and if we take the mapping $\Phi$ from Example 1, the equation (1) takes the form :

$$f(x)+\int_{0}^{\varphi(x)}\varphi(x,\ f(y))\,dy=p(x). \tag{10}$$

The operator $F$ and the mapping $\Phi$ satisfy the conditions (A), (B) and Condition 3 of Theorem 2. On the other hand, according to Lemma 3, the operator $K$ defined by

$$(Kf)(x)=\int_{0}^{\varphi(x)}\varphi(x,\ f(y))\,dy$$

is compact. Then according to theorem 2 the equation (10) has a solution.

*Remark 6.* Obviously, the condition (i) is satisfied if $\varphi(x,\ 0)\equiv 0,\ x\in\Omega$ and the function $\varphi$ satisfies the condition of Lipschitz with constant $L>0$. In this case the solution of equation (1) is unique.

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# APPROXIMATION OF THE SOLUTION OF THE DARBOUX PROBLEM FOR THE EQUATION $\frac{\partial^2 z}{\partial x\partial y}\in F(x,\ y,\ z)$

BY

GEORGETA TEODORU

In this paper we consider a sequence of approximating equations for the multivalued hyperbolic equation

$$\frac{\partial^2 z}{\partial x\partial y}\in F(x,\ y,\ z).$$

and we prove that the sequence of solutions of these equations is uniformly convergent on compact sets to a solution of the Darboux problem in the sense of the definition in [1], [5].

Assume that the following hypotheses hold:

($H_1$) $F: D\times\overline{\Omega}\to 2^{\mathbf{R}^n}$ is a multivalued function whose values are nonempty, convex and compact sets in $\mathbf{R}^n$; $F$ is upper semi-continuous and sublinear; here, $D=[0,\ a]\times[0,\ b]$ and $\Omega\subset\mathbf{R}^n$ is a bounded set in $\mathbf{R}^n$.

($H_2$) For each $z\in\overline{\Omega}$, the mapping $(x,\ y)\to F(x,\ y,\ z)$ is measurable with respect to the Lebesgue measure on $D$.

($H_3$) There exists a Lebesgue integrable function $g: D\to\mathbf{R}_+$ on $D$, such that $\|\xi\|\leq g(x,\ y),\ \forall\xi\in F(x,\ y,\ z),\ \forall(x,\ y)\in D,\ \forall z\in\overline{\Omega}$.

($H_4$) The functions $\sigma\in AC([0,\ a];\ \mathbf{R}^n)$, $\tau\in AC([0,\ b];\ \mathbf{R}^n)$ satisfy $\sigma(0)=\tau(0)$.

We consider the following Darboux problem

$$\frac{\partial^2 z}{\partial x\partial y}\in F(x,\ y,\ z),\ (x,\ y)\in D,\ z\in\overline{\Omega}, \tag{1}$$

with the boundary conditions

$$\begin{aligned} z(x,\ 0)&=\sigma(x),\ 0\leq x\leq a,\\ z(0,\ y)&=\tau(y),\ 0\leq y\leq b,\end{aligned}\qquad \sigma(0)=\tau(0). \tag{2}$$

**Definition** (A. Cellina [1]). *The function $Z: D\to\mathbf{R}^n$ is called a solution of the Darboux problem (1)+(2), if it is absolutely continuous in the Carathéodory sense on $D$ [3, §565—§568] and satisfies the equation (1) a.e. on $D$ and the conditions (2) for each $x\in[0,\ a]$, $y\in[0,\ b]$.*

Theorem 2 [1] on the approximation of multivalued functions can be extended to

**Theorem 1.** *Let $F : \Lambda \subset \mathbf{R}^{n+2} \to 2^{\mathbf{R}^n}$ be a multi-valued function defined on open sets $\Lambda$ and whose values are nonemply, convex and compact sets in $\mathbf{R}^n$. Assume that $F$ is upper semi-continuous on $\Lambda$. Then, there exists a sequence of single-valued continuous functions $f_k : \Lambda \to \mathbf{R}^n$, $k \in \mathbf{N}$, such that $f_k \to F$ uniformly on compact sets, in the sense of the definition 6* [1]. The proof is completely analogous with that of [1].

**Theorem 2.** *Assume that the hypotheses $(H_1)-(H_4)$ are satisfied. Let $f_k : D \times \overline{\Omega} \to \mathbf{R}^n$ be a sequence of single-valued continuous functions such that $f_k \to F$ uniformly on compact sets and let $z_k : D \to \mathbf{R}^n$, $k \in \mathbf{N}$, be the solutions of the Darboux problems*

$$\frac{\partial^2 z}{\partial x \partial y} = f_k(x, y, z), \ (x, y) \in D, \ z \in \overline{\Omega}, \quad k \in \mathbf{N}, \tag{3}$$

$$\begin{cases} z(x, 0) = \sigma(x), \ 0 \leqslant x \leqslant a, \\ z(0, y) = \tau(y), \ 0 \leqslant y \leqslant b, \end{cases} \qquad \sigma(0) = \tau(0). \tag{4}$$

*Then, there exists a solution $Z : D \to \mathbf{R}^n$ of the Darboux problem (1)+(2) and a sequence of positive integers $\{n_p\}_{p \in \mathbf{N}}$ such that $z_{n_p}(x, y) \to Z(x, y)$ uniformly on $D$.*

*Proof.* Suppose $D \times \overline{\Omega} \subset \Lambda \subset \mathbf{R}^{n+2}$. Taking into account the sublinearity of $F$ and the uniform convergence of $\{f_k\}_{k \in \mathbf{N}}$ to $F$ on compact sets, it follows that $f_k$, $k \in \mathbf{N}$, is sublinear. Given the compact set $K = D \times \overline{\Omega}$, by definition 6 [1] for each $\varepsilon > 0$ there exists a natural number $k(\varepsilon)$ such that

$$d^* (\text{graph } f_k, \text{graph } F) < \varepsilon, \quad \forall k > k(\varepsilon).$$

By the definition of the separation [2], the previous inequality yields

$$\sup d\{[(x, y, z, f_k(x, y, z)), \text{graph } F]\} < \varepsilon, \quad \forall k > k(\varepsilon),$$

$(x, y, z, f_k(x, y, z)) \in \text{graph } f_k$, and moreover

$$\sup_{\substack{(x, y, z, f_k) \in \\ \in \text{graph } f_k}} \inf_{\substack{(\bar{x}, \bar{y}, \bar{z}, \xi) \in \\ \in \text{graph } F}} \{d[(x, y, z, f_k(x, y, z)), (\bar{x}, \bar{y}, \bar{z}, \xi)]\}\} < \varepsilon, \quad \forall k > k(\varepsilon). \tag{5}$$

From (5), taking into account that [2],

$$d[(x, y, z, f_k(x, y, z)), (\bar{x}, \bar{y}, \bar{z}, \xi)] = \max \{d[(x, y, z), (\bar{x}, \bar{y}, \bar{z})], \ d[f_k(x, y, z), \xi]\},$$

we get:

(6) $d[(x, y, z), (\bar{x}, \bar{y}, \bar{z})] < \varepsilon$ and (7) $d[f_k(x, y, z), \xi] = \| f_k(x, y, z) - \xi \| < \varepsilon$.

Relation (6) yields $d(z, \bar{z}) = \| z - \bar{z} \| < \varepsilon$. Thus,

$$\| \bar{z} \| \leqslant \| z - \bar{z} \| + \| z \| < \varepsilon + \| z \|. \tag{8}$$

Since $F$ is sublinear, there exist $k_1 > 0$, $k_2 \in \mathbf{R}$, such that

$$\| \xi \| \leqslant k_1 \| \bar{z} \| + k_2, \quad \forall \xi \in F(x, y, z). \tag{9}$$



By (7), (8), (9), we deduce

$$\begin{aligned} \| f_k(x, y, z) \| &\leqslant \| f_k - \xi \| + \| \xi \| < \varepsilon + k_1 \| \bar{z} \| + k_2 < \varepsilon + k_1(\varepsilon + \| z \|) + k_2 = \\ &= k_1 \| z \| + k_1 \varepsilon + k_2 + \varepsilon = k_1 \| z \| + k_3, \ (x, y, z) \in D \times \overline{\Omega}, \end{aligned} \tag{10}$$

where $k_3 = k_1 \varepsilon + k_2 + \varepsilon$, $k_3 \in \mathbf{R}$.

By (10) we conclude that $f_k$, $k \in \mathbf{N}$, is sublinear.

Because $f_k$, $k \in \mathbf{N}$, is continuous and sublinear, by the Schauder Fixed Point Theorem, the Darboux problems (3)+(4) has at least a solution $z_k : D \to \mathbf{R}^n$, $k \in \mathbf{N}$.

Indeed, (3)+(4) is equivalent to the integral equation

$$\begin{aligned} z(x, y) &= \sigma(x) + \tau(y) - \sigma(0) + \int_0^x \int_0^y f_k(\xi, \eta, z(\xi, \eta)) \, d\xi d\eta = \\ &= \alpha(x, y) + \int_0^x \int_0^y f_k(\xi, \eta, z(\xi, \eta)) \, d\xi d\eta, \quad k \in \mathbf{N}, \end{aligned}$$

where $\alpha : D \to \mathbf{R}^n$ is the function defined by

$$\alpha(x, y) = \sigma(x) + \tau(y) - \sigma(0), \ (x, y) \in D.$$

Define the operator $T_k : C(D\,;\, \mathbf{R}^n) \to C(D\,;\, \mathbf{R}^n)$, $k \in \mathbf{N}$, by

$$T_k z(x, y) = \alpha(x, y) + \int_0^x \int_0^y f_k(\xi, \eta, z(\xi, \eta)) \, d\xi \, d\eta, \ (x, y) \in D, \ k \in \mathbf{N}.$$

The operator $T_k$, $k \in \mathbf{N}$, is continuous. The set

$$B[\alpha\,;\, r] = \{z \mid z \in C(D\,;\, \mathbf{R}^n), \ \| z - \alpha \| < r\}, \ r > 0$$

is convex and closed and it is carried by the operator $T_k$, $k \in \mathbf{N}$, into itself. Indeed, let $z \in B[\alpha\,;\, r]$. Then, taking into account (10) one gets

$$\begin{aligned} \| T_k z - \alpha \| &\leqslant \int_0^x \int_0^y \| f_k(\xi, \eta, z(\xi, \eta)) \| \, d\xi d\eta \leqslant \int_0^{x_0} \int_0^{y_0} \| f_k(\xi, \eta, z(\xi, \eta)) \| \, d\xi d\eta \leqslant \\ &\leqslant \int_0^{x_0} \int_0^{y_0} [k_1 \| z(\xi, \eta) \| + k_3] \, d\xi \, d\eta, \ 0 \leqslant x \leqslant x_0, \ 0 \leqslant y \leqslant y_0, \ (x_0, y_0) \in D. \end{aligned}$$

We have

$$\| z(\xi, \eta) \| \leqslant \| z(\xi, \eta) - \alpha(\xi, \eta) \| + \| \alpha(\xi, \eta) \| \leqslant r + \sup \| \alpha(\xi, \eta) \| = C,$$

$(\xi, \eta) \in [0, x_0] \times [0, y_0] = \Delta$, and thereby

$$\| T_k z - \alpha \| \leqslant \int_0^{x_0} \int_0^{y_0} (k_1 C + k_3) \, d\xi d\eta = (k_1 C + k_3) \, x_0 y_0. \tag{11}$$

Choose $(x_0, y_0) \in D$ such that the condition

$$(k_1 C + k_3) \, x_0 y_0 \leqslant r \ \text{holds.} \tag{12}$$

By (11), (12) we obtain $\| T_k z - \alpha \| \leqslant r$, i.e. $T_k z \in B[\alpha\,;\, r]$, or $T_k B[\alpha\,;\, r] \subseteq B[\alpha\,;\, r]$, $k \in \mathbf{N}$. The set $T_k B[\alpha\,;\, r]$ is relatively compact by the

Arzelà-Ascoli Theorem. By Schauder's Fixed Point Theorem, the operator $T_k$, $k\in\mathbf{N}$, has at least a fixed point $z_k$, $k\in\mathbf{N}$, which is a solution of the problem (3)+(4) on $\Delta$. This solution can be extended to all of $D$ [5], [6].

Then

$$z_k(x,y)=\alpha(x,y)+\int_0^x\int_0^y f_k(\xi,\eta,z_k(\xi,\eta))\,d\xi\,d\eta,\quad (x,y)\in D,\ k\in\mathbf{N}. \tag{13}$$

is equivalent to

$$\frac{\partial^2 z_k(x,y)}{\partial x\,\partial y}=f_k(x,y,z_k(x,y)),\quad (x,y)\in D,\ z_k\in\overline{\Omega}\subset\mathbf{R}^n, \tag{3'}$$

$$z_k(x,0)=\sigma(x),\ 0\leq x\leq a,\qquad z_k(0,y)=\tau(y),\ 0\leq y\leq b,\qquad \tau(0)=\sigma(0), \tag{4'}$$

where $z_k$, $k\in\mathbf{N}$, denotes the extended solution to $D$.

The family $\{z_k(x,y)\}_{k\in\mathbf{N}}$ of solutions is equicontinuous and equibounded in the Banach space $C(D;\mathbf{R}^n)$.

For $\bar h,\bar k\in\mathbf{R}$, $(x+\bar h,y+\bar k)\in D$, one gets

$$z_k(x+\bar h,y+\bar k)-z_k(x,y)=\int_0^{x+\bar h}\int_0^{y+\bar k} f_k(\xi,\eta,z_k(\xi,\eta))\,d\xi\,d\eta-$$

$$-\int_0^x\int_0^y f_k(\xi,\eta,z_k(\xi,\eta))\,d\xi\,d\eta=\int_x^{x+\bar h}\int_0^y f_k(\xi,\eta,z_k(\xi,\eta))\,d\xi\,d\eta+$$

$$+\int_0^x\int_y^{y+\bar k} f_k(\xi,\eta,z_k(\xi,\eta))\,d\xi\,d\eta+\int_x^{x+\bar h}\int_y^{y+\bar k} f_k(\xi,\eta,z_k(\xi,\eta))\,d\xi\,d\eta.$$

and consequently, from the absolute continuity of the integral, we have

$$\|z_k(x+\bar h,y+\bar k)-z_k(x,y)\|<\varepsilon\quad\text{for}\quad |\bar h|,\ |\bar k|<\delta(\varepsilon).$$

Thus, the family $\{z_k(x,y)\}_{k\in\mathbf{N}}$ is equicontinuous. Taking into account the sublinearity of $f_k$, $k\in\mathbf{N}$, by Gronwall's inequality [4] from (13) we get

$$\|z_k(x,y)\|\leq B\Big[1+k_1\int_0^x\int_0^y \exp\Big(\int_s^x\int_t^y k_1\,d\xi\,d\eta\Big)\,ds\,dt\Big]\leq$$

$$\leq B[1+k_1\,ab\exp(k_1ab)]=M,\ (x,y)\in D,\ k\in\mathbf{N},$$

where $B=\sup_{(x,y)\in D}\|\alpha(x,y)\|+|k_2|\,ab$, and thus $\{z_k(x,y)\}_{k\in\mathbf{N}}$ is equibounded.

By the Arzelà-Ascoli Theorem, the sequence $\{z_k(x,y)\}_{k\in\mathbf{N}}$ contains an uniformly convergent subsequence to a continuous function $Z\in C(D;\mathbf{R}^n)$, $z_{k_p}(x,y)\to Z(x,y)$, $(x,y)\in D$. We shall prove that the function $Z$ obtained above is a solution of the Darboux problem (1)+(2). To this end we shall show that the conditions of characterization of a solution is fulfilled [7]. The conditions (2) from (4') are obviously satisfied. We have to prove

$$Z(x_2,y_2)-Z(x_1,y_2)-Z(x_2,y_1)+Z(x_1,y_1)\in\int_{x_1}^{x_2}\int_{y_1}^{y_2}F(\xi,\eta,Z(\xi,\eta))\,d\xi\,d\eta \tag{14}$$



for the compact $[x_1,x_2]\times[y_1,y_2]\subseteq D$. Thus, we prove the inequality

$$d\Big(Z(x_2,y_2)-Z(x_1,y_2)-Z(x_2,y_1)+Z(x_1,y_1),\ \int_{x_1}^{x_2}\int_{y_1}^{y_2}F(\xi,\eta,Z(\xi,\eta))\,d\xi\,d\eta\Big)<\varepsilon. \tag{15}$$

Integrating (3') on $[x_1,x_2]\times[y_1,y_2]\subseteq D$, one gets

$$z_k(x_2,y_2)-z_k(x_1,y_2)-z_k(x_2,y_1)+z_k(x_1,y_1)=\int_{x_1}^{x_2}\int_{y_1}^{y_2}f_k(\xi,\eta,z_k(\xi,\eta))\,d\xi\,d\eta. \tag{16}$$

By (15), (16) we have

$$d\Big(Z(x_2,y_2)-Z(x_1,y_2)-Z(x_2,y_1)+Z(x_1,y_1),\ \int_{x_1}^{x_2}\int_{y_1}^{y_2}F(\xi,\eta,Z(\xi,\eta))\,d\xi\,d\eta\Big)\leq$$

$$\leq d\big(Z(x_2,y_2)-Z(x_1,y_2)-Z(x_2,y_1)+Z(x_1,y_1),\ z_k(x_2,y_2)-z_k(x_1,y_2)- \tag{17}$$

$$-z_k(x_2,y_1)+z_k(x_1,y_1)\big)+d\Big(\int_{x_1}^{x_2}\int_{y_1}^{y_2}f_k(\xi,\eta,z_k(\xi,\eta))\,d\xi\,d\eta,\ \int_{x_1}^{x_2}\int_{y_1}^{y_2}F(\xi,\eta,Z(\xi,\eta))\,d\xi\,d\eta\Big).$$

Denoting for simplicity $k=k_p$, by $z_k(x,y)\to Z(x,y)$ uniformly on the compact set $D$, we deduce

$$d\big(Z(x_2,y_2)-Z(x_1,y_2)-Z(x_2,y_1)+Z(x_1,y_1),\ z_k(x_2,y_2)-z_k(x_1,y_2)-$$
$$-z_k(x_2,y_1)+z_k(x_1,y_1)\big)<\varepsilon/2,\quad \forall k\geq N_1(\varepsilon). \tag{18}$$

Inasmuch as $d(\xi,A)=d(0,\xi-A)$, where $0$ is the null vector in $\mathbf{R}^n$ and $A\subset\mathbf{R}^n$ we deduce

$$d\Big(\int_{x_1}^{x_2}\int_{y_1}^{y_2}f_k(\xi,\eta,z_k(\xi,\eta))\,d\xi\,d\eta,\ \int_{x_1}^{x_2}\int_{y_1}^{y_2}F(\xi,\eta,Z(\xi,\eta))\,d\xi\,d\eta\Big)=$$
$$=d\Big(0,\ \int_{x_1}^{x_2}\int_{y_1}^{y_2}\{f_k(\xi,\eta,z_k(\xi,\eta))-F(\xi,\eta,Z(\xi,\eta))\}\,d\xi\,d\eta\Big). \tag{19}$$

For $(\xi,\eta)\in[x_1,x_2]\times[y_1,y_2]$, by the Lemma [1], there exists a natural number $N_2=N_2(\varepsilon,(\xi,\eta))$ such that [2], [7]

$$f_k(\xi,\eta,z_k(\xi,\eta))\in B[F(\xi,\eta,Z(\xi,\eta)),\varepsilon],$$

for $\forall k\geq N_2(\varepsilon,(\xi,\eta))$, and therefore

$$d(f_k(\xi,\eta,z_k(\xi,\eta)),\ F(\xi,\eta,Z(\xi,\eta)))<\varepsilon,$$

or

$$d(0,\ f_k(\xi,\eta,z_k(\xi,\eta))-F(\xi,\eta,Z(\xi,\eta)))<\varepsilon.$$

Thus, by the definition 2 [1]

$$0\in\underline{\lim}\ \{f_k(\xi,\eta,z_k(\xi,\eta))-F(\xi,\eta,Z(\xi,\eta))\}.$$

Then

$$0\in\int_{x_1}^{x_2}\int_{y_1}^{y_2}\underline{\lim}\ \{f_k(\xi,\eta,z_k(\xi,\eta))-F(\xi,\eta,Z(\xi,\eta))\}\,d\xi\,d\eta. \tag{20}$$

Using the Proposition 3.2, [5] (or Proposition 2 [7]), (20) yields

$$0\in\underline{\lim}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\{f_k(\xi,\eta,z_k(\xi,\eta))-F(\xi,\eta,Z(\xi,\eta))\}\,d\xi\,d\eta. \tag{21}$$

By (21) and the definition 2 [1] each neighbourhood of $\theta$ intersects the sets of the form

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} \{f_k(\xi, \eta, z_k(\xi, \eta)) - F(\xi, \eta, Z(\xi, \eta))\}\, d\xi\, d\eta$$

for $k$ large enough, $\forall k \geqslant N_2(\varepsilon, (\xi, \eta))$.

By (21) one gets

$$d(\theta, \int_{x_1}^{x_2}\int_{y_1}^{y_2} \{f_k(\xi, \eta, z_k(\xi, \eta)) - F(\xi, \eta, Z(\xi, \eta))\}\, d\xi\, d\eta < \varepsilon/2 \tag{22}$$

for $k \geqslant N_2(\varepsilon, (\xi, \eta))$, or by (19)

$$d(\int_{x_1}^{x_2}\int_{y_1}^{y_2} f_k(\xi, \eta, z_k(\xi, \eta))\, d\xi\, d\eta, \int_{x_1}^{x_2}\int_{y_1}^{y_2} F(\xi, \eta, Z(\xi, \eta))\, d\xi d\eta) < \varepsilon/2 \tag{23}$$

for $k \geqslant N_2(\varepsilon, (\xi, \eta))$.

Taking into account (18) and (23), (17) yields (15) for $\forall k \geqslant \max \{N_1(\varepsilon), N_2(\varepsilon, (\xi, \eta))\}$.

But $F(\xi, \eta, Z(\xi, \eta))$ is closed and then, by (15) for $\varepsilon \to 0$, one obtains

$$Z(x_2, y_2) - Z(x_1, y_2) - Z(x_2, y_1) + Z(x_1, y_1) \in \int_{x_1}^{x_2}\int_{y_1}^{y_2} F(\xi, \eta, Z(\xi, \eta))\, d\xi d\eta. \tag{24}$$

Recalling that (24) along with (2) are sufficient conditions in order that $Z$ be a solution of the Darboux problem (1)+(2), [5], [7], the proof is complete.

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# ON BOUNDED DETERMINISTIC AND STOCHASTIC INTEGRAL CONTRACTORS

BY

V. POPESCU and A. RĂŞCANU

In many papers the notion of deterministic or stochastic integral contractor is used (see e.g. [1], [3], [5]). The purpose of this paper is to prove that the existence of a bounded integral contractor is equivalent with a Lipschitz condition. Some advantages of the use of the approximations induced by an integral contractor are also given.

In the first section we consider the deterministic case (the case of ordinary differential equations and functional-differential equations) and in the second section we consider the stochastic case.

**§1. Differential Equations with Integral Contractors. 1.1. Ordinary Differential Equations.** Let $(B, \|\cdot\|)$ be a real Banach space and let $f: [0, T] \times B \to B$ be a continuous mapping. Consider the nonlinear evolution equation

$$dx/dt = f(t, x), \ 0 \leqslant t \leqslant T; \ x(0) = \xi. \tag{1.1}$$

The following definitions may be found in [3].

**Definition 1.1.** *Suppose* $\Gamma: [0, T] \times B \to L(B, B)$ *is a bounded continuous mapping and there exist positive numbers* $K$ *such that for any* $x, y \in C([0, T]; B)$ *one has*

$$\|f[t, x(t) + y(t) + \int_0^t \Gamma(s, x(s))\, y(s)\, ds] - f(t, x(t)) - \Gamma(t, x(t))\, y(t)\| \leqslant \tag{1.2}$$

$$\leqslant K \sup \{\|y(s)\| ; s \in [0, t]\}, \ 0 \leqslant t \leqslant T.$$

*Then we say that* $f(t, x)$ *has a bounded integral contractor* $\{I + \int_0^t \Gamma\}$.

**Definition 1.2.** *A bounded integral contractor is said to be regular if for any fixed* $x, z \in C([0, T]; B)$ *the integral equation*

$$y(t) + \int_0^t \Gamma(s, x(s))\, y(s)\, ds = z(t), \ 0 \leqslant t \leqslant T \tag{1.3}$$

*has a solution* $y \in C([0, T]; B)$.

The usual norm in $C([0, T]; B)$ is $\|x\| = \|x\|_T$ where

$$\|x\|_t = \sup \{\|x(s)\| ; s \in [0, t]\}, \ 0 \leqslant t \leqslant T. \tag{1.4}$$