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ON ORTHOGONALITY AND ON ADJOINT OPERATOR IN H-LOCALLY CONVEX SPACES

BY

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1. **Introduction.** Let X be an H-locally convex space, i.e. a locally convex vector space whose topology is generated by a system of separating Hilbertian seminorms $P=\{p_\alpha, \alpha\in\Delta\}$, where Δ is some index set [5]. Denote the semi-inner products by $(\ ,\ )_\alpha$, $\alpha\in\Delta$. P is a calibration since the sets $\{x\in X, p_\alpha(x)<\varepsilon, \varepsilon>0, \alpha\in\Delta\}$ constitute a neighborhood sub-base at 0 [2]. We shall study the notion of orthogonality in such a space, we shall prove the existence of vectors orthogonal to some elements in X and we will obtain some criteria of normability of such a space. In the last part we shall treat the notion of the adjoint operator in an H-locally convex space. We shall prove the existence of such an operator for some finite rank operators and we shall show that in a nonnormable H-locally convex space each finite rank operator does not have an adjoint operator.

2. **On the orthogonality of vectors.** Let X be an H-locally convex space. Two elements x, $y\in X$ are said to be orthogonal if $(x, y)_\alpha=0$ for all $\alpha\in\Delta$. This relation, which will be denoted by $\perp$, has similar properties as in the Hilbert space, except that there may be some vectors which are orthogonal only to the zero vector. With $M^\perp$ we shall denote the set of vectors which are orthogonal to alll vectors from the set M in X.

To a given vector $y\in X$ we associate the following set of linear functionals $\{f_y^\alpha, \alpha\in\Delta\}$, where

$$f_y^\alpha(x)=(x, y)_\alpha, \quad x\in X.$$

Theorem 2.1. *Let X be an H-locally convex space. Then for $y\in X$, $y\neq 0$ there exists at least one nonzero vector orthogonal to it, if and only if the system $\{f_y^\alpha, \alpha\in\Delta\}$ is not total in the dual space X'.*

Proof. Let there exists some $z\in X$, $z\neq 0$, $z\perp y$. Define $F(f)=f(z)$, $f\in X'$, then from the Hahn-Banach theorem also $F\neq 0$. Clearly, $F\in(X_b')'$ where X_b' is the strong dual of X. Now, we have

$$F(f_y^\alpha)=f_y^\alpha(z)=(z,y)_\alpha=0, \quad \alpha\in\Delta$$

and the above set of functionals is not total in X'.

Conversely, suppose the above system is not total in X'. Then there is some $F\in(X_b')'$, $F\neq 0$, such that $F(f_y^\alpha)=0$ for all $\alpha\in\Delta$. Since an H-locally convex space is semireflexive [6], there exists some $z\in X$, such that $F(f)=f(z)$, $f\in X'$ and clearly, $z\neq 0$. From $0=F(f_y^\alpha)=f_y^\alpha(z)=(z, y)_\alpha$ it follows that $z\perp y$.

Remark. This result can be found also in [1] when p_α, $\alpha\in\Delta$ are norms, the set P is countable and X is reflexive.

Let us now define the following set

$$F=\{e\in X,\ (x, e)_\alpha\, p_\beta^2(e)=(x, e)_\beta\, p_\alpha^2(e),\ \alpha,\beta\in\Delta,\ x\in X\}.$$

It is immediate that $e\in F$ if and only if, for each $x\in X$, the quotient $(x,e)_\alpha/p_\alpha^2(e)$ is independent on those $\alpha\in\Delta$ such that $p_\alpha(e)\neq 0$. We can verify easily that F is closed but in general it is not a subspace in X. The following theorem was proved in [7] for the case $F=X$ but the same proof can be also applied if F is a subspace in X.

Theorem 2.2. *If F is a subspace in an H-locally convex space X then there exists a norm $\|\cdot\|$ on F and $K_\alpha\geq 0$, $\alpha\in\Delta$, for which*

$$p_\alpha(x)=K_\alpha\|x\|,\ x\in F,\ \alpha\in\Delta.$$

Thus, in that case F is equivalent to a pre-Hilbert space. Since the above set depends on the choice of the family P of seminorms on X it will be denoted in the following theorem by F_P. For the barreled H-locally convex spaces we shall prove

Theorem 2.3. *Let X be complete barreled H-locally convex space. If there exist Hilbertian calibrations P_i on X such that F_{P_i} are subspaces of X, $i=1,2,\dots,m$, and*

$$X=F_{P_1}\oplus F_{P_2}\oplus\dots\oplus F_{P_m},$$

then X is normable by some Hilbertian norm.

Proof. Each of F_{P_i} is closed subspace of X and by the previous theorem all seminorms on it are proportional to some Hilbertian norm

$$p_\alpha^i(y)=K_\alpha^i\cdot\|y\|_i,\ y\in F_{P_i},\ \alpha\in\Delta_i,\ i=1,2,\dots,m.$$

An arbitrary $x\in X$ can be written in an unique way as $x=\sum_{i=1}^m x_i$, $x_i\in F_{P_i}$ and we define

(2.1) $$\|x\|^2=\sum_{i=1}^m\|x_i\|_i^2.$$

Clearly, $\|\cdot\|$ is a norm on X. Since all $\|\cdot\|_i$ are Hilbertian, this norm is obviously Hilbertian too. Choose now an arbitrary continuous seminorm q on X. Then for each i, $i=1,2,\dots,m$, there exist $C_i\geq 0$ and $p_{\alpha_j}^i\in P_i$, $j=1,2,\dots,n_i$, such that

$$q(x_i)\leq C_i\max\{p_{\alpha_j}^i(x_i),\ j=1,2,\dots,n_i\},\ x_i\in F_{P_i}.$$

Using the above proportionalities we obtain

$$q(x_i)\leq d_i\|x_i\|_i,\ x_i\in F_{P_i},\ i=1,2,\dots,m,$$

where $d_i=\max\{K_{\alpha_j}^i,\ j=1,2,\dots,n_i\}$. Now, we can estimate

$$q(x)\leq\sum_{i=1}^m q(x_i)\leq\sum_{i=1}^m d_i\|x_i\|_i\leq d\|x\|,$$

where we denoted $d=(\sum_{i=1}^m d_i^2)^{1/2}$. Thus, the topology τ' generated by the above norm is finer than the original one τ. Let us prove that X is also complete in the above norm. If $\{x^{(n)}\}$ is a τ'-Cauchy sequence in X and if $x^{(n)}=\sum_{i=1}^m x_i^{(n)}$, $n\in N$, then (2.1) implies that $\{x_i^{(n)}\}$ are Cauchy sequences in F_{P_i} which are complete. Hence we have $\lim_{n\to\infty}\|x_i^{(n)}-x_i\|_i=0$, where $x_i\in F_{P_i}$, $i=1,2,\dots,m$, and consequently $\lim_{n\to\infty}\|x^{(n)}-x\|=0$, where $x=\sum_{i=1}^m x_i$. The identity map from the Hilbert space (X,τ') onto the barreled space (X,τ) is continuous and, by the open mapping theorem [8], both topologies coincide.

Let $L(a_1,a_2,\dots,a_n)$ denote the linear span of the vectors $a_1,a_2,\dots,a_n\in X$. We shall connect the set F with the orthogonality relation.

Lemma 2.4. *In an H-locally convex space X the following statements are equivalent*

(i) $a\in F$, (ii) $X=L(a)\oplus L(a)^\perp$, (iii) *for arbitrary $x\in X$, $x\notin L(a)$, there exists $y\in L(a,x)$, $y\neq 0$, such that $y\perp a$.*

Proof. Let us first prove the implication (i)⇒(ii). Choose $a\in F$, $x\notin L(a)$ and $\beta\in\Delta$ such that $p_\beta(a)\neq 0$. Define $\lambda=(x,a)_\beta/p_\beta^2(a)$, then this number is independent on all $\beta\in\Delta$ such that $p_\beta(a)\neq0$. Put $y=x-\lambda a$; then obviously $(y,a)_\alpha=0$ for all $\alpha\in\Delta$ (if $\alpha\in\Delta$ is such that $p_\alpha(a)=0$ this equality holds trivially). Hence $y\in L(a)^\perp$ and we have $x=\lambda a+y$. This decomposition is unique since $L(a)\cap L(a)^\perp=\{0\}$. The case $x\in L(a)$ is trivial. In the same way we can prove that (i) implies (iii). Let us check that (ii)⇒(i). Choose first $x\notin L(a)$, then there exists $\lambda\neq0$ and $y\in L(a)^\perp$ such that $x=\lambda a+y$ and it follows

$$0=(y,a)_\alpha=(x,a)_\alpha-\lambda p_\alpha^2(a),\ \alpha\in\Delta.$$

This implies that $(x,a)_\alpha/p_\alpha^2(a)$ is independent on all $\alpha\in\Delta$ such that $p_\alpha(a)\neq0$. In the case $x\in L(a)^\perp$ this quotient is 0 for all $\alpha\in\Delta$. Thus $a\in F$. Let now (iii) be true. Choose a, $x\in X$ such that $a\neq0$ and $x\notin L(a)$. Then there exists $y\neq0$ of the form $y=\lambda a+\mu x$, $y\perp a$. By a short calculation we verify again that $a\in F$. Thus (iii) implies (i).

From the above lemma it follows immediately that each vector from F has a nonzero orthogonal vector which belongs even to a twodimensional subspace containing this vector. This situation is characteristic for the normability of an H-locally convex space.

Theorem 2.5. *If in an arbitrary twodimensional subspace of an H-locally convex space X (dim $X\geq2$) each vector has a nontrivial orthogonal vector, then X is equivalent to a pre-Hilbert space.*

Proof. Choose arbitrary $a\in X$, $a\neq 0$, and $x\notin L(a)$. Then there is som $y\neq 0$, $y\in L(a, x)$, such that $y\perp a$. By the above lemma $a\in F$, hence $F=X$ an by theorem 2.2 X is equivalent to a pre-Hilbert space.

3. **On the existence of the adjoint operator.** In [3] we defined fo a linear operator A on an H-locally convex space X the adjoint operator A by the relations

$$(3.1)\qquad (Ax,\ y)_\alpha=(x,\ A^0 y)_\alpha\ ;\ x,\ y\in X,\ \alpha\in\Delta.$$

We noted that the adjoint does not exist for each continuous linear operato Also, as far as continuity is concerned, there arise difficulties. If A is con tinuous in the sense $p_\alpha(Ax)\leq C_\alpha p_\alpha(x)$, $x\in X$, $\alpha\in\Delta$, then A^0 is continuous i this sense, too. The continuity of A^0, if A is continuous, can also be prove if the space is metrizable [3]. We shall prove a generalisation of this resul

Theorem 3.1. *Let X be a barreled H-locally convex space and A, B linear operators on X such that*

$$(Ax,\ y)_\alpha=(x,\ By)_\alpha\ ;\ x,\ y\in X,\ \alpha\in\Delta.$$

Then both A and B are continuous.

Proof. We may assume that $P=\{p_\alpha,\ \alpha\in\Delta\}$ is a calibration on X Choose arbitrary $\alpha\in\Delta$, denote $U_\alpha=\{x\in X,\ p_\alpha(x)\leq 1\}$ and define the se $\{f_y,\ y\in U_\alpha\}$ of linear functionals of the form

$$f_y(x)=(Ay,\ x)_\alpha,\ x\in X.$$

This set is pointwise bounded since $|f_y(x)|=|(y,\ Bx)_\alpha|\leq p_\alpha(Bx)$ for al $y\in U_\alpha$ and by the Banach-Steinhaus theorem it is uniformly bounded There exist $p_{\beta_i}\in P$, $i=1, 2, \ldots, n$ and $M_\alpha>0$ such that $|f_y(x)|\leq M_\alpha$ $\max\{p_{\beta_i}(x),\ i=1,\ 2,\ \ldots,\ n\}$, $x\in X$, $y\in U_\alpha$ and we have

$$|(y,\ Bx)_\alpha|\leq M_\alpha\cdot\max\ \{p_{\beta_i}(x),\ i=1,\ 2,\ \ldots,\ n\}\ ;\ x\in X,\ y\in U_\alpha.$$

Let first $x\in X$ be such that $p_\alpha(Bx)\neq 0$; then putting $y=Bx/p_\alpha(Bx)$ we obtai

$$p_\alpha(Bx)\leq M_\alpha\cdot\max\ \{p_{\beta_i}(x),\ i=1,\ 2,\ \ldots,\ n\},\ x\in X.$$

If on the other side $p_\alpha(Bx)=0$ then this inequality is trivially true. Thus B is continuous and by the simmetry this is true for A, too.

Corollary 3.2. *If in a barreled H-locally convex space X for a linea operator there exists the adjoint operator, then both are continuous.*

As usual, we shall denote by ker (A) and im (A) respectively the kerne and the image of A. We call A a finite rank operator if im(A) is finite dimen sional.

Lemma 3.3. *Let X be an H-locally convex space and A a linear conti nuous and finite rank operator. If the adjoint A^0 exists it is again a finite ran operator and*

$$\dim\ \mathrm{im}\ (A^0)=\dim\ \mathrm{im}\ (A).$$

Proof. Let us choose $z\in\mathrm{im}\ (A^0)\cap\ker\ (A)$. Then there is some $y\in X$ such that $z=A^0y$ and for all $\alpha\in\Delta$ we have

$$p_\alpha^2(z)=(z,\ z)_\alpha=(z,\ A^0y)_\alpha=(Az,\ y)_\alpha=0.$$

thus $z=0$ and it follows that im $(A^0)\cap\ker\ (A)=\{0\}$. Consequently, the natural mapping im $(A^0)\to X/\ker\ (A)$ is injective and we have $\dim\ \mathrm{im}\ (A^0)\leq$ $\leq\dim\ X/\ker\ (A)=\dim\ \mathrm{im}\ (A)$. Since $A^{00}=A$, we have also the opposite inequality $\dim\ \mathrm{im}\ (A)=\dim\ \mathrm{im}\ (A^{00})\leq\dim\ \mathrm{im}\ (A^0)$ and the proof is complete.

We shall connect the existence of the adjoint operator of some finite rank operators with the vectors from the set F.

Theorem 3.4. *Let X be an H-locally convex space. If for $A=a\otimes f$, where $a\in X$, $f\in X'$, $f\neq 0$ there exists the adjoint operator, then $a\in F$.*

Proof. To avoid triviality, we assume $a\neq 0$. By the above lemma A^0 is of the form $A^0=b\otimes g$, $b\in X$, $g\in X'$ and clearly $b\neq 0$ and $g\neq 0$. From the definition of the adjoint operator we obtain

$$f(x)\,(a,\ y)_\alpha=\bar{g}(y)\,(x,\ b)_\alpha\ ;\ x,\ y\in X,\ \alpha\in\Delta.$$

For $x=b$ we have

$$f(b)\,(a,\ y)_\alpha=\bar{g}(y)\,p_\alpha^2(b),\ y\in X,\ \alpha\in\Delta.$$

and it follows that $f(b)\neq 0$ since $g\neq 0$. Finally, fix $y=a$

$$f(b)\,p_\alpha^2(a)=\bar{g}(a)\,p_\alpha^2(b),\ \alpha\in\Delta.$$

From the last two equalities for arbitrary two indices $\alpha,\ \beta\in\Delta$ it follows: $|f(b)|^2.\ ((y,\ a)_\alpha\, p_\beta^2(a)-(y,\ a)_\beta\, p_\alpha^2(a))=0$, and consequently $a\in F$.

The opposite direction of the above theorem is in general not true. We shall obtain one result of this type for a special case. But, first we shall prove

Proposition 3.5. *If in an H-locally convex space X for a linear operator A the adjoint A^0 exists, then*

(i) $p_\alpha(Ae)=K\,.\,p_\alpha(e)$, $\alpha\in\Delta$, $e\in F$ *and* K *is independent of* $\alpha\in\Delta$.

(ii) F *and* $F^\perp$ *are invariant for* A.

Proof. Let $e\in F$ and put in the definition of the set F : $x=A^0Ae$. Then we obtain $p_\alpha(Ae)\,p_\beta(e)=p_\beta(Ae)\,p_\alpha(e)$ for all $\alpha,\ \beta\in\Delta$. We may suppose that $e\neq 0$, then there is some $\gamma\in\Delta$ such that $p_\gamma(e)\neq 0$ and (i) follows with $K=p_\gamma(Ae)/p_\gamma(e)$.

Let us prove (ii). Choosing arbitrary $e\in F$, $x\in X$ and $\alpha,\ \beta\in\Delta$ the statement (i) implies that : $(x,\ Ae)_\alpha p_\beta^2(Ae)=(A^0x,\ e)_\alpha\,.\,K^2\,.\,p_\beta^2(e)=(A^0x,\ e)_\beta\,.$ $.\,K^2\,.\,p_\alpha^2(e)=(x,\ Ae)_\beta p_\alpha^2(Ae)$ thus, $Ae\in F$.

It is clear that then also $A^0(F)\subset F$ since A^{00} exists. Hence for $e\in F^\perp$ it follows $(Ae,\ y)_\alpha=(e,\ A^0y)_\alpha=0$ for all $y\in F$, consequently $Ae\in F^\perp$.

Theorem 3.6. *Let for two elements a, b in an H-locally convex space X the following two conditions be true :*

(i) $a,\ b\in F$,

(ii) $p_\alpha(a)=K\,.\,p_\alpha(b)$, $\alpha\in\Delta$ *and* K *is independent of all* $\alpha\in\Delta$.

Then for the operator A of the form

$$Ax=(x,\ a)_\gamma b\ ;\ x\in X,\ \gamma\in\Delta.$$

the adjoint operator exists and is of the form

$$A^0x=(x,\ b)_\gamma a,\ x\in X.$$

Proof. We may assume that $a \neq 0$, $b \neq 0$ and then is $K \neq 0$, too. Let us choose first $\chi \in \Delta$ such that $p_\chi(a) \neq 0$, then also $p_\chi(b) \neq 0$ and we have

$$(Ax, y)_\chi = (x, a)_\gamma (b, y)_\chi = (x, a)_\chi \, p_\gamma^2(a) \, p_\chi^{-2}(a) \, (b, y)_\chi =$$

$$= (x, a)_\chi p_\gamma^2(b) \, p_\chi^{-2}(b) \, (b, y)_\chi = (x, a)_\chi (b, y)_\gamma = (x, A^* y)_\chi,$$

where we wrote $A^* y = (y, b)_\gamma a$. If $\chi \in \Delta$ is such that $p_\chi(a) = 0$ then also $p_\chi(b) = 0$ and the above relation holds trivially

$$(Ax, y)_\chi - (x, A^* y)_\chi = (x, a)_\gamma (b, y)_\chi - (b, y)_\gamma (x, a)_\chi = 0.$$

As a consequence each operator of the form $Ax = (x, e)_\gamma e$, $e \in F$, $\gamma \in \Delta$ has the adjoint operator. It is also clear that a finite linear combination of the operators of the above form again has the adjoint operator. Thus

Corollary 3.7. *For each operator A of the form*

$$Ax = \sum_{i=1}^{m} (x, a_i)_{\beta_i} b_i, \quad x \in X,$$

where $\beta_i \in \Delta$ and a_i, $b_i \in F$, $i = 1, 2, \ldots, n$ are such that $p_\chi(a_i) = K_i p_\chi(b_i)$, $\chi \in \Delta$, $i = 1, 2, \ldots, n$, there exists the adjoint operator.

Unfortunately, in a nonnormable H-locally convex space each finite rank operator does not have the adjoint operator, namely, we have

Theorem 3.8. *If in an H-locally convex space each finite rank operator has the adjoint operator, then it is equivalent to a pre-Hilbert space.*

Proof. Choose arbitrary $a \in X$, $f \in X'$. Since for the operator $a \otimes f$ there exists the adjoint operator, by theorem 3.4 it follows that $a \in F$. Consequently $F = X$ and by theorem 2.2, there is some $\gamma \in \Delta$ such that p_γ is a norm on X and all p_χ, $\chi \in \Delta$ are proportional to it.

Let us point out that we proved in [4] such a result for the case of an H-locally convex space with an orthogonal basis.

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PAIRWISE SEMI-REGULARIZATION AND SUBMAXIMALITY OF A BITOPOLOGICAL SPACE

BY

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1. Introduction. In the study of maximal topological spaces it is observed that some maximal spaces are submaximal (e.g. see [3], [4], [10]). B o u r b a k i [2], after associating a semiregular space to a topological space, introduces submaximal space and then gives a list of its properties. The associated semi-regular space to a topological space is termed semi-regularization of the space due to C a m e r o n [4]. Here in this article we generalize the related facts to bitopological spaces which were initiated by K e l l y [5].

In Section 2, our objective has been to investigate whether certain well-known bitopological properties are pairwise semiregular properties. In course of this study we have, at the same time, found improved versions of some of the results of [11]. We have also characterized pairwise ro-equivalent bitopological spaces.

In Section 3, we have introduced and studied, to some extent, pairwise submaximal bitopological spaces and have established the existence of a pairwise submaximal space which is finer than a given bitopological space. Ultimately we have inferred that every maximal R (minimal-R) bitopological space is pairwise submaximal (semi-regular) if R is a pairwise semi-regular property and this has enabled us to recognize certain minimal (maximal) bitopological spaces to be pairwise semiregular (submaximal). These latter results may find good applications in the study of maximal and minimal bitopological spaces.

By (X, Q_1, Q_2) or simply by X we shall mean a set X equipped with two topologies Q_1 and Q_2. The word „pairwise" shall be abbreviated as „p"- and Q_i-int A, Q_i-cl A shall denote the interior, and respectively closure of $A \subset X$, w.r.t. the topology Q_i ($i = 1$ or 2). A space (X, Q_1, Q_2) is called p-H a u s d o r f f [5] (p-U r y s o h n [12]) if for every pair of distinct points x, y of X, there exist Q_1-open set A containing x and Q_2-open set B containing y such that $A \cap B = \varnothing$ (Q_2-cl $A \cap Q_1$-cl $B = \varnothing$). A space (X, Q_1, Q_2) will be called finer (coarser) than the space (X, P_1, P_2) if Q_i is finer (coarser) than P_i, for $i = 1, 2$. (X, Q_1, Q_2) will be called an expansion of (X, P_1, P_2) if (X, Q_1, Q_2) is finer than (X, P_1, P_2).

2. Semiregularization of bitopological spaces. Let (X, Q_1, Q_2) be a space. A Q_i-open set U will be called ij-regularly open [11] or simply an ij-roset iff $U = Q_i$-int Q_j-cl U, where $i, j = 1, 2$ and $i \neq j$. A subset A of X is