But (2.12) leads to

$$\int_0^{t^*} u^*(t)\mathrm{d}t=\delta \text{ and so } a=y^*(t^*)=y_0e^{\delta}.$$

which is a contradiction, showing that (2.10) is valid.

Now, we have $y^*(t)=y_0e^{\delta t}$ on $[0, t^*]$.

From $a=y^*(t^*)=y_0e^{\delta t^*}$ it follows that $t^*=\gamma^{-1}\ln a/y_0$.

The control u^* is not identically zero on $[t^*, T]$ because

$$\int_{t^*}^{T} u^*(t)\mathrm{d}t=\gamma t^*=\ln a/y_0<\delta.$$

On this interval we can take any u^* such that :

$\int_{t^*}^{T} u^*(t)\mathrm{d}t=\delta-\ln a/y_0$, for example :

$$u^*(t)=\begin{cases}\gamma & \text{a.e. } t\in\,]t^*, t^*+\gamma^{-1}(\delta-\ln a/y_0)[\\ 0 & \text{a.e. } t\in\,]t^*+\gamma^{-1}(\delta-\ln a/y_0), T[.\end{cases}\tag{2.13}$$

Therefore, one of the optimal control is :

$$u(t)=\begin{cases}\gamma & \text{a.e. } t\in\,]0, \delta/\gamma[,\\ 0 & \text{a.e. } t\in\,]\delta/\gamma, T].\end{cases}\tag{2.14}$$

The optimal corresponding state is given by :

$$y^*(t)=\begin{cases}y_0e^{\delta t} & t\in[0, \gamma^{-1}\ln a/y_0],\\ a & t\in[\gamma^{-1}\ln a/y_0, T].\end{cases}\tag{2.15}$$

and the optimal value of the cost functional is :

$$\int_0^T y^*(t)\mathrm{d}t=\gamma^{-1}(a(1+T\gamma-\ln a/y_0)-y_0).\tag{2.16}$$

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Polytechnic Institute
Department of Mathematics
6600 – Iaşi, România

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THE MANIFOLD OF RIEMANNIAN METRICS ON A COMPACT MANIFOLD WITH BOUNDARY

BY

LILIANA MAXIM-RĂILEANU

0. **Introduction.** The manifold $\mathcal{M}$ of Riemannian structures, which a given smooth orientable manifold M, without boundary may carry is considered by D. Ebin in [3] and N. Koiso in [9]. The action of the group $\mathcal{D}$ of diffeomorphisms of the manifold on $\mathcal{M}$ is analysed and through this analysis one begins to know something about the space $\mathcal{M}/\mathcal{D}$ of equivalent Riemannian structures. The posibility of extension of these results to the case where the manifold M has boundary is suggested by D. Ebin in the note [3] of his paper. This extension entails the utilization of the technics of elliptic boundary value probleme and elliptic regularizations (see, e.g. [1], [8] and [6]).

In the present note we consider the situation $\partial M\neq\emptyset$ and we construct a slice for the action of the diffeomorphism group on the manifold $\mathcal{M}$. Moreover we consider the manifold $\mathcal{M}_0$ of the Riemannian metrics on M which induce the same Riemannian metric on ∂M and we construct a slice for the action of the group $\mathcal{D}_0=\{\eta\in D,\ \eta(x)=x$ for all $x\in\partial M\}$ on $\mathcal{M}_0$.

Notations. Let $(M, \partial M)$ be a n-dimensional compact, oriented, C^∞ manifold with boundary ∂M, TM be its tangent bundle, T^*M be the cotangent bundle, S^2T^* be the bundle of symmetric covariant 2-tensors. If E is a vector bundle over M, $C^k(E)(0\leqslant k<\infty)$ will be the Banach space of C^k-sections of E with C^k-topology and $C^\infty(E)=\bigcap_{k=0}^{\infty}C^k(E)$ will be the Fréchet space with the C^∞-topology. Also, for $s\geqslant 0$ an integer, we denote by $H^s(E)$ the Hilbert space of sections of E whose (distributional) derivatives up to order s are square integrable in charts (see [4], [11], [12]). By Sobolev embedding theorem, if $k\geqslant 0$ and $s>(n/2)+k$, $H^s(E)\subset C^k(E)$ and the inclusion is continuous.

The set $\mathcal{M}$ of Riemannian structures on M is, by definition, the open set of all sections in $C^\infty(S^2T^*)$ which induce a positive definite inner product on each tangent space T_p, $p\in M$. We denote by $\mathcal{M}_0$ the subset of the Riemannian structures of M which induce the same Riemannian structure on the boundary ∂M.

Let $\mathcal{D}$ denote the group of C^∞-diffeomorphisms on M and $\mathcal{D}_0=\{\eta\in\mathcal{D},\ \eta(x)=x$, for all $x\in\partial M\}$.

For $s>n/2+1$, let.

$$\mathcal{D}^s=\{\eta : M\to M,\ \eta \text{ is a } H^s\text{-map, is a bijection and } \eta^{-1} \text{ is a } H^s\text{-map}\},$$

$$D_0^s=\{\eta\in\mathcal{D}^s,\ \eta(x)=x \text{ for all } x\in\partial M\}.$$

In [4] Lemma 6.6 is proved that if M is embedded in its double $\tilde{M}$ then $D^s\subset H^s(M,\tilde{M})$ is a submanifold, as is $\mathcal{D}_0^s\subset\mathcal{D}^s$ for $s>n/2+1$. So, $\mathcal{D}$ and $\mathcal{D}_0$ have the structure of ILH-Lie groups with Lie algebras the vector fields on M tangent to $\partial M(X(x)\in T_x\partial M$ for all $x\in\partial M)$ and those zero on ∂M (denoted by $\mathcal{X}_{11}$ and $\mathcal{X}_0$ respectively) Also $\mathcal{D}_0$ is an ILH-subgroup of $\mathcal{D}$.

Let $\mathcal{X}^s=H^s(TM)$ denote the Hilbert space of H^s vector fields on M and

$$\mathcal{X}_{11}^s=\{X\in\mathcal{X}^s,\ X(x)\in T_x\partial M, \text{ for all } x\in\partial M\},$$

$$\mathcal{X}_0^s=\{X\in\mathcal{X}^s,\ X(x)=0 \text{ for all } x\in\partial M\}.$$

$\mathcal{D}$ acts on $C^\infty(S^2T^*)$ by „pull-back" and we denote this right action by the map :

$$A:\mathcal{D}\times C^\infty(S^2T^*)\to C^\infty(S^2T^*).$$

Let $A_0=A|_{\mathcal{D}_0\times C^\infty(S^2T^*)}$. $\mathcal{M}$ is invariant under A and, obvious, $\mathcal{M}_0$ is invariant under A_0 so that we have :

$$A:\mathcal{D}\times\mathcal{M}\to\mathcal{M},\quad A_0:\mathcal{D}_0\times\mathcal{M}_0\to\mathcal{M}_0.$$

If $g\in M$, let

$$\mathcal{D}_g=\{\eta\in\mathcal{D},\ A(\eta,g)=g\},\quad \mathcal{D}_g^0=\{\eta\in\mathcal{D}_0,\ A_0(\eta,g)=g\},$$

be the isotropy groups of the actions A and A_0. Let $A(\mathcal{D},g)=O_g$ be the orbit of $g\in\mathcal{M}$ under the action A so that O_g is the set of metrics isometric to g. Let $A_0(\mathcal{D}_0,g)=O_{g,0}$ be the orbit of $g\in\mathcal{M}_0$ under the action of $\mathcal{D}_0$.

If $\partial M=\varnothing$, D. Ebin proves in [3] that $O_g\subset\mathcal{M}$ is a smooth submanifold of $\mathcal{M}$. We shall prove the same for O_g and $O_{g,0}$ as well as the existence of a slice for the actions A and A_0.

§ 1. **The slice for the action A.** To avoid the technical difficulties related to the analysis on Fréchet manifolds, we work in the corresponding Sobolev spaces. We always assume that $s>n/2+1$. So $\mathcal{D}^s$ is a manifold and a topological group in which right multiplication is C^∞ and the action A extends to an action :

$$A^s:\mathcal{D}^{s+1}\times H^s(S^2T^*)\to H^s(S^2T^*)$$

and this extended action is continuous. Also it is linear in its second variable and, if $g\in C^\infty(S^2T^*)$, the mapping $A^s|_{\mathcal{D}^{s+1}\times\{g\}}$ is smooth. Since $\mathcal{M}^s=H^s(S^2T^*)\cap\cap C^0\mathcal{M}$ is an invariant subset of $H^s(S^2T^*)$ under the action A^s we can define an action $A^s:\mathcal{D}^{s+1}\times\mathcal{M}^s\to\mathcal{M}^s$ and we construct a slice for this action.

Theorem 1. *Let $g\in\mathcal{M}^{s+r}$, $r\geq 1$, and O_g^s, be its orbit under the action A^s. Then $O_g^s\subset\mathcal{M}^s$ is a C^r-closed submanifold with tangent space at g given by*

$$T_g(O_g^s)=\{L_Xg,\ X\in\mathcal{X}_{11}^{s+1}\}.$$

Proof. Let $\psi_g:\mathcal{D}^{s+1}\to\mathcal{M}^s$ be the orbit map through g, $\psi_g(\eta)=A^s(\eta,g)$. Then ψ_g is a C^r map and has tangent at the identity given by :

$$T_{id}\psi_g=\alpha_g:\mathcal{X}_{11}^{s+1}\to H^s(S^2T^*),\quad \alpha_g(X)=L_Xg,$$

and at $\eta\in\mathcal{D}^{s+1}$ is α_g extended to be right invariant. Now the following result, pointed out in [5] is useful :

Lemma. *Let $f:N\to P$ be a C^r-map of Banach manifolds, $r\geq 1$, such that $\operatorname{Ker} Tf\subset TN$ is a C^r-subbundle of TN and for each $x\in N$, Range T_xf is closed with a closed complement. Then $f(N)$ is a (locally) immersed submanifold of P and, if f is open onto its image, $f(N)\subset P$ is a submanifold.*

Since $O_g^s=\psi_g(\mathcal{D}^{s+1})$, we need to verify that ψ_g satisfies the conditions of this lemma. Firstly, we consider the following boundary value problem :

$$\mathcal{F}:C^\infty(TM)\to C^\infty(S^2T^*)+C^\infty(N(\partial M)),\quad \mathcal{F}(X)=(D(X),p(X)),$$

where $D(X)=L_Xg$, $N(\partial M)$ is the vector bundle over ∂M, normal to ∂M with respect to g, and $p:C^\infty(TM)\to C^\infty(N(\partial M))$ is the differential operator at the boundary of the order 0, $p(X)=\pi_n(X)|_{\partial M})$, with $\pi_n:C^\infty(TM|_{\partial M})\to\to N(\partial M)$ the canonical bundle projection. Let $D':C^\infty(S^2T^*)\to C^\infty(TN)$ be the formal adjoint of D. But $D(X)=L_Xg=2(\delta'\circ\theta_g(X)$, δ' being the formal adjoint of the operator $\delta:C^\infty(S^2T^*)\to C^\infty(T^*M)$, $(\delta T)_i=-\Sigma\nabla^l T_{li}$, $T\in C^\infty(S^2T^*)$ and $\theta_g:C^\infty(TM)\to C^\infty(T^*M)$, is the canonical isomorphism defined by g (see [2]). Hence $D'=2\theta_g^{-1}\circ\delta$.

We choose coordinates $x^1,\dots,x^{n-1}$, x^n for M so that $M=\{x^n\geq 0\}$ and $\partial M=\{x^n=0\}$ and we can make a volume element μ on M, $\mu=dx^1\wedge\dots\wedge dx^n$ and ν the unit normal covector, $\nu=dx^n$. In this local coordinate system the expression of the operator $E=D'\circ D$ is the following (we dont denote terms of lower degrees) :

$$[E(X)]^h=\alpha_k^{hlq}\frac{\partial^2X^k}{\partial x^l\partial x^q}+\dots$$

$$\alpha_k^{hlq}=2(g^{lq}\delta_k^h+g^{hl}\delta_k^q).$$

For the choise of the coordinates $\{X^1,\dots,X^n\}$ of the bundle TM agreeing with the decomposition $TM|\partial M=T(\partial M)+N(\partial M)$, the boundary operator p will be of the following form :

$$(p(X))^n=X^n.$$

Now we consider the following boundary value problem :

$$\mathcal{E}:C^\infty(TM)\to C^\infty(TM)\oplus C^\infty(N(\partial M),\quad \mathcal{E}(X)=(E(X),p(X)),$$

which is an elliptic boundary problem on the compact manifold M in the sense of [7] Ch. X, § 10.5 ($m=2$, $\mu=1$). So, for $s\geq 2$ $\mathcal{E}$ extends to a continuous linear map :

$$\mathcal{E}^s:H^s(TM)\to H^{s-2}(TM)\oplus H^{s-1/2}(N(\partial M))$$

and, applying the Theorem 10.5.1 from [7] we obtain that:

a) Ker $\mathcal{E}^s$ is finite dimensional and Ker $\mathcal{E}^s \subset C^r(TM)$;

b) The range of the operator $\mathcal{E}^s$ in $H^{s-2}(TM)\oplus H^{s-1/2}(N(\partial M))$ is closed with finite codimension.

Since Ker $\mathcal{E}^{s+1}$ = Ker $D^{s+1}|_{\mathfrak{X}^{s+1}_{1}}$ = Ker α_g we can conclude that Ker α_g is finite dimensional and hence $\mathcal{D}_g$ is a Lie group. Now, Ker $T_e\psi_g$ is Ker α_g extended to be right invariant over $\mathcal{D}^{s+1}$ so that by the arguments of D. Ebin and J. Marsden ([4], Appendix A) Ker ψ_g is a finite dimensional C^r-subbundle of $T\mathcal{D}^{s+1}$. On the other hand, the elliptic regularizations a), b) of E^s and the conclusions related to the equation (13) on page 447, [8], permit to prove, following the methods from [2], that the range of the operator

$D^{s+1}|_{\mathfrak{X}^{s+1}_1} = \alpha_g$ is closed with a closed complement

Ker $D'^{s+2}|_{\operatorname{Im} D^{s+1} \cap \mathcal{D}(D^{s-1})}$ where, as in [8], $\mathcal{D}_{(D^{s+2})^*}$

denote the domain of the Hilbert space adjoint $(D^{s+2})^*$ of D^{s+2}. To finish the proof we need to verify that ψ_g is open onto its image and has closed range. Since the proof proceeds in the same way as [3] prop. 6.13 we shall omit the details.

Now, in analogy with the slice theorem for the action of $\mathcal{D}^{s+1}$ on $\mathcal{M}$ in the case $\partial M = \varnothing$ ([3]), using the previous result and the Riemannian structures for the space of metrics defined in [3], we obtain the slice theorem for the action of $\mathcal{D}^{s+1}$ on $\mathcal{M}^s$. We can also allow $s=\infty$.

§ 2. The slice for the action A_0. First we show that $\mathcal{M}_0$ is a closed submanifold in $\mathcal{M}$. Indeed, let $\mathcal{M}^0$ be the manifold of riemannian structures on ∂M and $f: M\to M^0$ be the mapping defined by $f(g) = g|_{\partial M}$. Then $\mathcal{M}_0 = f^{-1}(g_0)$ where g_0 is a fixed Riemannian metric on ∂M. Since the tangent at a point $g\in\mathcal{M}$ of the mapping f is obviously a surjection of $C^r(S^2T^*)\to C^r(S^2(T\partial M)^*)$ we obtain that $\mathcal{M}_0$ is a closed submanifold in $\mathcal{M}$. As in § 1 we consider the extended action of A_0:

$$A_0^s: \mathcal{D}_0^{s+1}\times\mathcal{M}_0^s \to \mathcal{M}_0^s$$

and we construct a slice for this action.

Theorem 2. *Let $g\in\mathcal{M}_0^{s+r}$, $r\geq 1$, and $O^s_{g,0}$ be its orbit under the action A_0^s. Then $O^s_{g,0}\subset M_0^s$ is a C^r closed submanifold with tangent space at g given by:*

$$T_g(O^s_{g,0}) = \{L_Xg,\ X\in\mathfrak{X}_0^{s+1}\}.$$

Proof. Let $\psi_g^0: \mathcal{D}_0^{s+1}\to\mathcal{M}_0^s$ be the orbit map through g, $\psi_g^0(\eta) = A_0^s(\eta, g)$. ψ_g^0 is a C^r map and has tangent at the identity given by

$$T_{id}\psi_g^0 = \alpha_g^0: \mathfrak{X}_0^{s+1}\to H^s(S^2T^*),\quad \alpha_g^0(X) = L_Xg.$$

and at $\eta\in\mathcal{D}_0^{s+1}$ is ψ_g^0 extended to be right invariant. As in the proof of the Theorem 1, to verify that ψ^0_g satisfies the conditions of the previous lemma, we consider the following boundary value problem:

$$F_0: C^\infty(TM)\to C^\infty(S^2T^*)\oplus C^\infty(N(\partial M))\oplus C^\infty(T(\partial M)),$$

$$\mathcal{F}_0(X) = (D(X), p(X), q(X)).$$

with the operators D and p as in the proof of Theorem 1 and $q: C^\infty(TM)\to C^\infty(T(\partial M))$ the boundary operator of the order 0, $q(X) = \pi_t(X)|_{\partial M}$, $\pi_t: C^\infty(TM|_{\partial M})\to C^\infty(T(\partial M))$ being the mapping induced by the canonical bundle projection.

Let

$$\mathcal{E}^0: C^\infty(TM)\to C^\infty(TM)\oplus C^\infty(N(\partial M))\oplus C^\infty(T(\partial M))$$

$$\mathcal{E}^0(X) = (E(X), p(X), q(X)).$$

be the associated boundary problem, $E = D'\circ D$.

We shall show that this boundary value problem is an elliptic boundary value problem in the sense from [6]. Using the notations from the paper [6] we have:

1) $\alpha_k^{hnn} = a_k^h = 2(g^{nn}\delta_k^h + g^{hn}\delta_k^n)$, 3) $\alpha_k^{hln} = b_k^{hl} = 2(g^{ln}\delta^h_k + g^{hl}\delta^n_k)$ and

2) $a_{kj} = g_{hj}a_k^h = 2(g^{nn}g_{kj} + \delta_k^n\delta_j^n)$, 4) $b^l_{kj} = b_k^{hl}g_{hj} = 2(g^{ln}g_{kj}+\delta^l_j\delta^n_k)$.

So, the second and the fifth condition are obvious.

Since the symbol $\sigma_D(\xi)$ is injective, the first condition follows from [11] (Corollary, p. 75). The condition 3 is satisfied because $a_{\alpha\beta}(\alpha,\beta = \overline{1,n-1})$ is the representative of the metric on $N(\partial M)$ and a_{nn} is the representative of the metric on $T(\partial M)$, while $a_{\alpha n}=0$. The condition 4 which says that one can integrate by parts to find a bilinear form $Q(X,Y)$ such that:

$$\ll E(X), Y\gg = \int_M \langle E(X), Y\rangle\mu = Q(X,Y) \text{ when } q(X)=p(Y)=0$$

is also obvious.

By a direct calculus there results that the self-adjoint elliptic boundary value problem $\mathcal{E}^0$ satisfies a persuasive estimate or subelliptic inequality:

$$|X|_0^2 \leq \mathrm{Re} \ll EX, X\gg + \|X\|_0^2$$

when $p(X)=q(X)=0$. (In the previous inequality $\|X\|_n$ is L_2 norm over M of X and its partial derivatives of degree n or less, $|X|_n$ is the Sobolev norm on ∂M which measures the L_2 norm over ∂M of $X|_{\partial M}$ and its partial derivatives of degree n or less in directions tangent to the boundary only and $\leq$ is the convention of Kohn and Folland).

Now we can apply for the problem $\mathcal{E}^0$ the theorem, proved by J. Kohn and L. Nirenberg in [8], in the formulation from [6], §3.5 and we obtain that $\mathcal{E}^0$ has finite dimensional kernel and closed range with finite codimension. Moreover dim Ker $\mathcal{E}^0$ = Codim Im $\mathcal{E}^0$. Using the theorems 2.6.4 and 2.6.5 from [7] we obtain also that for all $s\geq 2$, $\mathcal{E}^0$ extends to a continuous linear map:

$$\mathcal{E}^{0,s}: H^s(TM)\to H^{s-2}(TM)\oplus H^{s-1/2}(N(\partial M))\oplus H^{s-1/2}(T(\partial M)),$$

so that $\mathcal{E}^{0,s}$ has finite dimensional kernel and closed range with finite codimension. Since Ker $\mathcal{E}^{0,s+1}$ = Ker $\mathcal{D}^{s+1}|_{\mathfrak{X}_0^{s+1}}$ = Ker α_g^0 we can conclude that Ker α_g^0 is finite dimensional and hence $\mathcal{D}_g^0$ is a Lie group. Ker $T_e\psi_g^0$ is Ker α_g^0 extended to be right invariant over $\mathcal{D}_0^{s+1}$ so that Ker ψ_g^0 is a finite dimensional C^r-subbundle of $T\mathcal{D}_0^{s+1}$.

The previous elliptic regularization and the statement involving the equation (13) from [8] permit to prove that the range of the operator $\mathcal{D}^{s+1}|_{\mathcal{X}_0^{s-1}} = \alpha_g^0$ is closed with a closed complement $\mathrm{Ker}\, D'^{s+2}|_{\mathrm{Im} D^{s+2}|_{\mathcal{X}_0^{s+3}}}$.

The utilisation of the manifold $\mathcal{M}_0^s$ instead of $\mathcal{M}^s$ in Prop. 6.13 from [3] causes no difficulty in the proof so that this completes the proof of the theorem.

Now, proceeding along the standard lines of the Ebin Slice theorem it is straightforward to show that the action of $\mathcal{D}_\bullet^{s+1}$ on $\mathcal{M}_0^s$ has a slice.

Remarks : 1. The usual consequences of the existence of a slice, such as generically trivial isotropy groups, follow directly.

2. A slice for the action of the conformorphism group $\mathcal{C}$ on $\mathcal{M}$ in the case $\partial M = \varnothing$ is constructed by A. Fischer and J. Marsden in [5]. Now it is not difficult to extend this result to the case where M has boundary.

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Seminarul Matematic „Al. Myller"
Universitatea „Al. I. Cuza" Iaşi
6600, Iaşi, România

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RULED REAL HYPERSURFACES IN QUATERNIONIC PROJECTIVE SPACE

BY

A. MARTINEZ

1. **Introduction.** Let QP^m be the $4m$-dimensional quaternionic projective space and M a real hypersurface of QP^m. We say that M is a *ruled* real hypersurface if there is a foliation of M by quaternionic hyperplanes QP^{m-1} (see Definition 1). This type of submanifolds has the following property :

(*) All the quaternionic sectional curvatures are constant 4.

In this paper we obtain examples of minimal ruled real hypersurfaces of QP^m and we study the real hypersurfaces of QP^m with constant quaternionic sectional curvatures.

2. **Preliminaries.** Let Q be the algebra of quaternions. In Q^{m+1} we consider the usual symplectic product given by

$$(a, b) = \Sigma_{i=0}^{m} \bar{a}_i b_i \tag{2.1}$$

for any $a(a_0, \ldots, a_m)$, $b(b_0, \ldots, b_m)$ in Q^{m+1}, where by $\bar{a}_i$ we denote the quaternionic conjugate of a_i.

Let $S^{4m+3} = \{q \in Q^{m+1} | g_0(q, q) = 1\}$ be the unit sphere in Q^{m+1}, where $g_0 = \mathrm{Re}(\,,\,)$ is the usual Euclidean metric of Q^{m+1}. It is known, [2, 4], that the $4m$-dimensional quaternionic projective space can be obtained from S^{4m+3} by identifying q to λq, $\lambda \in Q$, $|\lambda| = 1$. Thus S^{4m+3} is a fibre bundle over QP^m with structural group S^3 and projection π. If $q \in S^{4m+3}$, the horizontal subspace of $T_q S^{4m+3}$ is $T'_q = \{p \in Q^{m+1} | (p, q) = 0\}$ and the vertical subspace is spanned by $j_1 q$, $j_2 q$ and $j_3 q$, where j_1, j_2 and j_3 are the unit quaternions.

We shall consider on QP^m the metric given by

$$g\,(X, Y) = g_0(X', Y') \tag{2.2}$$

and the connection

$$\bar{\nabla}_X Y = (\pi)_*(D_{X'} Y'). \tag{2.3}$$

for nay $X, Y \in TQP^m$, where $'$ denotes the corresponding horizontal lift and D is the covariant differentiation of S^{m+3}.

If over QP^m we consider the 3-dimensional vector bundle V of tensors of type (1, 1) with local basis of almost Hermitian structures $\{J_1, J_2, J_3\}$ given by