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A P-COMPLETE MENGER STRUCTURE FOR DISTRIBUTION FUNCTIONS SPACE

BY

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1. **Introduction.** Let (X, F, T) be a Menger space (see [5]); for every $\lambda\in[0, 1)$ we define $d_\lambda : X\times X\to R_+$ letting $d_\lambda(x, y)=\inf\{a>0 : F_{xy}(a)\geq\lambda\}$. Let $\delta=\{d_\lambda : \lambda\in[0, 1)\}$; then (δ, T) is a probabilistic pseudometric on X. Conversely, for every probabilistic pseudometric (δ, T) which separates the points of a set X, there exists a Menger metric F which induces (δ, T) (see [3]). If $F_{xy}(a)>\lambda$ then $d_\lambda(x, y)<a$ and $d_\lambda(x, y)<a$ implies $F_{xy}(a)\geq\lambda$, for every $x, y\in X$, $\lambda\in[0, 1)$, $d_\lambda\in\delta$, $a>0$. For every $\lambda\in(0, 1]$ let $\mathcal{U}_\lambda$ be the semiuniformity induced by the family $\{d_\mu : \mu<\lambda\}$ and let $\mathcal{U}=\{\mathcal{U}_\lambda : \lambda\in I=[0, 1]\}$, where $\mathcal{U}_0=\{X\times X\}$. Then $(\mathcal{U}, T)$ is the probabilistic uniformity induced by (F, T) (see [2]). We remark that $\mathcal{U}_1$ is the uniformity defined by B. Schweizer, A. Sklar and E. Thorp in [7].

Let us denote by $\mathcal{C}$ the set of all sequences on X. For every $\lambda\in[0, 1]=I$, we say that $\chi\in\mathcal{C}$ is λ-convergent (λ-Cauchy) if χ converges in $(X, \mathcal{U}_\lambda)$ (χ is a Cauchy sequence in $(X, \mathcal{U}_\lambda)$).

For every $\chi\in\mathcal{C}$ and $x\in X$ let $p(\chi, x)=\sup\{\lambda\in I : \chi \text{ is } \lambda\text{-conv.}\}$ and $c(\chi)=\sup\{\lambda\in I : \chi \text{ is } \lambda\text{-Cauchy}\}$. Then $p(\chi, x)=\lim\limits_{a\to 0}\underline{\lim}_n F_{x_n x}(a)$ and $c(\chi)=\lim\limits_{a\to 0}\underline{\lim}_{m, n} F_{x_m x_n}(a)$ (see [4,5,7]).

For every $\chi\in\mathcal{C}$ and $x\in X$ we have:

$$T(p(\chi, x), p(\chi, x))\leq c(\chi).$$

We say that a Menger space (X, F, T) is *probabilistic complete* (p-complete) if for every $\chi\in\mathcal{C}$ there exists $x\in X$ such that:

$$T(c(\chi), c(\chi))\leq p(\chi, x).$$

We remark that a Menger space (X, F, T) is p-complete if and only if for every $\chi\in\mathcal{C}$ there exists $x\in X$ such that if χ is λ-Cauchy then χ is $T(\lambda, \lambda)$-convergent to x. Therefore, if (X, F, T) is p-complete then $\mathcal{U}_1$ is a complete uniformity for X. Hence, every p-complete Menger space is a complete Menger space in Sherwood's sense (see [8], [9]).

The purpose of this paper is to define a p-complete Menger structure on the distribution functions space $\mathcal{D}$ (section 2.). Using this Menger structure, we obtain, in section 3, some results concerning the continuity of a probabilistic metric (every probabilistic metric is $\mathcal{D}$-valued). These results generalize the theorem 5.4 of [4] and the theorem 8.2 of [6].

2. The Menger structure of distribution functions space. Let $\mathcal{D}$ be the set of all distribution functions, i.e. $F\in\mathcal{D}$ iff F is a nondecreasing, left-continuous function on R which satisfies

$$F(-\infty)=\lim_{a\to-\infty} F(a)=0 \quad\text{and}\quad F(+\infty)=\lim_{a\to+\infty} F(a)=1.$$

For every $\lambda\in[0, 1]$ we define $L_\lambda:\mathcal{D}\times\mathcal{D}\to R_+$ letting:

$$\begin{aligned}(1)\qquad L_\lambda(F, G)=\inf\{\varepsilon>0:\ & F(a)\leqslant G(a+\varepsilon)+\varepsilon+1-\lambda,\\ & G(a)\leqslant F(a+\varepsilon)+\varepsilon+1-\lambda,\ \forall a\in R\}.\end{aligned}$$

Let $L=\{L_\lambda:\lambda\in[0, 1)\}$ and $T_m: I\times I\to I$, $T_m(a, b)=\max(a+b-1, 0)$.

Theorem 1. *(L, T_m) is a left-continuous probabilistic pseudometric on $\mathcal{D}$.*

Proof. First, we remark that, for every $a\in R$, $\lambda\in[0, 1)$, $F(a)\leqslant G(a+\lambda)+1$ and $G(a)\leqslant F(a+\lambda)+1$, hence $L_\lambda(F, G)\leqslant\lambda$ for every $\lambda\in[0, 1)$. It follows directely from definition that L is a family of semipseudometrics on X and $L_0(F, G)=0$, for every $F, G\in\mathcal{D}$. Obviously, $L_\lambda\leqslant L_\mu$ for every $\lambda\leqslant\mu$. Let $\lambda_n\uparrow\lambda$ and $\delta_1>\sup\limits_n L_{\lambda_n}(F, G)$; for every $n\in N$ and $a\in R$, $F(a)\leqslant G(a+\delta_1)+\delta_1+1-\lambda_n$ and $G(a)\leqslant F(a+\delta_1)+\delta_1+1-\lambda_n$. Then, for every $a\in R$, we have $F(a)\leqslant G(a+\delta_1)+\delta_1+1-\lambda$ and $G(a)\leqslant F(a+\delta_1)+\delta_1+1-\lambda$, hence $\delta_1\geqslant L_\lambda(F, G)$. Therefore $L(F, G)$ is a left-continuous function, for every $F, G\in\mathcal{D}$.

Now, for every $\lambda, \mu\in[0, 1)$ with $\lambda+\mu\geqslant 1$, $F, G, H\in\mathcal{D}$ and for every $\varepsilon_1>L_\mu(F, G)$, $\varepsilon_2>L_\lambda(G, H)$ we have $F(a)\leqslant G(a+\varepsilon_1)+\varepsilon_1+1-\mu$, $G(a)\leqslant F(a+\varepsilon_1)+\varepsilon_1+1-\mu$ and $G(a)\leqslant H(a+\varepsilon_2)+\varepsilon_2+1-\lambda$, $H(a)\leqslant G(a+\varepsilon_2)+\varepsilon_2+1-\lambda$, for every $a\in R$. Then $F(a)\leqslant H(a+\varepsilon_1+\varepsilon_2)+\varepsilon_1+\varepsilon_2+1-T_m(\lambda, \mu)$ and $H(a)\leqslant F(a+\varepsilon_1+\varepsilon_2)+\varepsilon_1+\varepsilon_2+1-T_m(\lambda, \mu)$, so that $\varepsilon_1+\varepsilon_2\geqslant L_{T_m(\lambda,\mu)}(F, H)$. Therefore $L_{T_m(\lambda,\mu)}(F, H)\leqslant L_\mu(F, G)+L_\lambda(G, H)$. If $\lambda+\mu<1$ then $L_{T_m(\lambda,\mu)}(F, H)=L_0(F, H)=0\leqslant L_\mu(F, G)+L_\lambda(G, H)$. Hence (L, T_m) is a left-continuous probabilistic pseudometric on X (see [3]).

Remarks. 1. L_1 is Lévy's metric on $\mathcal{D}$.

2. If $(\mathcal{U}, T_m)$, $\mathcal{U}=\{\mathcal{U}_\lambda:\lambda\in I\}$ is the probabilistic uniformity induced by (L, T_m), then $\mathcal{U}_1$ induces the weak topology on $\mathcal{D}$.

3. If $L_\lambda(F, G)=0$, for every $\lambda\in[0, 1)$, then $L_1(F, G)=0$, hence $F=G$. So, (L, T_m) satisfies the condition (PP3) from [3, prop. 1]. Then there exists a Menger metric on $\mathcal{D}$ which induces (L, T_m) (see [3, theorem 1]); it follows immediately that this metric is defined by:

$$(2)\qquad \mathcal{F}(F, G)(a)=\min\{\inf_{b\in R}\min\{G(a+b)-F(b), F(a+b)-G(b)\}+a+1, 1\}$$

for every $a>0$ and $\mathcal{F}(F, G)(a)=0$ for every $a\leqslant 0$. We remark that, if $\mathcal{F}(F, G)(a)=1$ then $L_1(F, G)\leqslant a$ and if $L_1(F, G)<a$ then $\mathcal{F}(F, G)(a)=1$. Taking into account the probabilistic interpretation of Menger metrics, we conclude that Lévy's metric L_1 is an ideal metric: the semipseudometrics L_λ, $\lambda\in[0, 1)$, give the nearness degree between functions of $\mathcal{D}$ in comparison with this ideal metric. A similar probabilistic metric on $\mathcal{D}$ is introduced by U. Höhle in [10]; see also [11].

Theorem 2. *For every $\lambda\in[0, 1]$, $(F_n)_{n\in N}\subseteq\mathcal{D}$, $F\in\mathcal{D}$ we have $L_\lambda(F_n, F)\to 0$ if and only if*

$$(3)\qquad F(a)-(1-\lambda)\leqslant \varliminf_n F_n(a)\leqslant \varlimsup_n F_n(a)\leqslant F(a)+(1-\lambda),\ \textit{for all continuity}$$

points a of F.

Proof. First, suppose that $L_\lambda(F_n, F)\to 0$. Then, for every $\varepsilon>0$, there exists $n_0\in N$ such that, for every $n\geqslant n_0$ and $a\in R$, we have $F_n(a)\leqslant F(a+\varepsilon)+\varepsilon+1-\lambda$ and $F(a-\varepsilon)\leqslant F_n(a)+\varepsilon+1-\lambda$, so that $\varlimsup\limits_n F_n(a)\leqslant F(a+\varepsilon)+\varepsilon+1-\lambda$ and $\varliminf\limits_n F_n(a)\geqslant F(a-\varepsilon)-\varepsilon-(1-\lambda)$. If a is a continuity point of F then we take the limit of both inequalities as $\varepsilon\to 0$ and we obtain (3).

Conversely, suppose that (3) is fulfiled. Because $F\in\mathcal{D}$, for every $\varepsilon>0$ there exist $a, b\in R$, continuity points of F, such that $F(b)>1-\varepsilon/2$, $F(a)<\varepsilon/2$. Now, let $I_k=(c_k-\varepsilon, c_k)$, where c_k is a continuity point of F, for every $k=1, 2, \ldots, l$, such that $[a, b]\subseteq\bigcup\limits_{k=1}^{l} I_k$. From (3), there exists $n_0\in N$ such that for every $n\geqslant n_0$ we have $F(c)\leqslant F_n(c)+\varepsilon/2+1-\lambda$ and $F_n(c)\leqslant F(c)+\varepsilon/2+1-\lambda$, for every $c\in\{a, b, c_1, c_2, \ldots, c_l\}$. Let $n\geqslant n_0$ and $c\in R$; we have the following three situations:

a) If $c\leqslant a$ then $F(c)\leqslant F(a)<\varepsilon/2\leqslant F_n(c+\varepsilon)+\varepsilon+1-\lambda$ and $F_n(c)\leqslant F_n(a)\leqslant F(a)+\varepsilon/2+(1-\lambda)<\varepsilon+1-\lambda\leqslant F(c+\varepsilon)+\varepsilon+1-\lambda$.

b) If $a<c<b$ then there exists k such that $c\in(c_k-\varepsilon, c_k)$. Hence $F(c)\leqslant F(c_k)\leqslant F_n(c_k)+\varepsilon/2+1-\lambda<F_n(c+\varepsilon)+\varepsilon+1-\lambda$ and $F(c+\varepsilon)\geqslant F(c_k)\geqslant F_n(c_k)-\varepsilon/2-(1-\lambda)\geqslant F_n(c)-\varepsilon/2-(1-\lambda)>F_n(c)-\varepsilon-(1-\lambda)$.

c) If $c\geqslant b$ then $F_n(c+\varepsilon)\geqslant F_n(b)\geqslant F(b)-\varepsilon/2-(1-\lambda)>1-\varepsilon-(1-\lambda)\geqslant F(c)-\varepsilon-(1-\lambda)$ and $F(c+\varepsilon)\geqslant F(c)\geqslant F(b)>1-\varepsilon/2\geqslant F_n(c)-\varepsilon-(1-\lambda)$. Therefore, for every $n\geqslant n_0$, $L_\lambda(F_n, F)\leqslant\varepsilon$, so that $L_\lambda(F_n, F)\to 0$.

We say that a set $\mathcal{A}\subseteq\mathcal{D}$ is a *λ-tight*, $\lambda\in[0, 1]$, if for every $\varepsilon>0$ there exist $a, b\in R$, $a<b$, such that:

$$F(a)<1-\lambda+\varepsilon \text{ and } F(b)>\lambda-\varepsilon, \text{ for every } F\in\mathcal{A}.$$

We remark that a 1-tight is a tight (see [1, p. 37]).

The following is a generalization of theorem of relative compactness in $\mathcal{D}$ with the weak topology.

Theorem 3. *Let $\mathcal{A}\subseteq\mathcal{D}$ and $\lambda\in[0, 1]$; $\mathcal{A}$ is λ-tight if and only if every sequence of elements in $\mathcal{A}$ has a subsequence which is L_λ-convergent.*

Proof. First, we suppose that $\mathcal{A}$ is a λ-tight in $\mathcal{D}$ and let $(F_n)_{n\in N}\subseteq\mathcal{A}$ be a sequence of distribution functions. From Helly's theorem there exist a subsequence $(F_{k_n})_{n\in N}$ of $(F_n)_{n\in N}$ and a bounded, nondecreasing, left-continuous function $\tilde{F}$ such that $(F_{k_n})_{n\in N}$ converges to $\tilde{F}(c)$ as $n\to\infty$, for every continuity point c of $\tilde{F}$. Let $\tilde{F}(+\infty)=\lim\limits_{a\to+\infty}\tilde{F}(a)$ and $\tilde{F}(-\infty)=\lim\limits_{a\to-\infty}\tilde{F}(a)$. It is obvious that $0\leqslant\tilde{F}(-\infty)\leqslant\tilde{F}(+\infty)\leqslant 1$. For every $\varepsilon>0$ there exist $a, b\in R$, $a<b$, such that $F_{k_n}(a)<1-\lambda+\varepsilon$ and $F_{k_n}(b)>\lambda-\varepsilon$, for all $n\in N$. Hence $\tilde{F}(-\infty)\leqslant 1-\lambda$ and $\tilde{F}(+\infty)\geqslant\lambda$. We remark that $\lambda\leqslant\min(\tilde{F}(+\infty), 1-\tilde{F}(-\infty))$.

1) If $\tilde{F}(+\infty)=\tilde{F}(-\infty)=k$ then, for every $F\in\mathcal{D}$ and for every $c\in R$ we have:

$F(c)-(1-\lambda)\leqslant\lambda\leqslant\tilde F(+\infty)=k=\lim\limits_n F_{k_n}(c)\leqslant 1-\lambda\leqslant F(c)+1-\lambda$. From the theorem 2 it follows that $L_\lambda(F_{k_n}, F)\to 0$. 2) If $\tilde F(+\infty)>\tilde F(-\infty)$ then we define $F: R\to I$ letting $F(a)=[\tilde F(a)-\tilde F(-\infty)]/[\tilde F(+\infty)-\tilde F(-\infty)]$, for every $a\in R$. Then $F\in\mathcal{D}$. If c is a continuity point of F then c is a continuity point of $\tilde F$, so that $F_{k_n}(c)\to\tilde F(c)$. Then $F(c)-(1-\lambda)\leqslant F(c)-(1-\lambda)F(c)=\lambda F(c)\leqslant$ $\tilde F(+\infty)F(c)+\tilde F(-\infty)[1-F(c)]=\tilde F(c)=\varliminf\limits_n F_{k_n}(c)=\lim\limits_n F_{k_n}(c)=\varlimsup\limits_n F_{k_n}(c)=$ $=F(c)[\tilde F(+\infty)-\tilde F(-\infty)]+\tilde F(-\infty)\leqslant F(c)+1-\lambda$. Hence $L_\lambda(F_{k_n}, F)\to 0$.

Conversely, if $\mathcal{A}$ is not a λ-tight then there exist an $\varepsilon>0$ and $(F_n)_{n\in N}\subseteq\mathcal{A}$ such that $F_n(-n)\geqslant 1-\lambda+\varepsilon$ or $F_n(n)\leqslant\lambda-\varepsilon$. Let $(F_{k_n})_{n\in N}$ be a subsequence of $(F_n)_{n\in N}$ L_λ-convergent to a function $F\in\mathcal{D}$. Let $(q_m)_{m\in N}$ and $(r_m)_{m\in N}$ be two sequences of continuity points of F such that $q_m\to+\infty$ and $r_m\to-\infty$. Then, for every $m\in N$, $F(q_m)-1+\lambda\leqslant\varliminf\limits_n F_{k_n}(q_m)\leqslant\varlimsup\limits_n F_{k_n}(k_n)$ and $F(r_m)+1-\lambda\geqslant\lim\limits_n F_{k_n}(r_m)\geqslant\varlimsup\limits_n F_{k_n}(-k_n)$. Therefore, for every $m\in N$, $F(q_m)-1+\lambda\leqslant\lambda-\varepsilon$ or $F(r_m)+1-\lambda\geqslant 1-\lambda+\varepsilon$. Hence $F(+\infty)\leqslant 1-\varepsilon$ or $F(-\infty)>\varepsilon$. But this is a contradiction because $F\in\mathcal{D}$.

Remark. For every $\mathcal{A}\subseteq\mathcal{D}$ let $l(\mathcal{A})=\sup\{\lambda:\mathcal{A}=\lambda\text{-tight}\}$ and $c(\mathcal{A})=$ $=\sup\{\lambda:\mathcal{A}$ is L_λ-sequentially relatively compact$\}$. From the previous theorem $l(\mathcal{A})=c(\mathcal{A})$. In addition, if $\mathcal{A}''$ is the set of all nondecreasing, left-continuous functions on R, $\tilde F$, such that there exists $(F_n)_{n\in N}\subseteq\mathcal{A}$ $F_n(c)\to\tilde F(c)$ for every continuity point of $\tilde F$, and $a(\mathcal{A})=\inf\{\min(\tilde F(+\infty), 1-\tilde F(-\infty)):$ $:\tilde F\in\mathcal{A}''\}$ then $l(\mathcal{A})=c(\mathcal{A})=a(\mathcal{A})$.

Theorem 4. *Let $\mathcal{F}$ be the Menger metric defined on $\mathcal{D}$ by (2); then $(\mathcal{D}, \mathcal{F}, T_m)$ is p-complete.*

Proof. Let $(F_n)_{n\in N}$ be a sequence of $\mathcal{D}$ and $\mathcal{A}=\{F_n: n\in N\}$. Then $\mathcal{A}$ is $l(\mathcal{A})$-tight. From theorem 3, there exist $F\in\mathcal{D}$ and a subsequence $(F_{k_n})_{n\in N}$ such that $L_{l(\mathcal{A})}(F_{k_n}, F)\to 0$.

Now, assume that $(F_n)_{n\in N}$ is λ-Cauchy: then, for every $\mu<\lambda$, $L_\mu(F_n, F_m)\to 0$ as $n, m\to\infty$. If $\lambda\leqslant 1/2$ then $T_m(\lambda, \lambda)=0$, hence $(F_n)_{n\in N}$ is $T_m(\lambda, \lambda)$-convergent to F. If $\lambda>1/2$ and $\mu<T_m(\lambda, \lambda)=2\lambda-1$, then there exists $\gamma<\lambda$ such that $\mu<T_m(\gamma, \gamma)$. So, $L_\gamma(F_n, F_m)\to 0$, hence, for every $\varepsilon>0$ there exists $n_0\in N$ such that for every $m, n\geqslant n_0$ $L_\gamma(F_n, F_m)<\varepsilon/4$. There exist $a', b'\in R$, $a'<b'$, such that for every $n\leqslant n_0$ $F_n(a')<\varepsilon/4$ and $F_n(b')>1-\varepsilon/4$. If $n>n_0$ and $c\in R$ then $F_n(c)\leqslant F_{n_0}(c+\varepsilon/4)+\varepsilon/4+1-\gamma$ and $F_{n_0}(c)\leqslant F_n(c+\varepsilon/4)+\varepsilon/4+$ $+1-\gamma$. Hence $F_n(a'-\varepsilon/4)\leqslant F_{n_0}(a')+\varepsilon/4+1-\gamma<\varepsilon/2+1-\gamma$ and $F_n(b'+\varepsilon/4)\geqslant$ $\geqslant F_{n_0}(b')-\varepsilon/4-1+\gamma>\gamma-\varepsilon/2$. Let $a=a'-\varepsilon/4$ and $b=b'+\varepsilon/4$. Then, for every $n\in N$, $F_n(a)<\varepsilon+1-\gamma$ and $F_n(b)>\gamma-\varepsilon$. It follows that $(F_n)_{n\in N}$ is a γ-tight. Hence $\gamma\leqslant l(\mathcal{A})$. Then $L_\gamma(F_{k_n}, F)\to 0$.

From $L_{T_m(\gamma,\gamma)}(F_n, F)\leqslant L_\gamma(F_n, F_{k_n})+L_\gamma(F_{k_n}, F)$, we have that $(F_n)_{n\in N}$ is $L_{T_m(\gamma,\gamma)}$-convergent to F. Therefore, $(F_n)_{n\in N}$ is L_μ-convergent to F, for every $\mu<T_m(\lambda, \lambda)$, so that $(F_n)_{n\in N}$ is $T_m(\lambda, \lambda)$-convergent to F.

3. **The continuity of Menger metrics.** Let (X, F, T_m) be a Menger space. Then $F: X\times X\to(\mathcal{D}, \mathcal{F}, T_m)$. Let $\delta=\{d_\lambda:\lambda\in[0, 1)\}$ be the probabilistic pseudometric induced by F.

Theorem 5. *For every $x, y, u, v\in X$ and for every $a, b>0$ if $F_{xu}(a)>\lambda$ and $F_{yv}(b)>\mu$ then $L_{T_m(\lambda,\mu)}(F_{xy}, F_{uv})<a+b$, where $L=\{L_\lambda:\lambda\in[0, 1)\}$ is the probabilistic pseudometric on $\mathcal{D}$.*

Proof. Because $L_0(F_{xy}, F_{uv})=0$, it suffices to consider the case when $\lambda+\mu>1$. F_{xu} and F_{yv} are left-continuous in a and b, hence there exist $a_1<a$ and $b_1<b$ such that $F_{xu}(a_1)>\lambda$ and $F_{yv}(b_1)>\mu$. For every $c\in R$ we have $F_{uv}(c+a_1+b_1)\geqslant T_m(F_{xu}(a_1), F_{xv}(c+b_1))\geqslant F_{xu}(a_1)+F_{xv}(c+b_1)-1\geqslant F_{xu}(a_1)+$ $+F_{xy}(c)+F_{yv}(b_1)-2\geqslant F_{xy}(c)+\lambda+\mu-2=F_{xy}(c)+T_m(\lambda, \mu)-1\geqslant F_{xy}(c)+$ $+T_m(\lambda, \mu)-1-a_1-b_1$. Hence $F_{xy}(c)\leqslant F_{uv}(c+a_1+b_1)+a_1+b_1+1-T_m(\lambda, \mu)$.

Similarly, $F_{uv}(c)\leqslant F_{xy}(c+a_1+b_1)+a_1+b_1+1-T_m(\lambda, \mu)$.

Therefore, $L_{T_m(\lambda,\mu)}(F_{xy}, F_{uv})\leqslant a_1+b_1<a+b$.

Corollary 1. *For every $x, y, u, v\in X$ and $a, b>0$ we have*

$$\mathcal{F}(F_{xy}, F_{uv})(a+b)\geqslant T_m(F_{xu}(a), F_{yv}(b)). \tag{4}$$

Proof. If $\mathcal{F}(F_{xy}, F_{uv})(a+b)<T_m(F_{xu}(a), F_{yv}(b))$, then there exist $\lambda<$ $<F_{xu}(a)$ and $\mu<F_{yv}(b)$ such that $\mathcal{F}(F_{xy}, F_{uv})(a+b)<T_m(\lambda, \mu)$. But, from the previous theorem, it follows that $L_{T_m(\lambda,\mu)}(F_{xy}, F_{uv})<a+b$, so that $\mathcal{F}(F_{xy}, F_{uv})(a+b)\geqslant T_m(\lambda, \mu)$.

A similar result is obtained in [10, Satz 1].

The following corollary shows that theorem 5 generalizes theorem 5.4 from [4] and thus, theorem 8.2. from [6].

Corollary 2. *Let (X, F, T_m) be a Menger space. Then, for every sequences $\alpha=(x_n)_{n\in N}$, $\beta=(y_n)_{n\in N}\in\mathcal{X}$ and $x, y\in X$, we have:*

$$F_{xy}(a)-(2-p(\alpha, x)-p(\beta, y))\leqslant\varliminf_n F_{x_ny_n}(a)\leqslant\varlimsup_n F_{x_ny_n}(a)\leqslant$$
$$\leqslant F_{xy}(a)+2-p(\alpha, x)-p(\beta, y), \tag{5}$$

for all continuity points $a>0$ of F_{xy}.

Proof. For every $\varepsilon>0$, $p(\alpha, x)=\inf\limits_{a>0}\sup\limits_n\inf\limits_{k\geqslant n}F_{x_kx}(a)>p(\alpha, x)-\varepsilon$. It follows that, for every $a>0$, there exists $n_0\in N$ such that, for all $n\geqslant n_0$ we have $F_{x_nx}(a)>p(\alpha, x)-\varepsilon=\lambda$ and $F_{y_ny}(a)>p(\beta, y)-\varepsilon=\mu$. From theorem 5, it follows that $L_{T_m(\lambda,\mu)}(F_{x_ny_n}, F_{xy})<2a$. Therefore $(F_{x_ny_n})_{n\in N}$ is $L_{T_m(\lambda,\mu)}$-convergent to F_{xy}. From the theorem 2, for all continuity point $a>0$ of F_{xy} we have $F_{xy}(a)-(1-T_m(\lambda, \mu))\leqslant\varliminf\limits_n F_{x_ny_n}(a)\leqslant\varlimsup\limits_n F_{x_ny_n}(a)\leqslant F_{xy}(a)+1-T_m(\lambda, \mu)$. When $\varepsilon\to 0$ we obtain (5).

Because $(x_n)_{n\in N}$ is λ-convergent to x if and only if $p((x_n), x)\geqslant\lambda$, we obtain from the previous corollary:

Corollary 3. *If $(x_n)_{n\in N}$ is λ-convergent to x and $(y_n)_{n\in N}$ is μ-convergent to y in (X, F, T_m) then $(F_{x_ny_n})_{n\in N}$ is $T_m(\lambda, \mu)$-convergent to F_{xy} in $(\mathcal{D}, \mathcal{F}, T_m)$.*

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SOME CLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

BY

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1. Introduction. Let A denote the class of functions of the form

$$f(z)=z+\sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

that are analytic in the unit disk $E=\{z:|z|<1\}$. A function $f(z)$ in A is said to be in the subclass K_n, $n\in N_0=\{0, 1, 2, \ldots\}$ if

$$\operatorname{Re}\frac{D^{n+1}f(z)}{D^n f(z)}>\frac{1}{2}, \qquad (z\in E) \tag{2}$$

where $D^n f(z)=\dfrac{z}{(1-z)^{n+1}}*f(z)$. ($*$ denotes the Hadamard product or convolution of two regular functions). The class K_n was introduced by Ruscheweyh [2]. Let T be the subclass of A consisting of functions of the form $f(z)=z-\sum\limits_{k=2}^{\infty}|a_k|z^k$ that are univalent in E, and $Q_n[\alpha]$, $0\leqslant\alpha<1$, $n\in N_0$ denote the subclass of T whose members satisfy $\operatorname{Re}(D^n f(z))'>\alpha$ for $z\in E$. Let $T^*(\alpha)$ and $C(\alpha)$ be the subclasses of T that are respectively starlike of order α and convex of order α. In this paper we investigate some properties of $Q_n[\alpha]$.

2. The class $Q_n[\alpha]$. Let $c(n, k)=\left(\prod\limits_{m=1}^{k-1}(n+m)\right)\Big/(k-1)!$ so that

$$\frac{z}{(1-z)^{n+1}}=z+\sum_{k=2}^{\infty}c(n, k)\,z^k, \quad \text{and} \quad D^n f(z)=z-\sum_{k=2}^{\infty}|a_k|\,c(n, k)\,z^k.$$

Theorem 1. *Let $f(z)=z-\sum\limits_{k=2}^{\infty}|a_k|z^k$ be in T. Then $f(z)\in Q_n[\alpha]$ if and only if $\sum\limits_{k=2}^{\infty}k\,c(n, k)\,|a_k|\leqslant 1-\alpha$.*

Proof. Suppose

$$\sum_{k=2}^{\infty}k\,c(n, k)\,|a_k|\leqslant 1-\alpha. \tag{3}$$