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SOME CLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

BY

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1. Introduction. Let A denote the class of functions of the form

$$f(z)=z+\sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

that are analytic in the unit disk $E=\{z:|z|<1\}$. A function $f(z)$ in A is said to be in the subclass K_n, $n\in N_0=\{0, 1, 2, \ldots\}$ if

$$\operatorname{Re}\frac{D^{n+1}f(z)}{D^n f(z)}>\frac{1}{2}, \qquad (z\in E) \tag{2}$$

where $D^n f(z)=\dfrac{z}{(1-z)^{n+1}} * f(z)$. ($*$ denotes the Hadamard product or convolution of two regular functions). The class K_n was introduced by Ruscheweyh [2]. Let T be the subclass of A consisting of functions of the form $f(z)=z-\sum\limits_{k=2}^{\infty}|a_k|z^k$ that are univalent in E, and $Q_n[\alpha]$, $0\leqslant\alpha<1$, $n\in N_0$ denote the subclass of T whose members satisfy $\operatorname{Re}(D^n f(z))'>\alpha$ for $z\in E$. Let $T^*(\alpha)$ and $C(\alpha)$ be the subclasses of T that are respectively starlike of order α and convex of order α. In this paper we investigate some properties of $Q_n[\alpha]$.

2. The class $Q_n[\alpha]$. Let $c(n, k)=\left(\prod\limits_{m=1}^{k-1}(n+m)\right)\Big/(k-1)!$ so that

$$\frac{z}{(1-z)^{n+1}}=z+\sum_{k=2}^{\infty} c(n, k)\, z^k, \quad \text{and} \quad D^n f(z)=z-\sum_{k=2}^{\infty}|a_k|\, c(n, k)\, z^k.$$

Theorem 1. *Let $f(z)=z-\sum\limits_{k=2}^{\infty}|a_k|z^k$ be in T. Then $f(z)\in Q_n[\alpha]$ if and only if $\sum\limits_{k=2}^{\infty} k\, c(n, k)\,|a_k|\leqslant 1-\alpha$.*

Proof. Suppose

$$\sum_{k=2}^{\infty} k\, c(n, k)\,|a_k|\leqslant 1-\alpha. \tag{3}$$

It is sufficient to show that $(D^n f(z))'$ lies in a circle with centre at $w=1$ and radius $1-\alpha$. we have

$$|(D^n f(z))'-1| \leqslant \sum_{k=2}^{\infty} k\,|a_k|\,c(n,k).$$

The last expression is bounded above by $1-\alpha$ if (3) is satisfied.

Conversely suppose that

$$\operatorname{Re}(D^n f(z))' = \operatorname{Re}\left\{1-\sum_{k=2}^{\infty} k\,|a_k|\,c(n,k)\,z^{k-1}\right\} > \alpha;$$

choose values of z on the real axis so that $(D^n f(z))'$ is real. Letting $z\to 1$ along the real axis we obtain (1).

Corollary 1. *If* $f(z)=z-\sum\limits_{k=2}^{\infty}|a_k|\,z^k$ *is in* $Q_n[\alpha]$ *then* $|a_k| \leqslant (1-\alpha)/kc(n,k)$ *with equality for the function* $f(z)=z-[(1-\alpha)/kc(n,k)]\,z^k$.

Theorem 2. $Q_{n+1}[\alpha] \subset Q_n[\alpha]$ *for each* $n\in N_0$.

Proof. Let $f(z)=z-\sum\limits_{k=2}^{\infty}|a_k|\,z^k \in Q_{n+1}[\alpha]$; then $\sum\limits_{k=2}^{\infty} k\,|a_k|\,c(n+1,k) \leqslant 1-\alpha$ and since $c(n,k)\leqslant c(n+1,k)$ for $k=2,3,\ldots$ we have $\sum\limits_{k=2}^{\infty} k\,|a_k|\,c(n,k)\leqslant 1-\alpha$. The result follows from Theorem 1.

Theorem 3. *Let* $f(z)=z-\sum\limits_{k=2}^{\infty}|a_k|\,z^k$ *be a member of* $Q_n[\alpha]$ *and let* F *be defined by*

$$F(z)=\frac{1+c}{z^c}\int_0^z t^{c-1} f(t)\,dt = z-\sum_{k=2}^{\infty}\frac{1+c}{k+c}\,|a_k|\,z^k. \tag{4}$$

Then

(i) *for every* $c>-1$, $F\in Q_n[\alpha]$ *and*

(ii) *for every* c, $-1<c\leqslant n$, $F\in Q_{n+1}[\alpha]$.

Proof. (i) In view of Theorem 1 it is sufficient to prove that

$$\sum_{k=2}^{\infty} k\,\frac{1+c}{k+c}\,|a_k|\,c(n,k)\leqslant 1-\alpha. \tag{5}$$

Since $f\in Q_n[\alpha]$, $\sum\limits_{k=2}^{\infty} k\,|a_k|\,c(n,k)\leqslant 1-\alpha$ and the inequality (5) follows.

(ii) Again in view of Theorem 1 it is sufficient to show that

$$\sum_{k=2}^{\infty} k\,\frac{1+c}{k+c}\,|a_k|\,c(n+1,k)\leqslant 1-\alpha.$$

Since $c(n,k)-(1+c)/(k+c)\;c(n+1,k)\geqslant 0$ if $-1<c\leqslant n$ $(k=2,3,\ldots)$ the result follows from Theorem 1.

Putting $c=0$ in Theorem 3 we get the following

Corollary 2. *Let* $f(z)=z-\sum\limits_{k=2}^{\infty}|a_k|\,z^k$ *be a member of* $Q_n[\alpha]$ *and let* F *be defined by*

$$F(z)=\int_0^z \frac{f(t)}{t}\,dt. \tag{6}$$

Then $F(z)\in Q_{n+1}[\alpha]$.

Theorem 4. *Let* $F(z)=z-\sum\limits_{k=2}^{\infty}|a_k|\,z^k$ *be a member of* $Q_n[\alpha]$, $f(z)=[1/(1+c)]\,z^{1-c}\,[z^cF(z)]'$, $c>-1$. *Then* $\operatorname{Re}(D^n f(z))'>\beta$ $(0\leqslant\beta<1)$ *for* $|z|<r_0$ *where*

$$r_0=\inf_k\,[(1+c)(1-\beta)/(k+c)(1-\alpha)]^{1/(k-1)} \quad \text{for } k=2,3,\ldots$$

Proof. It is sufficient to show that

$$|(D^n f(z))'-1|\leqslant 1-\beta \text{ for } |z|<r_0. \tag{7}$$

We have

$$|(D^n f(z))'-1|\leqslant \sum_{k=2}^{\infty} k\,\frac{k+c}{1+c}\,|a_k|\,c(n,k)\,|z|^{k-1}.$$

Hence the inequality (7) will be satisfied if

$$\sum_{k=2}^{\infty} k\,\frac{k+c}{1+c}\,|a_k|\,c(n,k)\,|z|^{k-1}\leqslant 1-\beta. \tag{8}$$

Since $F\in Q_n[\alpha]$, from Theorem 1, $\sum\limits_{k=2}^{\infty} k\,|a_k|\,c(n,k)\leqslant 1-\alpha$ and the inequality (8) will be satisfied if

$$k\,\frac{k+c}{1+c}\,\frac{|a_k|\,c(n,k)\,|z|^{k-1}}{1-\beta}\leqslant\frac{k\,|a_k|\,c(n,k)}{1-\alpha}.$$

Solving it for $|z|$ we obtain

$$|z|\leqslant[(1+c)(1-\beta)/[(k+c)(1-\alpha)]]^{1/(k-1)} \quad \text{for } k=2,3,4,\ldots$$

Writing $|z|=r_0$ the result follows. The estimate is sharp for the function

$$F(z)=z-\frac{1-\alpha}{k\,c(n,k)}\,z^k \text{ for some } k.$$

Theorem 4, for $\alpha=\beta$, $n=0$ and $c=1$ yields a result of the authors [3].

Theorem 5. *Let* $f(z)=z-\sum\limits_{k=2}^{\infty}|a_k|\,z^k$ *be a member of* $T^*(\alpha)$ *and let*

$$F(z)=\frac{1+c}{z^c}\int_0^z t^{c-1} f(t)\,dt = z-\sum_{k=2}^{\infty}\frac{1+c}{k+c}\,|a_k|\,z^k. \qquad)$$

Then (i) $F(z)\in T^*(\alpha)$ *for* $c>-1$ *and* (ii) $F(z)\in C(\alpha)$ *for* $-1<c\leqslant 0$.

Proof. (i) $f(z)=z-\sum_{k=2}^{\infty}|a_k|z^k\in T^*(\alpha)$ iff $\sum_{k=2}^{\infty}|a_k|(k-\alpha)\leqslant 1-\alpha$ [4].

For $c>-1$, $\sum_{k=2}^{\infty}\frac{1+c}{k+c}|a_k|(k-\alpha)\leqslant\sum_{k=2}^{\infty}|a_k|(k-\alpha)\leqslant 1-\alpha$. Hence $F\in T^*(\alpha)$.

(ii) Let $-1<c\leqslant 0$. From (9) we obtain $zF'(z)=(1+c)f(z)-cF(z)$. Since $T^*(\alpha)$ is closed under convex linear combinations [4], $zF'(z)\in T^*(\alpha)$. That is $F\in C(\alpha)$.

Theorem 5 is comparable to the following result of Bernardi for a wider class of functions [1]: Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be a member of S^*. Then $F(z)=(c+1)\,z^{-c}\int_0^z t^{c-1}f(t)\,dt$ is also a member of S^* for $c=1,2,3,\dots$.

Theorem 6. *If* $f(z)=z-\sum_{k=2}^{\infty}|a_k|z^k$ *is a member of* $Q_n[\alpha]$ *then* $f(z)$ *is convex for* $|z|<r=r(\alpha)=\inf_k\,[c(n,k)/(k(1-\alpha))]^{1/(k-1)}$ $k=2,3,4,\dots$ *The estimate is sharp for the function* $f(z)=z-[(1-\alpha)/(kc(n,k))]\,z^k$ *for some* k.

Proof. It is omitted.

Theorem 7. *If* $f(z)=z-\sum_{k=2}^{\infty}|a_k|z^k$, $g(z)=z-\sum_{k=2}^{\infty}|b_k|z^k$ *are elements of* $Q_n[\alpha]$ *then* $h(z)=f(z)*g(z)=z-\sum_{k=2}^{\infty}|a_k||b_k|z^k$ *is an element of* $Q_n\left[\frac{2n+1+2\alpha-\alpha^2}{2(n+1)}\right]$.

Proof. From Theorem 1 we know that

$$\sum_{k=2}^{\infty}k|a_k|c(n,k)\leqslant 1-\alpha \text{ and } \sum_{k=2}^{\infty}k|b_k|c(n,k)\leqslant 1-\alpha.$$

We wish to find the largest $\beta=\beta(\alpha)$ such that $\sum_{k=2}^{\infty}k|a_k||b_k|c(n,k)\leqslant 1-\beta$ or, equivalently, we want to show that

(10) $\sum_{k=2}^{\infty}\frac{k|a_k|c(n,k)}{1-\alpha}\leqslant 1$ and (11) $\sum_{k=2}^{\infty}\frac{k|b_k|c(n,k)}{1-\alpha}\leqslant 1$ should imply that

(12) $\sum_{k=2}^{\infty}\frac{k|a_k||b_k|c(n,k)}{1-\beta}\leqslant 1$ for all $\beta=\beta(\alpha)\leqslant\frac{2n+1+2\alpha-\alpha^2}{2(n+1)}$.

From (10) and (11) we obtain by means of the Cauchy-Schwartz inequality:

(13) $$\sum_{k=2}^{\infty}\frac{k\sqrt{|a_k||b_k|}\,c(n,k)}{1-\alpha}\leqslant 1.$$

Hence it will be sufficient to show that

$k|a_k||b_k|c(n,k)/(1-\beta)\leqslant k\sqrt{|a_k||b_k|}\,c(n,k)/(1-\alpha)$, $\beta\leqslant\beta(\alpha)$, $k=2,3,\dots$, i.e.,

(14) $$\sqrt{|a_k||b_k|}\leqslant(1-\beta)/(1-\alpha).$$

From (13) we get $\sqrt{|a_k||b_k|}\leqslant(1-\alpha)/(kc(n,k))$. Hence (14) is satisfied if

(15) $$(1-\alpha)/(kc(n,k))\leqslant(1-\beta)/(1-\alpha),\quad\text{for all } n.$$

Inequality (15) is equivalent to

(16) $$\beta\leqslant 1-(1-\alpha)^2/(kc(n,k)).$$

The right side of (16) is an increasing function of k, $k=2,3,\dots$ Hence, putting $k=2$ in (16), we obtain $(2n+1+2\alpha-\alpha^2)/(2(n+1))$. The estimate is sharp for the function

$$f(z)=g(z)=z-(1-\alpha)/(2(n+1))\,z^2.$$

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