Theorem 3.8. (a) *A bitopological space which is either minimal p-Hausdorff or minimal p-Urysohn or minimal p — E_1, is p-semiregular (spaces which are minimal w.r.t. other p-semiregular properties, are not mentioned because of triviality.*
(b) *A bitopological space which is either maximal p-QHC or maximal p-feebly compact or maximal p-S-closed or maximal p-connected, is a p-submaximal space (to avoid triviality other cases are not mentioned).*

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ON AN APPLICATION OF FREDHOLM'S ALTERNATIVE FOR NONLINEAR ABSTRACT INTEGRAL EQUATIONS

BY

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The present paper studies an abstract nonlinear analogue of the Volterra integral equation. With the help of the technique developed in [1] a fundamental theory of the equation considered is constructed.

Let Ω be a topological space with Borelian measure μ and $\Phi : x \to M_x$ be a mapping which associates every element $x \in \Omega$ to a closed subset M_x of Ω, and let denote by $\mathcal{M}$ the set $\{M_x | x \in \Omega\}$.

We will say that the conditions (A) are fulfilled if the following conditions hold :

A1. For $M_x \in \mathcal{M}$ the following is fulfilled :

$$\omega = \sup_{x \in \Omega} \mu(M_x) < +\infty.$$

A2. For any $\varepsilon > 0$ and $x \in \Omega$ there exists a neighbourhood $V = V(x, \varepsilon)$ such that for every $y \in V$ we have

$$\mu(M_x \Delta M_y) = \mu(\{M_x \setminus M_y\} \cup \{M_y \setminus M_x\}) < \varepsilon.$$

A3. For any $x \in \Omega$ and $y \in M_x$ one has $M_y \subseteq M_x$.

A4. For every $x \in \Omega$ and for every nonempty relatively open subset O of M_x there exists a neighbourhood $W = W(x, O)$ such that for every $y \in W$ the following holds

$$M_y \cap O \neq \varnothing.$$

A5. There exists a point $x_0 \in \Omega$ for which $\mu(M_{x_0}) = 0$.

Example 1. Let $\Omega = [0, 1]$ and let $a : [0, 1] \to [0, 1]$. We define the mapping $\Phi : x \to [0, a(x)]$.

Then the conditions A2 to A5 are fulfilled if and only if $0 \leqslant a(x) \leqslant x$ and the function $a(x)$ is continuous for $x \in [0, 1]$.

Let B be a real Banach space with norm $\|\cdot\|_B$. Consider the linear spaces

$$C^{-1} = C^{-1}(\Omega, B) = \{f \mid f : \Omega \to B, \sup_{x \in \Omega} \| f(x) \|_B < \infty\},$$

$$C = C(\Omega, B) = \{f \mid f : \Omega \to B \text{ is continuous and } \sup_{x \in \Omega} \|f(x)\|_B < \infty\}.$$

Remark 1. The space C^{-1} is a Banach space with respect to the norm $\|f\|=\sup_{x\in\Omega}\|f(x)\|_B$.

Definition 1. *We say that the function $f\in C^{-1}$ is $\mathcal{M}$-almost separably-valued if for any $M_x\in\mathcal{M}$ there exists a set $E_x\subseteq M_x$, $\mu(E_x)=0$ such that $f(M_x\setminus E_x)$ is a separable subset of B.*

The operator $F:\Omega\times C^{-1}\to C^{-1}$ will be said to satisfy the conditions (B) when the following conditions are fulfilled:

B1. For every $\varepsilon>0$ and $f_0\in C^{-1}$ there exists $\delta=\delta(\varepsilon, f_0)>0$, such that for every $f\in C^{-1}$ for which $\|f-f_0\|<\delta$ the following inequality holds for $x\in\Omega$

$$\|F(x, f)-F(x, f_0)\|<\varepsilon.$$

B2. For every $\varepsilon>0$ and $x_0\in\Omega$ there exists a neighbourhood $U=U(x_0, \varepsilon)$ of the point x_0 such that for every $x\in U$ the following inequality holds for $f\in C^{-1}$

$$\|F(x, f)-F(x_0, f)\|<\varepsilon.$$

B3. For every fixed element $x\in\Omega$ and every fixed function $f\in C^{-1}$ the mapping $\varphi_f(y)=F(x, f)(y)$ is $\mathcal{M}$-almost separably-valued and weakly measurable with respect to M_x for $x\in\Omega$.

B4. For every $x\in\Omega$ and $f\in C^{-1}$ the following inequality holds

$$\|F(x, f)\|\leq A\|f\|,$$

where the constant A does not depend on x and f.

Remark 2. The condition B3 is satisfied if for example B is separable and $\varphi_f(y)$ is continuous with respect to y.

Consider the equation

$$f+Kf=p, \tag{1}$$

where f, $p\in C^{-1}$ while the operator K is defined by the equality

$$(Kf)(x)=\int_{M_x}F(x, f)(y)\,d\mu_y, \quad x\in\Omega. \tag{2}$$

Since for every linear and continuous functional φ^* defined on B the map $\langle\varphi^*, \varphi_f\rangle:\Omega\to R$ is measurable on every $M_x\in\mathcal{M}$ then the existence of the integral in (2) is guaranteed by the condition (B) and the theorems 3.7.4, p. 94 and 3.5.3, p. 86 in [2].

Remark 3. The condition A3 allows to consider for every point $a\in\Omega$ the restrictions $K_a:C^{-1}(M_a, B)\to C^{-1}(M_a, B)$ $(K_a:C^{-1}(M_a, B)\to C(M_a, B))$ where the operator K_a is defined by the equality (2), $C^{-1}(M_0, B)$ is the space of the bounded functions $f:M_a\to B$ and $C(M_a, B)$ is the space of the bounded and continuous functions $f:M_a\to B$. In this case the restriction $f\mid M_a$ of the solution $f\in C^{-1}(\Omega, B)$ of the equation $f+Kf=p$, $p\in C^{-1}(\Omega, B)$ is a solution of the equation $\tilde{\varphi}+K_a\tilde{\varphi}=p\mid M_a$, $\tilde{\varphi}\in C^{-1}(M_a, B)$.

Lemma 1. *Let the conditions* A1, B3 *and* B4 *hold. Then the operator K maps C^{-1} in C^{-1}.*

Proof. Let $f\in C^{-1}$ be an arbitrary element. Then A1 and B4 imply $\|Kf\|\leq A\omega\|f\|$, i.e. $Kf\in C^{-1}$.

Lemma. 2 *Let the conditions* A1, B1, B3 *and* B4 *hold. Then K is a continuous map from C^{-1} into C^{-1}.*

If, in addition the conditions A2 *and* B2 *hold, then K maps C^{-1} into C continuously.*

Proof. Let $\lim_{n\to\infty} f_n=f_0$, $\{f_n\}\subset C^{-1}$, $f_0\in C^{-1}$ and let $\varepsilon>0$ be an arbitrary number. Condition B1 implies that there exists a number $n_0=n_0(\varepsilon, f_0)$ such that if $n\geqslant n_0$ then

$$\|F(x, f_n)-F(x, f_0)\|<\varepsilon/\omega, \text{ for } x\in\Omega.$$

The above inequality and A1 yield $\|Kf_n-Kf_0\|<\varepsilon$.
Now assume that the conditions A2 and B2 are fulfilled and let $f\in C^{-1}$ be an arbitrary element. We will show that the function Kf is continuous. Let $\varepsilon>0$ and $x_0\in\Omega$. From A2 and B2 it follows that there exists a neighbourhood $W=W(\varepsilon, x_0)=U(\varepsilon, x_0)\cap V(\varepsilon, x_0)$ such that for $x\in W$ the following inequalities hold

$$\|F(x, f)-F(x_0, f)\|<\varepsilon/(3\omega), \quad \mu(M_x\Delta M_{x_0})<\varepsilon/(3A\|f\|). \tag{3}$$

Then, for $x\in W$, (3) implies

$$\|(Kf)(x)-(Kf)(x_0)\|_B=\Big\|\int_{M_x}F(x, f)(y)\,d\mu_y-\int_{M_{x_0}}F(x_0, f)(y)\,d\mu_y\Big\|_B\leqslant$$

$$\begin{aligned}&\leqslant\Big\|\int_{M_x\cap M_{x_0}}[F(x, f)(y)-F(x_0, f)(y)]\,d\mu_y\Big\|_B+\Big\|\int_{M_x\setminus M_{x_0}}F(x, f)(y)\,d\mu_y\Big\|_B+\\ &\quad+\Big\|\int_{M_{x_0}\setminus M_x}F(x_0, f)(y)\,d\mu_y\Big\|_B<[\varepsilon/(3\omega)]+\\ &\quad+2A\|f\|\,\varepsilon/(3A\|f\|)\,\omega=\varepsilon.\end{aligned} \tag{4}$$

The following lemma treats the problem of the compactness of the operator K in the compactly open topology.

Lemma 3. *Let the conditions* A1—A4 *be fulfilled, the conditions* (B) *hold and let $B(0, R)$ be the closed ball with center 0 and radius R. Then the set $K(B(0, R))$ is uniformly bounded and equicontinuous.*

Proof. Let $f\in B(0, R)$. Conditions A1 and B4 yield

$$\|Kf\|=\sup_{x\in\Omega}\Big\|\int_{M_x}F(x, f)(y)\,d\mu_y\Big\|_B\leqslant A\omega R.$$

i.e. the set $K(B(0, R))$ is uniformly bounded.

Let $x_0\in\Omega$ be an arbitrary fixed element, $f\in B(0, R)$ and let $\varepsilon>0$ be an arbitrary number. A2 and B2 imply that there exists a neighbourhood $W=W(x_0, \varepsilon)=U(x_0, \varepsilon)\cap V(x_0, \varepsilon)$ such that for $x\in W$ the following inequalities are satisfied

$$\mu(M_x\Delta Mx_0)<\varepsilon/(3AR), \quad \|F(x_0, f)-F(x, f)\|<\varepsilon/(3\omega). \tag{5}$$

Then for every $x\in W$ the relations (4) and (5) imply that the following inequality holds

$$\| (Kf)(x)-(Kf)(x_0) \|_B<\varepsilon.$$

This completes the proof of the lemma.

Remark 4. The Ascoli-Arzelà theorem that the necessary and sufficient condition for the set $K(B(0, R))$ to be relatively compact with respect to the compactly open topology is the demand for the sets $H_x=\{(Kf)(x) \mid f\in B(0, R)\}$ to be relatively compact in B for every $x\in\Omega$ and Ω to be a regular locally compact topological space.

Theorem 1. *Suppose that the following conditions hold:*

1. *The conditions* (A) *and* (B) *are fulfilled.*
2. *The space* Ω *is connected.*
3. *For every element* $b\in\Omega$ *and* $f\in C$ *the following inequality holds*

$$\sup_{x\in M_b}\ \sup_{y\in M_x} \| F(x, f)(y) \|_B\leqslant A_1(f) \sup_{y\in M_b} \| f(y) \|_B,$$

where $A_1(f)$ *is independent on* M_x.

Then the equation

$$f=\lambda Kf, \tag{6}$$

has only the trivial solution $f(x)\equiv 0$ *for* $\lambda\in R$.

Proof. Let $f_0\in C^{-1}$ be a solution of (6). Lemma 3 implies that f_0 is a continuous solution of (6) and by $\mathcal{N}_{f_0}$ denote the set

$$\mathcal{N}_{f_0}=\{x \mid x\in\Omega,\ f(y)=0,\ y\in M_x\}.$$

The condition A5 implies that there exists an element $x_0\in\Omega$ such that $\mu(M_{x_0})=0$. Assume that $M_{x_0}\neq\varnothing$ (The case $M_{x_0}=\varnothing$ is trivial). Then A3 yields that $\mu(M_x)=0$ for any $x\in M_{x_0}$ and from (6) we obtain that $f_0(x)=0$ for $x\in M_{x_0}$. Hence $x_0\in\mathcal{N}_{f_0}$ i.e. $\mathcal{N}_{f_0}\neq\varnothing$.

The set $\mathcal{N}_{f_0}$ is closed. Let $\{x_\alpha\}_{\alpha\in D}\subset\mathcal{N}_{f_0}$ where D is an arbitrary oriented set, be an arbitrary convergent generalized sequence and denote $\lim\limits_{\alpha} x_\alpha=x^*$. We will prove that $x^*\in N_{f_0}$.

Let $z\in M_{x^*}$ be a fixed point and let $\varepsilon>0$ be an arbitrary number. There exists a neighbourhood $W(\varepsilon, z)$ of the point z such that for $y\in W$ we have

$$\| f(y)-f(z) \|_B<\varepsilon. \tag{7}$$

From A4 it follows that for the set $O_{x^*}=W\cap M_{x^*}$ there exists a neighbourhood U_x of x^* for which $M_y\cap O_{x^*}\neq\varnothing$ for $y\in U_{x^*}$.

There exists an element $\alpha_0=\alpha_0(\varepsilon, x^*)$ with the property

$$x_\alpha\in U_{x^*},\quad M_{x_\alpha}\cap O_{x^*}\neq\varnothing,$$

for $\alpha>\alpha_0$. Let $y^*\in M_{x_\alpha}\cap O_{x^*}$ for some $\alpha>\alpha_0$.

Then obviously $f(y^*)=0$ and (7) yields $\| f(z) \|_B<\varepsilon$ whence we conclude that $f(z)=0$ therefore $x^*\in\mathcal{N}_{f_0}$ i.e. $\mathcal{N}_{f_0}$ is closed.

We will prove that $\mathcal{N}_{f_0}$ is open. Let $a\in\mathcal{N}_{f_0}$ be any element while $\varepsilon>0$ satisfies the inequality $\varepsilon \mid\lambda\mid A_1(f_0)<1/2$. A2 implies that there exists a neighbourhood $V=V(a, \varepsilon)$ of the point a such that for $y\in V$ we have

$$\mu(M_a\Delta M_y)<\varepsilon. \tag{8}$$

Let $b\in V$ and denote by $\varphi(x)$ the restriction of $f_0(x)$ over M_b i.e. $\varphi(x)=f_0(x) \mid M_b$. Then from the conditions (A) and the condition 3 of Theorem 1 the following inequalities hold

$$\sup_{x\in M_b} \| \varphi(x) \|_B=\mid\lambda\mid \sup_{x\in M_b} \Big\| \int_{M_x} F(x, \varphi)(y)\, d\mu_y \Big\|_B\leqslant$$

$$\leqslant\mid\lambda\mid \sup_{x\in M_b}\int_{M_x\setminus M_a} \| F(x, \varphi)(y) \|_B\, d\mu_y\leqslant\mid\lambda\mid A_1(f_0) \sup_{x\in M_b}\mu(M_x/M_a) \sup_{y\in M_b} \|\varphi(y)\|_B\leqslant$$

$$\leqslant\frac{1}{2}\sup_{x\in M_b} \| \varphi(x) \|.$$

Hence $\varphi(x)\equiv 0$ for $x\in M_b$ i.e. $f_0(x)\equiv 0$ for $x\in M_b$. Then $b\in\mathcal{N}_{f_0}$ which implies that $\mathcal{N}_{f_0}$ is an open set.

Since Ω is connected, then $\mathcal{N}_{f_0}=\Omega$ and hence $f_0(x)=0$ for every $x\in\Omega$. This means that (6) possesses the trivial solution only for any $\lambda\in R$. This completes the proof of Theorem 1.

Remark 5. It is not difficult to see that the assertion of Theorem 1 remains valid if we replace the condition A1 with the following condition:

A1'. For $x\in\Omega$ the sets M_x are closed, connected, and every one contains a point $y=y(x)$ for which $\mu(M_y)=0$ and $\bigcup\limits_{x\in\Omega} M_x=\Omega$.

Theorem 2. *Suppose that the following conditions are fulfilled:*

1. *The assumptions of Theorem 1 are satisfied.*
2. *For every element* $f\in C^{-1}$ *and* $\lambda\in R$ *the following equality holds*

$$F(x, \lambda f)=\lambda F(x, f). \tag{9}$$

3. *The operator* K *is compact with respect to the topology generated by* $\|\cdot\|$.

Then the equation (1) *has a solution for every* $p\in C^{-1}$.

Proof. Since the identity is a (1.1.1) homeomorphism (see [1], p. 54), then Theorem 2 is implied by Theorem 1 and Theorem 3.2 from [1], p. 61.

Corollary 1. *Suppose that the following conditions hold;*

1. *The conditions 1 and 2 of Theorem 2 are fulfiled.*
2. Ω *is a compact regular topological space*
3. *For every closed ball with center 0 and radius* R, $B(0, R)$, *the sets* $H_x=\{(Kf)(x)/f\in B(0, R)\}$ *are relatively compact subsets of* B *for* $x\in\Omega$.

Then the equation (1) *has a solution for* $p\in C^{-1}$.

The proof of this corollary follows immediately from Lemma 3 and Theorem 2.

Example 2. Let $\Omega=[0, c]$, $c>0$ and let the function $\varphi:\Omega\times R^n\to R^n$ be continuous and satisfy the conditions:

(i) There is a constant $L>0$ such that for every $x\in\Omega$ and $u\in R^n$ the following inequality holds: $\| \varphi(x, u) \|_{R^n}\leqslant L \| u \|_{R^n}$.

(ii) $\varphi(x, \lambda u)=\lambda\varphi(x, u)$, $x\in\Omega$, $u\in R^n$, $\lambda\in R$.

Then, if we set

$$F(x, f)(y)=\varphi(x, f(y))$$

and if we take the mapping Φ from Example 1, the equation (1) takes the form :

$$f(x)+\int_0^{\alpha(x)}\varphi(x,\ f(y))\,dy=p(x). \tag{10}$$

The operator F and the mapping Φ satisfy the conditions (A), (B) and Condition 3 of Theorem 2. On the other hand, according to Lemma 3, the operator K defined by

$$(Kf)\,(x)=\int_0^{\alpha(x)}\varphi(x,\ f(y))\,dy$$

is compact. Then according to theorem 2 the equation (10) has a solution.

Remark 6. Obviously, the condition (i) is satisfied if $\varphi(x,\ 0)\equiv 0,\ x\in\Omega$ and the function φ satisfies the condition of Lipschitz with constant $L>0$. In this case the solution of equation (1) is unique.

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APPROXIMATION OF THE SOLUTION OF THE DARBOUX PROBLEM FOR THE EQUATION $\frac{\partial^2 z}{\partial x\partial y}\in F(x,\ y,\ z)$

BY

GEORGETA TEODORU

In this paper we consider a sequence of approximating equations for the multivalued hyperbolic equation

$$\frac{\partial^2 z}{\partial x\partial y}\in F(x,\ y,\ z),$$

and we prove that the sequence of solutions of these equations is uniformly convergent on compact sets to a solution of the Darboux problem in the sense of the definition in [1], [5].

Assume that the following hypotheses hold:

(H_1) $F:D\times\overline{\Omega}\to 2^{\mathbf{R}^n}$ is a multivalued function whose values are nonempty, convex and compact sets in $\mathbf{R}^n$; F is upper semi-continuous and sublinear ; here, $D=[0,\ a]\times[0,\ b]$ and $\Omega\subset\mathbf{R}^n$ is a bounded set in $\mathbf{R}^n$.

(H_2) For each $z\in\overline{\Omega}$, the mapping $(x,\ y)\to F(x,\ y,\ z)$ is measurable with respect to the Lebesgue measure on D.

(H_3) There exists a Lebesgue integrable function $g:D\to\mathbf{R}_+$ on D, such that $\|\xi\|\leqslant g(x,\ y),\ \forall\xi\in F(x,\ y,\ z),\ \forall(x,\ y)\in D,\ \forall z\in\overline{\Omega}$.

(H_4) The functions $\sigma\in AC([0,\ a]\,;\ \mathbf{R}^n),\ \tau\in AC([0,\ b]\,;\ \mathbf{R}^n)$ satisfy $\sigma(0)=\tau(0)$.

We consider the following Darboux problem

$$\frac{\partial^2 z}{\partial x\partial y}\in F(x,\ y,\ z),\ (x,\ y)\in D,\ z\in\overline{\Omega}, \tag{1}$$

with the boundary conditions

$$\begin{aligned} &z(x,\ 0)=\sigma(x),\ 0\leqslant x\leqslant a \\ &z(0,\ y)=\tau(y),\ 0\leqslant y\leqslant b, \end{aligned}\qquad \sigma(0)=\tau(0). \tag{2}$$

Definition (A. Cellina [1]). *The function $Z:D\to\mathbf{R}^n$ is called a solution of the Darboux problem (1)+(2), if it is absolutely continuous in the Carathéodory sense on D [3, §565—§568] and satisfies the equation (1) a.e. on D and the conditions (2) for each $x\in[0,\ a]$, $y\in[0,\ b]$.*

Theorem 2 [1] on the approximation of multivalued functions can be extended to