of extension of these results to the case where the manifold M has boundary is suggested by D. Ebin in the note [3] of his paper. This extension entails the utilization of the technics of ellip- tic boundary value probleme and elliptic regularizations (see, e.g. [1], [8] and [6]).

In the present note we consider the situation $\partial M \neq \emptyset$ and we construct a slice for the action of the diffeomorphism group on the manifold $\mathcal{M}$. Moreo- ver we consider the manifold $\mathcal{M}_0$ of the Riemannian metrics on M which induce the same Riemannian metric on ∂M and we construct a slice for the action of the group $\mathcal{D}_0 = \{\eta \in D, \eta(x) = x$ for all $x \in \partial M\}$, on $\mathcal{M}_0$.

Notations. Let $(M, \partial M)$ be a n-dimensional compact, oriented, C^∞ manifold with boundary ∂M, TM be its tangent bundle, T^*M be the cotan- gent bundle, S^2T^* be the bundle of symmetric covariant 2-tensors. If E is a vector bundle over M, $C^k(E)(0 \leqslant k < \infty)$ will be the Banach space of C^k-sections of E with C^k-topology and $C^\infty(E) = \bigcap_{k=0}^{\infty} C^k(E)$ will be the Fréchet space with the C^∞-topology. Also, for $s \geqslant 0$ an integer, we denote by $H^s(E)$ the Hilbert space of sections of E whose (distributional) derivatives up to order s are square integrable in charts (see [4], [11], [12]). By Sobolev embedding theo- rem, if $k \geqslant 0$ and $s > (n/2) + k$, $H^s(E) \subset C^k(E)$ and the inclusion is continuous.

The set $\mathcal{M}$ of Riemannian structures on M is, by definition, the open set of all sections in $C^\infty(S^2T^*)$ which induce a positive definite inner product on each tangent space T_p, $p \in M$. We denote by $\mathcal{M}_0$ the subset of the Rieman- nnian structures of M which induce the same Riemannian structure on the boundary ∂M.

Let $\mathcal{D}$ denote the group of C^∞-diffeomorphisms on M and $\mathcal{D}_0 = \{\eta \in \mathcal{D}$ $\eta(x) = x$, for all $x \in \partial M\}$.

For $s > n/2 + 1$, let.