Proof. We may assume that $a \neq 0$, $b \neq 0$ and then is $K \neq 0$, too. Let us choose first $\chi \in \Delta$ such that $p_\chi(a) \neq 0$. then also $p_\chi(b) \neq 0$ and we have

$$(Ax, y)_\chi = (x, a)_\gamma (b, y)_\chi = (x, a)_\chi p_\gamma^2(a) p_\chi^{-2}(a) (b, y)_\chi =$$

$$= (x, a)_\chi p_\gamma^2(b) p_\chi^{-2}(b) (b, y)_\chi = (x, a)_\chi (b, y)_\gamma = (x, A^\circ y)_\chi,$$

where we wrote $A^\circ y = (y, b)_\gamma a$. If $\chi \in \Delta$ is such that $p_\chi(a) = 0$ then also $p_\chi(b) = 0$ and the above relation holds trivially

$$(Ax, y)_\chi - (x, A^\circ y)_\chi = (x, a)_\gamma (b, y)_\chi - (b, y)_\gamma (x, a)_\chi = 0.$$

As a consequence each operator of the form $Ax = (x, e)_\gamma e$, $e \in F$, $\gamma \in \Delta$ has the adjoint operator. It is also clear that a finite linear combination of the operators of the above form again has the adjoint operator. Thus

Corollary 3.7. *For each operator A of the form*

$$Ax = \sum_{i=1}^{m} (x, a_i)_{\beta_i} b_i, \quad x \in X,$$

where $\beta_i \in \Delta$ and a_i, $b_i \in F$, $i = 1, 2, \ldots, n$ are such that $p_\chi(a_i) = K_i p_\chi(b_i)$, $\chi \in \Delta$, $i = 1, 2, \ldots, n$, there exists the adjoint operator.

Unfortunately, in a nonnormable H-locally convex space each finite rank operator does not have the adjoint operator, namely, we have

Theorem 3.8. *If in an H-locally convex space each finite rank operator has the adjoint operator, then it is equivalent to a pre-Hilbert space.*

Proof. Choose arbitrary $a \in X$, $f \in X'$. Since for the operator $a \otimes f$ there exists the adjoint operator, by theorem 3.4 it follows that $a \in F$. Consequently $F = X$ and by theorem 2.2, there is some $\gamma \in \Delta$ such that p_γ is a norm on X and all p_χ, $\chi \in \Delta$ are proportional to it.

Let us point out that we proved in [4] such a result for the case of an H-locally convex space with an orthogonal basis.

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PAIRWISE SEMI-REGULARIZATION AND SUBMAXIMALITY OF A BITOPOLOGICAL SPACE

BY

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1. Introduction. In the study of maximal topological spaces it is observed that some maximal spaces are submaximal (e.g. see [3], [4], [10]). Bourbaki [2], after associating a semiregular space to a topological space, introduces submaximal space and then gives a list of its properties. The associated semi-regular space to a topological space is termed semi-regularization of the space due to Cameron [1]. Here in this article we generalize the related facts to bitopological spaces which were initiated by Kelly [5].

In Section 2, our objective has been to investigate whether certain well-known bitopological properties are pairwise semiregular properties. In course of this study we have, at the same time, found improved versions of some of the results of [11]. We have also characterized pairwise ro-equivalent bitopological spaces.

In Section 3, we have introduced and studied, to some extent, pairwise submaximal bitopological spaces and have established the existence of a pairwise submaximal space which is finer than a given bitopological space. Ultimately we have inferred that every maximal R (minimal-R) bitopological space is pairwise submaximal (semi-regular) if R is a pairwise semi-regular property and this has enabled us to recognize certain minimal (maximal) bitopological spaces to be pairwise semiregular (submaximal). These latter results may find good applications in the study of maximal and minimal bitopological spaces.

By (X, Q_1, Q_2) or simply by X we shall mean a set X equipped with two topologies Q_1 and Q_2. The word „pairwise" shall be abbreviated as „p"- and Q_i-int A, Q_i-cl A shall denote the interior, and respectively closure of $A \subset X$, w.r.t. the topology Q_i $(i = 1$ or $2)$. A space (X, Q_1, Q_2) is called p-Hausdorff [5] (p-Urysohn [12]) if for every pair of distinct points x, y of X, there exist Q_1-open set A containing x and Q_2-open set B containing y such that $A \cap B = \emptyset$ (Q_2-cl $A \cap Q_1$-cl $B = \emptyset$). A space (X, Q_1, Q_2) will be called finer (coarser) than the space (X, P_1, P_2) if Q_i is finer (coarser) than P_i, for $i = 1, 2$. (X, Q_1, Q_2) will be called an expansion of (X, P_1, P_2) if (X, Q_1, Q_2) is finer than (X, P_1, P_2).

2. Semiregularization of bitopological spaces. Let (X, Q_1, Q_2) be a space. A Q_i-open set U will be called ij-regularly open [11] or simply an ij-roset iff $U = Q_i$-int Q_j-cl U, where $i, j = 1, 2$ and $i \neq j$. A subset A of X is called an ij-rc set iff $X-A$ is an ij-roset. The collection of all ij-rosets forms a base for a topology Q_i^s which is coarser than $Q_i (i, j=1, 2$ and $i \neq j)$. The space (X, Q_1^s, Q_2^s) is pairwise semi-regular [11] (i.e., a space such that the set of all ij-ro sets of (X, Q_1^s, Q_2^s) forms a base for Q_i^s) and we call it the p-semiregularization of (X, Q_1, Q_2). In the last sentence and in subsequent statements wherever i and j both occur, we assume that $i \neq j$ and $i, j=1, 2$.

Theorem 2.1. *Let (X, Q_1, Q_2) be a space and (X, Q_1^s, Q_2^s) be its p-semiregularitzation. Then for each $U \in Q_i$, Q_j-cl $U = Q_j^s$-cl U and Q_i-int Q_j-cl $U = Q_i^s$-int Q_j^s-cl U.*

Proof. Let $U \in Q_i$, then Q_j-cl $U \subset Q_j^s$-cl U as $Q_j \supset Q_j^s$. Now we find that Q_j-int Q_i-cl $(X - Q_j$-cl $U) = Q_j$-int Q_i-cl Q_j-int $(X-U) = Q_j$-int $(X-U) = X - Q_j$-cl U which implies that $X - Q_j$-cl U is ji-roset, hence Q_j-cl U is Q_j^s-closed and therefore Q_j^s-cl $U \subset Q_j$-cl U. Thus Q_j-cl $U = Q_j^s$-cl U.

Again Q_i-int $(Q_j$-cl $U) \supset Q_i^s$-int $(Q_j^s$-cl $U)$, since $Q_i \supset Q_i^s$ $(i=1, 2)$ and Q_j-cl $U = Q_j^s$-cl U. But if $V = Q_i$-int $(Q_j$-cl $U)$, then $V = Q_i$-int $(Q_j$-cl $V)$ so V is Q_i^s-open and since V is contained in Q_j^s-cl U, we have Q_i-int $(Q_j$-cl $U) = Q_i^s$-int $(Q_j^s$-cl $U)$.

Corollary 2. 2. *A space (X, Q_1, Q_2) and its p-semiregularization have the same collection of ij-rosets and hence have the same collection of ij-rosets.*

Note 2.3. In Theorem 5 of [11] it is proved that the family of ij-rosets of (X, Q_1, Q_2) and that of (X, Q_1^s, Q_2^s) generate the same topology. Now Corollary 2.2. presents a much clearer insight to the result.

Theorem 2.4. *If (X, Q_1^s, Q_2^s) is the p-semiregularization of (X, Q_1, Q_2) and (X, Q_1^0, Q_2^0) is a p-semiregular space with $Q_i^s \subset Q_i^0 \subset Q_i$ (for $i=1, 2$) then $Q_i^s = Q_i^0$ (for $i=1, 2$).*

Proof. Let U be an ij-roset in (X, Q_1^0, Q_2^0), then Q_j^s-cl $U = Q_j^0$-cl U since $Q_i^s \subset Q_i^0 \subset Q_i$ and Q_j^s-cl $U = Q_j$-cl U. Also Q_i^s-int $(Q_j^s$-cl $U) = Q_i$-int $(Q_j$-cl $U)$, therefore, Q_i-int $(Q_j$-cl $U) = Q_i^0$-int $(Q_j^0$-cl $U) = U$. Thus U is an ij-roset of (X, Q_1, Q_2) and hence $U \in Q_i^s$. So we have $Q_i^s = Q_i^0$ for $i=1, 2$.

Remark 2.5. The above theorem implies that given any space (X_1, Q_1, Q_2), its p-semiregularization is the finest p-semiregular space among all the p-semiregular spaces which are coarser than (X, Q_1, Q_2).

Theorem 2.6. *If (X, Q_1, Q_2) is p-Hausdorff then (X, Q_1^s, Q_2^s) is p-Hausdorff and conversely.*

Proof. Let (X, Q_1, Q_2) be p-Hausdorff and x, y be two distinct points of X. Take $U_i \in Q_i$ and $V_j \in Q_j$ such that $x \in U_i$, $y \in V_j$ and $U_i \cap V_j = \varnothing$. We have, Q_j-cl $U_i \cap V_j = \varnothing$ and so Q_i-int $(Q_j$-cl $U_i) \cap V_j = \varnothing$. Consequently Q_i-int $(Q_j$-cl $U_i) \cap Q_j$-int $(Q_i$-cl $V_j) = \varnothing$. But $x \in Q_i$-int $(Q_j$-cl $U_i) \in Q_i^s$ and $y \in Q_j$-int $(Q_i$-cl $V_j) \in Q_j^s$. Hence (X, Q_1^s, Q_2^s) is p-Hausdorff.

The converse part is clear since $Q_i^s \subset Q_i$, for $i=1, 2$.

Theorem 2.7. *If (X, Q_1, Q_2) is p-Urysohn then (X, Q_1^s, Q_2^s) is p-Urysohn and conversely.*

Proof. Let (X, Q_1, Q_2) be p-Urysohn and x, y be distinct points of X. Choose $U_i \in Q_i$ and $V_j \in Q_j$ such that $x \in U_i$, $Y \in V_j$ and Q_j-cl $U_i \cap Q_i$-cl $V_j = \varnothing$. Let $T_i = Q_i$-int $(Q_j$-cl $U_i)$ and $S_j = Q_j$-int $(Q_i$-cl $V_j)$, then $x \in T_i \in Q_i^s$ and $y \in S_j \in Q_j^s$. Also we have (using Theorem 2.1) Q_j-cl $U_i = Q_j$-cl $T_i = Q_j^s$-cl T_i and Q_i-cl $V_j = Q_i$-cl $S_j = Q_i^s$-cl S_j. Hence (X, Q_1^s, Q_2^s) is p-Urysohn. The converse part is obvious.

Note 2.8. It is proved in Theorem 7 and 8 of [11] that if (X, Q_1, Q_2) is p-Hausdorff (p-Urysohn) then (X, Q_1^s, Q_2^s) is pairwise T_1 (respectively p-Hausdorff). Since p-Urysohn $\Rightarrow$ p-Hausdorff $\Rightarrow$ p-T_1, the results in Theorems 2.6 and 2.7 have strengthened and improved respectively Theorem 7 and 8 of [11].

Definition 2.9. *A space (X, Q_1, Q_2) is called p-E_1 iff each point x in* **X** *is the countable intersection of the Q_1-closures of Q_2-open sets containing x and is also that of the Q_2-closures of Q_1-open sets containing x.*

Theorem 2.10. *If (X, Q_1, Q_2) is p-E_1, then (X, Q_1^s, Q_2^s) is also so and conversely.*

Proof. For the first part we are only to observe that if $U \in Q_i$ then $V = Q_i$-int $(Q_j$-cl $U) \in Q_i^s$, $U \subset V$ and by Theorem 2.1, Q_j-cl $U = Q_j$-cl $V = Q_j^s$-cl V. The converse part also follows from Theorem 2.1.

Definition 2.11. *A space (X, Q_1, Q_2) is called ij-QHC (termed ij-almost compact in [7]) iff any arbitrary Q_i-open cover of X has a finite subfamily, the union of whose members is Q_j-dense in X. We define (X, Q_1, Q_2) to be ij-feebly compact if in the above definition only countable Q_i-open cover is taken instead of arbitrary Q_i-open cover. X is called p-QHC (p-feebly compact) iff it is 12-QHC (12-feebly compact) and 21-QHC (21-feebly compact).*

Theorem 2.12 [8]. *A space (X, Q_1, Q_2) is ij-QHC (p-QHC) iff (X, Q_1^s, Q_2^s) is ij-QHC (p-QHC).*

We now easily obtain.

Theorem 2.13. *(X, Q_1, Q_2) is ij-feebly compact (p-feebly compact) iff (X, Q_1^s, Q_2^s) is respectively so.*

Definition 2.14. *A subset A in a space (X, Q_1, Q_2) is said to be ij-semiopen [I] iff there exists $B \in Q_i$ such that $B \subset A \subset Q_j$-cl B and hence iff* $A \subset Q_j$-cl $(Q_i$-int $A)$.

A space (X, Q_1, Q_2) is ij-S-closed [6] iff every ij-semiopen cover of X has a finite subfamily whose Q_j-closures cover the space. X is called p-S-closed [6] iff it is 12-S-closed as well as 21-S-closed.

Theorem 2.15. *(X, Q_1, Q_2) is ij-S-closed (p-S-closed) iff (X, Q_1^s, Q_2^s) is respectively so.*

Proof. The theorem follows from Corollary 2.2 and the following lemma.

Lemma 2.16. *A space (X, Q_1, Q_2) is ij-S-closed iff every cover of* **X** *by ji-rc sets has a finite subfamily which is a cover of X.*

Proof. If C is any ji-rc set in X then it is also ij-semiopen and Q_j-closed, and hence necessity part follows.

Now let the condition hold and $\{G_\alpha : \alpha \in I\}$ be a cover of X by ij-semiopen sets. Then Q_j-cl $G_\alpha = Q_j$-cl $(Q_i$-int $G_\alpha)$ is a ji-rc set for each $\alpha \in I$ and also $\{Q_j$-cl $G_\alpha : \alpha \in I\}$ covers X. Hence by hypothesis, X is ij-S-closed.

P e r v i n [9] defines a space (X, Q_1, Q_2) to be p-connected iff X cannot be expressed as the union of two nonempty disjoint open sets, one of which is Q_1-open and the other Q_2-open. Now we have

Theorem 2.17. *(X, Q_1, Q_2) is p-connected iff (X, Q_1^s, Q_2^s) is so.*

Proof. We only observe that if $A \in Q_i$ and $B \in Q_j$ such that $X = A \cup B$ and $A \cap B = \varnothing$, then $A \in Q_i^s$ and $B \in Q_j^s$.

Definition 2.18. *A bitopological property R is said to be a p-semiregular property provided a space (X, Q_1, Q_2) has the property R iff (X, Q_1^s, Q_2^s) has the property R.*

From Theorems 2.6, 2.7, 2.10, 2.12, 2.13, 2.15 and 2.17 we can now say

Theorem 2.19. *For a bitopological space, the property of being p-Hausdorff or p-Urysohn or $p-E_1$ or p-QHC or p-feebly compact or p-S-closed or p-connected is a p-semiregular property.*

Definition 2.20. *Two spaces (X, Q_1, Q_2) and (X, P_1, P_2) are said to be p-ro-equivalent iff $Q_i^s = P_i^s$, for $i = 1, 2$.*

Theorem 2.21. *An expansion (X, P_1, P_2) of a space (X, Q_1, Q_2) is p-ro-equivalent to (X, Q_1, Q_2) iff P_i-cl $U = Q_i$-cl U, for all $U \in P_j$.*

Proof. Let $U \in P_j$, then P_i-cl $U \subset Q_i$-cl U, since $P_i \supset Q_i$. Now let $V = X - P_i$-cl $U = P_i$-int $(X - U)$, then P_i-int $(P_j$-cl $V) = V$ and therefore $V \in P_i^s$ which is same as Q_i^s. Hence $V \in Q_i$ i.e., P_i-cl U is Q_i-closed, so Q_i-cl $U \subset P_i$-cl U. Thus P_i-cl $U = Q_i$-cl U. Conversely, let (X, P_1, P_2) be an expansion of (X, Q_1, Q_2) and P_i-cl $U = Q_i$-cl U for all $U \in P_j$. To show $P_i^s = Q_i^s$ (for $i = 1, 2$) we observe that the base $\mathcal{B}(P_i^s) = \{V \in P_i : V = P_i$-int $(P_j$-cl $V)\}$ for P_i^s is identical with the family $\{X - P_i$-cl $U : U \in P_j\}$. In fact if $U \in P_j$ then $X - P_i$-cl $U \in \mathcal{B}(P_i^s)$ and if $V = P_i$-int $(P_j$-cl $V)$, then $V = X - P_i$-cl U where $U = X - P_j$-cl V. By our supposition P_i-cl $U = Q_i$-cl U, for all $U \in P_j$. Hence $\mathcal{B}(P_i^s) \subset Q_i$, which implies that $P_i^s \subset Q_i$ $(i = 1, 2)$ and therefore by Remark 2.5, $P_i^s \subset Q_i^s$. Again $Q_i \subset P_i$ implies $Q_i^s \subset P_i$ and so $Q_i^s \subset P_i^s$, for $i = 1, 2$. Hence we get $Q_i^s = P_i^s$ $(i = 1, 2)$.

3. **Pairwise submaximality of a bitopological space.** **Definition 3.1.** *Let (X, Q_1^0, Q_2^0) be a p-semiregular space and $\mathcal{C}$ be the collection of all spaces (X, Q_1^x, Q_2^x) which are p-ro-equivalent to (X, Q_1^0, Q_2^0), that is, p-semiregularization of each (X, Q_1^x, Q_2^x) is (X, Q_1^0, Q_2^0). Then $\mathcal{C}$ is partially ordered set under the relation „$\leq$" such that $(X, Q_1^x, Q_2^x) \leq (X, Q_1^\beta, Q_2^\beta)$ iff $Q_i^x \subset Q_i^\beta$ (for $i = 1, 2$, where (X, Q_1^x, Q_2^x) and $(X, Q_1^\beta, Q_2^\beta)$ are in $\mathcal{C}$ Then a maximal element of $\mathcal{C}$ is defined to be p-submaximal bitopological space.*

We now prove the following existence theorem for p-submaximal spaces.

Theorem 3.2. *Every bitopological space is coarser than some p-submaximal space.*

Proof. Let (X, Q_1, Q_2) be a bitopological space and let (X, Q_1^s, Q_2^s) be its p-semiregularization. Let $\mathcal{C}$ be the collection of all spaces (X, Q_1^x, Q_2^x) such that $Q_i^{xs} = Q_i^s$ (for $i = 1, 2$). $\mathcal{C}$ is a poset with the relation '$\leq$' such that $(X, Q_1^x, Q_2^x) = (X, Q_1^\beta, Q_2^\beta)$ iff $Q_i^x \subset Q_i^\beta$, for $i = 1, 2$.

Let $\mathcal{C}_1 = \{(X, Q_1^x, Q_2^x) \in \mathcal{C} : Q_i \subset Q_i^x$ for $i = 1, 2\}$. Then $\mathcal{C}_1$ is also a poset under the same relation as in $\mathcal{C}$. Let $\mathcal{C}_0$ be a chain in $\mathcal{C}_1$. Let $\mathcal{B}_1 = \cup Q_1^x$ and $\mathcal{B}_2 = \cup Q_2^x$ where $(X, Q_1^x, Q_2^x) \in \mathcal{C}_0$. It is clear that $\mathcal{B}_i$ forms a base for some topology Q_i', for $i = 1, 2$. Obviously, $Q_i \subset Q_i'$, for $i = 1, 2$. We show that $Q_i^s = Q_i'^s$ (for $i = 1, 2$), i.e., (X, Q_1, Q_2) and (X, Q_1', Q_2') are p-ro-equivalent. Let $U \in Q_j'$. Then Q_i-cl $U \supset Q_i'$-cl U as $Q_i \subset Q_i'$ $(i = 1, 2)$. Let $x \in Q_i'$-cl U and let W be any Q_i-open set containing x. Let $W_i \in \mathcal{B}_i$ such that $x \in W_i \subset W$, then $W_i \in Q_i^x$ for some $(X, Q_1^x, Q_2^x) \in \mathcal{C}_0$. Let $W_i^* = Q_i^x$-int Q_j^x-cl W_i, then $W_i^* \in Q_i$ (since $W_i^* \in Q_i^{xs} = Q_i^s \subset Q_i$) and $U_y \in W_i^*$ and therefore $W_i^* \cap U \neq \emptyset$. Let $y \in W_i^* \cap U$ and $y \in U_y \subset U$, where $y \in \mathcal{B}_j$. Then $U_y \in Q_j^\beta$ for some $(X, Q_1^\beta, Q_2^\beta) \in \mathcal{C}_0$. Now if possible, let $W \cap U = \emptyset$, then $W_i \cap U_y = \emptyset$ and so

Q_j^β-cl $W_i \cap U_y = \emptyset$. But (X, Q_1^x, Q_2^x) and $(X, Q_1^\beta, Q_2^\beta)$ are comparable and p-ro-equivalent. Hence by Theorem 2.21, Q_j^x-cl $W_i = Q_j^\beta$-cl W_i and therefore $W_i^* \subset Q_j^\beta$-cl W_i which implies $W_i^* \cap U_y = \emptyset$ i.e., $y \notin W_i^*$. This contradicts that $y \in W_i^* \cap U$. Hence $W \cap U \neq \emptyset$ i.e., $x \in Q_i$-cl U, so Q_i-cl $U = Q_i'$-cl U. Thus $(X, Q_1', Q_2') \in \mathcal{C}_0$ and clearly it is an upper bound of $\mathcal{C}_0$. Hence by Zorn's Lemma, $\mathcal{C}_1$ has a maximal element, say (X, Q_1^*, Q_2^*). This maximal element of $\mathcal{C}_1$ is obviously a maximal element of $\mathcal{C}$ and therefore (X, Q_1^*, Q_2^*) is a p-submaximal space which is an expansion of (X, Q_1, Q_2).

Theorem 3.3. *A space (X, Q_1, Q_2) is p-submaximal if every Q_j-dense subset of X is Q_i-open.*

Proof. Let (X, P_1, P_2) be an expansion of (X, Q_1, Q_2) such that $P_i^s = Q_i^s$, for $i = 1, 2$. Let $U \in P_i$, then $U_0 = P_i$-int P_j-cl $U \in P_i^s$ and so $U_0 \in Q_i$. Also let $V = U \cup (X - Q_j$-cl $U)$, then V is Q_j-dense. Therefore by hypothesis V is Q_i-open. But $U = U_0 \cap V$ (since P_j-cl $U = Q_j$-cl U, by Theorem 2.21), hence $U \in Q_i$ which implies $Q_i = P_i$. Thus (X, Q_1, Q_2) is a p-submaximal space.

Definition 3.4. (a) *A subset A of a space (X, Q_1, Q_2) is said to be p-dense* [11] *iff every non-empty subset of X, which is the intersection of a Q_1-open and a Q_2-open subset of X, has non-void intersection with A.*

(b) [3] *For a topological space (X, Q) and for any subset A of X, the space $(X, Q(A))$, where $Q(A)$ is the topology generated by the subbase $Q \cup \{A\}$, is called the simple expansion of Q by A.*

Remark 3.5. As a simple consequence of the above definition we see that a subset A of (X, Q_1, Q_2) is p-dense in X iff A is dense in X w.r.t. the topology $Q_1(U)$, for each $U \in Q_2$, or equivalently iff A is dense in X w.r.t. the topology $Q_2(V)$ for each $V \in Q_1$.

Theorem 3.6. *In a p-submaximal space (X, Q_1, Q_2) every p-dense subset is p-open (i.e., Q_1-open as well as Q_2-open).*

Proof. Let (X, Q_1, Q_2) be a p-submaximal space and D be a p-dense subset of X. Let $P_i = Q_i(D)$ be the simple expansion of Q_i by D, for $i = 1, 2$. Then $P_i = \{V \cup (U \cap D) / U, V \in Q_i\}$, for $i = 1, 2$. We show that $P_i^s = Q_i^s$ for $i = 1, 2$. Let $W \in P_i$, then Q_j-cl $W \supset P_j$-cl W as $Q_j \subset P_j$. Let $x \in Q_j$-cl W and $T \in P_j$ such that $x \in T$. Let $W = V_i \cup (U_i \cap D)$, $V_i, U_i \in Q_i$ and $T = V_j \cup (U_j \cap D)$, $V_j, U_j \in Q_j$. If $x \in V_j$, then clearly $W \cap T \neq \emptyset$. So let $x \in U_j \cap D$. We have $U_j \cap W \neq \emptyset$ & so, $U_j \cap (V_i \cup U_i) \neq \emptyset$, which implies that $U_j \cap (V_i \cup U_i)) \cap D \neq \emptyset$. Then $(U_j \cap D) \cap ((V_i \cup U_i) \cap D) \neq \emptyset$ and hence $(U_j \cap D) \cap (V_i \cup (U_i \cap D)) \neq \emptyset$, so that $T \cap W \neq \emptyset$. Thus $x \in P_j$-cl W and we get ultimately Q_j-cl $W = P_j$-cl W. Then P_i and Q_i are ro-equivalent (by Theorem 2.21), but (X, Q_1, Q_2) is p-submaximal. Hence $P_i = Q_i$ or $U \in Q_i$.

Theorem 3.7. *Every maximal-R (minimal-R) bitopological space is a p-submaximal (p-semiregular) space if R is a p-semiregular property.*

Proof. If a maximal-R (minimal-R) space (X, Q_1, Q_2) is not p-submaximal (p-semiregular) then there exists a p-submaximal space (X, P_1, P_2) finer than (X, Q_1, Q_2) (p-semiregular space (X, Q_1^s, Q_2^s) coarser than (X, Q_1, Q_2)) which has the property R. This contradicts that (X, Q_1, Q_2) is maximal-R (minimal-R).

In Theorem 2.19 we found some p-semiregular properties. It now finally follows by virtue of the above theorem that

Theorem 3.8. (a) *A bitopological space which is either minimal p-Hausdorff or minimal p-Urysohn or minimal* $p-E_1$, *is p-semiregular (spaces which are minimal w.r.t. other p-semiregular properties, are not mentioned because of triviality.*
(b) *A bitopological space which is either maximal p-QHC or maximal p-feebly compact or maximal p-S-closed or maximal p-connected, is a p-submaximal space (to avoid triviality other cases are not mentioned).*

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ON AN APPLICATION OF FREDHOLM'S ALTERNATIVE FOR NONLINEAR ABSTRACT INTEGRAL EQUATIONS

BY

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The present paper studies an abstract nonlinear analogue of the Volterra integral equation. With the help of the technique developed in [1] a fundamental theory of the equation considered is constructed.

Let Ω be a topological space with Borelian measure μ and $\Phi : x\to M_x$ be a mapping which associates every element $x\in\Omega$ to a closed subset M_x of Ω, and let denote by $\mathcal{M}$ the set $\{M_x | x\in\Omega\}$.

We will say that the conditions (A) are fulfilled if the following conditions hold :

A1. For $M_x\in\mathcal{M}$ the following is fulfilled :

$$\omega=\sup_{x\in\Omega}\mu(M_x)<+\infty.$$

A2. For any $\varepsilon>0$ and $x\in\Omega$ there exists a neighbourhood $V=V(x, \varepsilon)$ such that for every $y\in V$ we have

$$\mu(M_x\Delta M_y)=\mu(\{M_x\setminus M_y\}\cup\{M_y\setminus M_x\})<\varepsilon.$$

A3. For any $x\in\Omega$ and $y\in M_x$ one has $M_y\subseteq M_x$.

A4. For every $x\in\Omega$ and for every nonempty relatively open subset O of M_x there exists a neighbourhood $W=W(x, O)$ such that for every $y\in W$ the following holds

$$M_y\cap O\neq\emptyset.$$

A5. There exists a point $x_0\in\Omega$ for which $\mu(M_{x_0})=0$.

Example 1. Let $\Omega=[0, 1]$ and let $a:[0, 1]\to[0, 1]$. We define the mapping $\Phi : x\to[0, a(x)]$.

Then the conditions A2 to A5 are fulfilled if and only if $0\leqslant a(x)\leqslant x$ and the function $a(x)$ is continuous for $x\in[0, 1]$.

Let B be a real Banach space with norm $\|\cdot\|_B$. Consider the linear spaces

$$C^{-1}=C^{-1}(\Omega, B)=\{f\,|\,f:\Omega\to B,\ \sup_{x\in\Omega}\|f(x)\|_B<\infty\},$$

$$C=C(\Omega, B)=\{f\,|\,f:\Omega\to B \text{ is continuous and } \sup_{x\in\Omega}\|f(x)\|_B<\infty\}.$$