By (21) and the definition 2 [1] each neighbourhood of 0 intersects the sets of the form

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} \{f_k(\xi, \eta, z_k(\xi, \eta)) - F(\xi, \eta, Z(\xi, \eta))\}\, d\xi\, d\eta$$

for k large enough, $\forall k \geq N_2(\varepsilon, (\xi, \eta))$.

By (21) one gets

$$d(0, \int_{x_1}^{x_2}\int_{y_1}^{y_2} \{f_k(\xi, \eta, z_k(\xi, \eta)) - F(\xi, \eta, Z(\xi, \eta))\}\, d\xi\, d\eta < \varepsilon/2 \tag{22}$$

for $k \geq N_2(\varepsilon, (\xi, \eta))$, or by (19)

$$d(\int_{x_1}^{x_2}\int_{y_1}^{y_2} f_k(\xi, \eta, z_k(\xi, \eta))\, d\xi\, d\eta, \int_{x_1}^{x_2}\int_{y_1}^{y_2} F(\xi, \eta, Z(\xi, \eta))\, d\xi d\eta) < \varepsilon/2 \tag{23}$$

for $k \geq N_2(\varepsilon, (\xi, \eta))$.

Taking into account (18) and (23), (17) yields (15) for $\forall k \geq \max\{N_1(\varepsilon), N_2(\varepsilon, (\xi, \eta))\}$.

But $F(\xi, \eta, Z(\xi, \eta))$ is closed and then, by (15) for $\varepsilon \to 0$, one obtains

$$Z(x_2, y_2) - Z(x_1, y_2) - Z(x_2, y_1) + Z(x_1, y_1) \in \int_{x_1}^{x_2}\int_{y_1}^{y_2} F(\xi, \eta, Z(\xi, \eta))\, d\xi d\eta. \tag{24}$$

Recalling that (24) along with (2) are sufficient conditions in order that Z be a solution of the Darboux problem (1)+(2), [5], [7], the proof is complete.

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ON BOUNDED DETERMINISTIC AND STOCHASTIC INTEGRAL CONTRACTORS

BY

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In many papers the notion of deterministic or stochastic integral contractor is used (see e.g. [1], [3], [5]). The purpose of this paper is to prove that the existence of a bounded integral contractor is equivalent with a Lipschitz condition. Some advantages of the use of the approximations induced by an integral contractor are also given.

In the first section we consider the deterministic case (the case of ordinary differential equations and functional-differential equations) and in the second section we consider the stochastic case.

§1. Differential Equations with Integral Contractors. 1.1. Ordinary Differential Equations. Let $(B, \|\cdot\|)$ be a real Banach space and let $f : [0, T] \times B \to B$ be a continuous mapping. Consider the nonlinear evolution equation

$$dx/dt = f(t, x), \ 0 \leq t \leq T ; \ x(0) = \eta. \tag{1.1}$$

The following definitions may be found in [3].

Definition 1.1. *Suppose* $\Gamma : [0, T] \times B \to L(B, B)$ *is a bounded continuous mapping and there exist positive numbers* K *such that for any* $x, y \in C([0, T] ; B)$ *one has*

$$\|f[t, x(t) + y(t) + \int_0^t \Gamma(s, x(s))\, y(s)\, ds] - f(t, x(t)) - \Gamma(t, x(t))\, y(t)\| \leq \tag{1.2}$$

$$\leq K \sup\{\|y(s)\| ; s \in [0, t]\}, \ 0 \leq t \leq T.$$

Then we say that $f(t, x)$ *has a bounded integral contractor* $\{I + \int_0^t \Gamma\}$.

Definition 1.2. *A bounded integral contractor is said to be regular if for any fixed* $x, z \in C([0, T] ; B)$ *the integral equation*

$$y(t) + \int_0^t \Gamma(s, x(s))\, y(s)\, ds = z(t), \ 0 \leq t \leq T \tag{1.3}$$

has a solution $y \in C([0, T] ; B)$.

The usual norm in $C([0, T] ; B)$ is $\|x\| = \|x\|_T$ where

$$\|x\|_t = \sup\{\|x(s)\| ; s \in [0, t]\}, \ 0 \leq t \leq T. \tag{1.4}$$

Another useful norm in $C([0, T]; B)$ may be defined as follows:

$$|||x||| = \sup\{\exp(-\lambda t) \cdot \|x\|_t : t \in [0, T]\}, \tag{1.5}$$

where $\lambda > 0$ is a fixed number. Since

$$\exp(-\lambda T) \cdot \|x\| \leq |||x||| \leq \|x\|, \tag{1.6}$$

the above two norms on $C([0, T]; B)$ are equivalent.

Lemma 1.1. *Assume that* $\Gamma : [0, T] \times B \to L(B, B)$ *is a bounded continuous mapping. Then for every* $x, z \in C([0, T]; B)$ *there exists a unique* $y \in C([0, T]; B)$, *solution of the equation* (1.3). *Furthermore, there exist constants* $\tilde{L}$, *independent on* x *and* z, *such that*

$$\|y\| \leq \tilde{L} \|z\|. \tag{1.7}$$

Proof. For fixed $x, z \in C([0, T]; B)$ let us define

$$(Sy)(t) = z(t) - \int_0^t \Gamma(s, x(s))\, y(s)\, ds, \quad y \in C([0, T]; B). \tag{1.8}$$

Notice that S is an affine continuous operator from $C([0, T]; B)$ into $C([0, T]; B)$. Furthermore, for every $y_1, y_2 \in C([0, T]; B)$ and for every $0 \leq s \leq t \leq T$ we have

$$\|(Sy_1)(s) - (Sy_2)(s)\| \leq \int_0^s \|\Gamma(\tau, x(\tau))\| \, \|y_1(\tau) - y_2(\tau)\| \, d\tau \leq$$

$$\leq M \int_0^t e^{\lambda\tau} e^{-\lambda\tau} \|y_1 - y_2\|_\tau \, d\tau \leq M\, |||y_1 - y_2||| \int_0^t \exp(\lambda\tau)\, d\tau \leq$$

$$\leq (M/\lambda)\, e^{\lambda t} \cdot |||y_1 - y_2|||;$$

which implies

$$\exp(-\lambda t)\, \|Sy_1 - Sy_2\|_t \leq (M/\lambda)\, |||y_1 - y_2|||.$$

For $\lambda \geq 2M + 1$ we deduce

$$|||Sy_1 - Sy_2||| \leq (1/2)\, |||y_1 - y_2||| \tag{1.9}$$

for all $y_1, y_2 \in C([0, T]; B)$.

Hence by the Banach fixed point theorem, the equation (1.3) has a unique solution which may be obtained as the uniform limit of the following sequence

$$\begin{cases} y_0(t) = z(t), \quad 0 \leq t \leq T, \\ y_n(t) = z(t) - \int_0^t \Gamma(s, x(s))\, y_{n-1}(s)\, ds, \quad 0 \leq t \leq T;\ n = 0, 1, 2, \ldots \end{cases} \tag{1.10}$$

Let us note that

$$|||y_n||| \leq |||y_n - y_{n-1}||| + |||y_{n-1} - y_{n-2}||| + \ldots + |||y_1 - y_0||| + |||y_0||| \leq$$

$$\leq 2\, |||y_1 - y_0||| + |||y_0||| \leq 2\, \|y_1 - z\| + \|z\|.$$

Therefore, for $n \to \infty$ we have

$$|||y||| \leq 2\, \|y_1 - z\| + \|z\|.$$

The above inequality easily gives the estimation (1.7) and Lemma 1.1 is completely proved.

Remark 1.1. In fact, Lemma 1.1 says that every bounded integral contractor is a regular one.

Theorem 1.1. *The function* $f(t, x)$ *has a bounded integral contractor* $\{I + \int_0^t \Gamma\}$ *if and only if* $f(t, .)$ *is Lipschitzian, uniformly with respect to* $t \in [0, T]$.

Proof. Let $\{I + \int_0^t \Gamma\}$ be a bounded integral contractor. For every $x, z \in B$ let us consider the constant functions $x(t) = x$, $z(t) = z$, $t \in [0, T]$. From Lemma 1.1 we deduce that the equation

$$y(t) + \int_0^t \Gamma(s, x)\, y(s)\, ds = z, \quad 0 \leq t \leq T \tag{1.11}$$

has a unique solution $y \in C([0, T]; B)$ and

$$\|y\| \leq \tilde{L} \|z\|. \tag{1.7'}$$

Using the relations (1.11) and (1.7′) we obtain

$$\|f(t, x+z) - f(t, x)\| = \|f(t, x + y(t) + \int_0^t \Gamma(s, x)\, y(s)\, ds) - f(t, x) -$$

$$- \Gamma(t, x)\, y(t) + \Gamma(t, x)\, y(t)\| \leq K \|y\|_t + \|\Gamma(t, x)\, y(t)\| \leq$$

$$\leq (K + M)\, \|y\| \leq \tilde{L}(K + M)\, \|z\|.$$

Conversely, assume that there exist positive constants L such that

$$\|f(t, x+z) - f(t, x)\| \leq L \|z\|$$

for all $t \in [0, T]$ and every $x, z \in B$. If we put $\Gamma(s, x) = 0$ for all $(s, x) \in [0, T] \times B$ then the relation (1.2) holds and Theorem 1.1 is completely proved.

Remark 1.2. Let us return now to the initial value problem (1.1), where f is a continuous function. The problem (1.1) is equivalent to the following integral equation

$$x(t) = \eta + \int_0^t f(s, x(s))\, ds, \quad 0 \leq t \leq T. \tag{1.1'}$$

If we assume that $f(t, x)$ is Lipschitzian in x, uniformly with respect to $t \in [0, T]$, say

$$\|f(t, x) - f(t, y)\| \leq L \|x - y\|, \quad 0 \leq t \leq T, \ x, y \in B, \tag{1.12}$$

then the integral equation (1.1′) has a unique solution $x \in C([0, T]; B)$. Furthermore, this solution is the uniform limit of the successive approximation sequence

$$(1.13)\qquad \begin{cases} x_0(t)=\eta,\ 0\leq t\leq T \\ x_n(t)=\eta+\int_0^t f(s,\ x_{n-1}(s))\ ds,\ 0\leq t\leq T;\quad n=1,\ 2,\ \ldots \end{cases}$$

On the other hand, Theorem 1.1 means that this condition is equivalent with the existence of integral contractors $\{I+\int_0^t \Gamma\}$ (i.e. the inequality (1.2) is satisfied). Let us consider the following sequences (see also [3]);

$$(1.14)\qquad \begin{cases} x_0(t)=\eta,\ 0\leq t\leq T \\ x_{n+1}(t)=x_n(t)-y_n(t)-\int_0^t \Gamma(s,\ x_n(s))\ y_n(s)\ ds,\ 0\leq t\leq T;\ n=0,\ 1,\ \ldots \\ y_n(t)=x_n(t)-\int_0^t f(s,\ x_n(s))\ ds-\eta,\quad 0\leq t\leq T;\quad n=0,\ 1,\ \ldots \end{cases}$$

It may be proved that the unique solution of the initial value problem (1.1) is the uniform limit of the sequence $\{x_n\}$ from (1.14), while the error sequence $\{y_n\}$ is uniformly convergent to zero (see [3]).

Noticing that for $\Gamma=0$ the sequences (1.14) reduce to the sequence (1.13), we also remark that if $f(t,\cdot)$ is Lipschitzian, uniformly with respect to $t\in[0,\ T]$, then for every bounded continuous mapping $\Gamma:[0,\ T]\times B\to L(B,\ B)$, $\{I+\int_0^t \Gamma\}$ is a bounded integral contractor.

The following question arises: is the method (1.14) better than the method (1.13)? This will be true whenever there are integral contractors such that $K<L$ (recall that K and L are the constants which appear in (1.2) and (1.12), respectively) (see, e.g. [3]). We do not think that this happens (see also the following remark).

Remark 1.3. Here, for the sake of simplicity, we restrict ourselves to the case $B=R$ (the real line). Our opinion is that the natural definition of integral contractor is as follows: We will say that $\{I+\int_0^t \Gamma\}$ is an integral contractor for the function $f:[0,\ T]\times R\to R$ if there exist a bounded continuous function $\Gamma:\{(t,\ s);\ 0\leq s\leq t\leq T\}\times R\to R$ and a positive constant K such that:

$$(1.15)\qquad \left|\int_0^t [f(s,\ x(s)+y(s)+\int_0^s \Gamma(s,\ \tau,\ x(\tau))\ y(\tau)\ d\tau)-f(s,\ x(s))-\Gamma(t,\ s,\ x(s))\ y(s)]\ ds\right|\leq K\int_0^t |y(s)|\ ds,\quad 0\leq t\leq T.$$

for all $x,\ y\in C([0,\ T])$.

Once again, the integral equation

$$(1.3')\qquad y(t)+\int_0^t \Gamma(t,\ s,\ x(s))\ y(s)\ ds=z(t),\ 0\leq t\leq T$$

has a unique solution $y\in C([0,\ T])$ for every fixed $x,\ z\in C([0,\ T])$. Furthermore, in this case we have

$$(1.7'')\qquad |y(t)|\leq|z(t)|+M\exp(MT)\cdot\int_0^t |z(s)|\ ds,\ 0\leq t\leq T.$$

Indeed, from the equation (1.3') we obtain $|y(t)-z(t)|\leq M\int_0^t ||y(s)-z(s)|\ ds+M\int_0^t |z(s)|\ ds$ and, using Gronwall's lemma, we have $|y(t)-z(t)|\leq M\exp(MT)\cdot\int_0^t |z(s)|\ ds,\ 0\leq t\leq T$ and hence the inequality (1.7'').

From these estimates it may be obtained that the inequality (1.15) is equivalent with the following „integral Lipschitz condition":

$$\left|\int_0^t [f(s,\ x(s)+z(s))-f(s,\ x(s))]\ ds\right|\leq L\int_0^t |z(s)|\ ds,\quad 0\leq t\leq T$$

for all $x,\ z\in C([0,\ T])$, which is equivalent with Lipschitz condition (1.12), too.

For a fixed $z\in C([0,\ T])$ let us consider the continuous function

$$(1.16)\qquad J(z)(t)=z(t)-\int_0^t f(s,\ z(s))\ ds-\eta,\quad 0\leq t\leq T$$

and the linear operator $A(z):C([0,\ T])\to C([0,\ T])$ given by

$$(1.17)\qquad A(z)(y)(t)=y(t)+\int_0^t \Gamma(t,\ s,\ z(s))\ y(s)\ ds,\quad 0\leq t\leq T.$$

Assuming that $f(t,\cdot)$ is continuously differentiable with $\partial f/\partial x$ uniformly bounded and uniformly Hölder continuous in x, then the nonlinear operator $J:C([0,\ T])\to C([0,\ T]$ is Fréchet differentiable and

$$(1.18)\qquad (J'(z)\ y)(t)=y(t)-\int_0^t \frac{\partial f}{\partial x}(s,\ z(s))\ y(s)\ ds,\quad 0\leq t\leq T;\quad y\in C([0,\ T]).$$

It is easy to obtain also that

$$(1.19)\qquad (J'(z))^{-1}(y)(t)=y(t)+\int_0^t \frac{\partial f}{\partial x}(s,\ z(s))\exp\left(\int^t \frac{\partial f}{\partial x}(\tau,\ z(\tau))\ d\tau\right)\cdot y(s)\ ds$$

Now, the sequences (1.14) are equivalent with the sequence

$$(1.20)\qquad x_0(t)=\eta,\ x_{n+1}(t)=x_n(t)-A(x_n)(J(x_n))(t),\ 0\leq t\leq T;\quad y=0,\ 1,\ \ldots$$

Let $x\in C([0,\ T])$ be the unique solution of the initial value problem (1.1). Since $J(x)=0$, after some calculation, we obtain:

$$(1.21)\qquad ||x_{n+1}-x||\leq\left[||A(x_n)\ J'(x)-I||+\frac{||\omega(x,\ x_n-x)||}{||x_n-x||}\right]||x_n-x||,$$

where $\lim\limits_{y\to 0} \omega\|(x,\ y)\|/\|y\|=0$.

If we may choose an integral contractor such that $\|A(x_n)\,J'(x)-I\|\to 0$ as $n\to\infty$ then the approximations (1.14) (equivalent with (1.20)) will be better than the approximations (1.13). From (1.17) and (1.19) it seems that the best choice of the function Γ is as follows:

$$\Gamma(t,\ s,\ z(s))=\frac{\partial f}{\partial x}(s,z(s))\exp\left(\int_s^t \frac{\partial f}{\partial x}(\tau,\ z(\tau))\,d\tau\right) \tag{1.22}$$

In such a case, the relations (1.20) become

$$\begin{cases} x_0(t)=\eta, & 0\le t\le T \\ x_{n+1}(t)=x_n(t)-(J'(x_n))^{-1}(J(x_n))(t), & 0\le t\le T;\quad n=0,\ 1,\ \ldots \end{cases} \tag{1.20'}$$

which is the Newton-Kantorovich method for the integral equation (1.1)

1.2. **Functional-Differential Equations.** Let us consider the following initial value problem

$$\begin{cases} dx/dt = f(t,\ x_t), & 0\le t\le T, \\ x_0=\varphi \end{cases} \tag{1.23}$$

where $f:[0,\ T]\times C([-h,\ 0];\ R^n)\to R^n$ is a continuous function and $\varphi\in C([-h,\ 0];\ R^n)$ is a given element; $h>0$ is the retardation. As usual, by $x_t\in C([-h,\ 0];\ R^n)$ we have denoted the function defined by

$$x_t(s)=x(t+s),\quad s\in[-h,\ 0];\quad 0\le t\le T. \tag{1.24}$$

The initial value problem (1.23) is equivalent with the following equation

$$\begin{cases} x(t)=\varphi(0)+\int_0^t f(s,\ x_s)\,ds, & 0\le t\le T, \\ x_0=\varphi. \end{cases} \tag{1.25}$$

A natural extension of the notion of integral contractors given in [3] is as follows (see also [4]):

Definition 1.3. *Suppose $\Gamma:[-h,\ T]\times R^n\to L(R^n,\ R^n)$ is a bounded measurable matrix-valued function and there exist positive constants K such that for any $x,\ y\in C([-h,\ T];\ R^n)$ one has*

$$\begin{aligned} &\left|f\Big(t,\ x_t(\cdot)+y_t(\cdot)+\int_0^{t\cdot}\Gamma(s,\ x(s))\,y(s)\,ds\Big)-f(t,\ x_t)-\Gamma(t,\ x(t))\,y(t)\right|\le \\ &\le K\sup\{|y(s)|:\ -h\le s\le t\};\quad 0\le t\le T. \end{aligned} \tag{1.26}$$

where

$$\Gamma(s,\ x(s))=0\ \text{ for every }\ -h\le s\le 0. \tag{1.27}$$

Then we say that the function $f(t,\ \Psi)$ has a bounded integral contractor $\{I+\int_0^t\Gamma\}$.

Remark 1.4. Every bounded integral contractor is *regular* (i.e. the below equation (1.28) has a solution). Indeed, we have to prove that for every

$x,\ z\in C([-h,\ T];\ R^n)$ there exists a (unique) function $y\in C([-h,\ T];\ R^n)$ such that

$$y_t(\cdot)+\int_0^{t\cdot}\Gamma(s,\ x(s))\,y(s)\,ds=z_t(\cdot),\quad 0\le t\le T. \tag{1.28}$$

But, the equation (1.28) is equivalent to the equation

$$\begin{cases} y(t)+\int_0^t\Gamma(s,\ x(s))\,y(s)\,ds=z(t), & 0\le t\le T, \\ y(s)=z(s), & -h\le s\le 0 \end{cases} \tag{1.29}$$

and, using Lemma 1.1. we may obtain the stated result. Furthermore, from (1.7) we also deduce that there exist positive numbers $\tilde{L}$ such that

$$\sup\{|y(t)|:\ -h\le t\le T\}\le \tilde{L}\sup\{|z(t)|;\ -h\le t\le T\}. \tag{1.30}$$

Theorem 1.2. *The continuous function $f(t,\ \Psi)$ has a bounded integral contractor if and only if $f(t,\ .)$ is Lipschitzian, uniformly with respect to $t\in[0,\ T]$.*

Proof. Let $\{I+\int_0^t\Gamma\}$ be a bounded integral contractor; let us fix $t\in[0,\ T]$. For every $\varphi,\ \Psi\in C([-h,\ 0];\ R^n)$ let us consider the functions $x_\varphi,\ z_\Psi\in C([-h,\ T];\ R^n)$ defined by

$$x_\varphi(s)=\begin{cases} \varphi(-h), & s\le t-h \\ \varphi(s-t), & t-h\le s\le t, \\ \varphi(0), & s\ge t. \end{cases}$$

and

$$z_\Psi(s)=\begin{cases} \Psi(-h), & s\le t-h, \\ \Psi(s-t), & t-h\le s\le t, \\ \Psi(0), & s\ge t. \end{cases}$$

From Remark 1.4 we can obtain that the integral equation

$$\begin{cases} y(s)+\int_0^s\Gamma(\tau,\ x(\tau))\,y(\tau)\,d\tau=z(s), & 0\le s\le T, \\ y(\tau)=z(\tau), & -h\le\tau\le 0, \end{cases}$$

has a unique solution $y\in C([-h,\ T];\ R^n)$ which satisfies the condition (1.30). Particularly, from (1.28) for $s=t$, we obtain

$$y_t+\int_0^{t\cdot}\Gamma(\tau,\ x(\tau))\,y(\tau)\,d\tau=\Psi$$

such that the relation (1.26) becomes:

$$|f(t,\ \varphi+\Psi)-f(t,\ \varphi)-\Gamma(t,\ x(t))\,y(t)|\le K\sup\{|y(s)|;\ -h\le s\le t\}.$$

From this and (1.30) we easily obtain that $f(t,\ .)$ is Lipschitzian, uniformly with respect to $t\in[0,\ T]$.

Since the converse part of the theorem is obvious (we may put $\Gamma=0$), Theorem 1.2 is completely proved.

§2. **Stochastic Differential Equations with Integral Contractors. 2.1. Preliminaries.** To begin with, we shall recall some classical aspects of stochastic analysis. In the sequel we shall consider a fixed complete stochastic base $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in R})$, which means that $(\Omega, \mathcal{F}, P)$ is a complete probability space, $(\mathcal{F}_t)_{t\geq 0}$ is an increasing family of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_0$ contains every P-negligible subset of Ω, and $\mathcal{F}_s=\mathcal{F}_0$ for all $s\leq 0$.

Let $(H, |\cdot|)$ and $(H_0, |\cdot|_0)$ be two real separable Hilbert space with the inner products $(.,.)$ and $(.,.)_0$, respectively. We denote by $\mathcal{B}$ and $\mathcal{B}_0$ the σ-algebras of Borel sets of H and H_0, respectively.

Let $T>0$ and $\tau\geq 0$ be given. We denote by $L^2_{ad}(\Omega; C([-\tau, T]; H))$ the space of the stochastic processes $f\in L^2(\Omega; C([-\tau, T]; H)) \stackrel{def}{=} L^2(\Omega, \mathcal{F}, P; C([-\tau, T]; H))$ with the following property: there exists a representative $\bar{f}$ of the equivalence class f such that for all $t\in[-\tau, T]$ the function $\omega\to\bar{f}(\omega, t):\Omega\to H$ is $(\mathcal{F}_t, \mathcal{B})$-measurable (Borel measurable = weakly measurable = strongly measurable since H is separable). We can easily show that every representative of f has the same property as the process $\bar{f}$, and $L^2_{ad}(\Omega; C([-\tau, T]; H))$ is a closed linear subspace of $L^2(\Omega; C([-\tau, T]; H))$. Hence $L^2_{ad}(\Omega; C([-\tau, T]; H))$ is a Banach space with the usual norm $\|f\|=\|f\|_T$ where

$$\|f\|_t^2=E(\sup\{|f(s)|^2;\ -\tau\leq s\leq t\}). \tag{2.1}$$

Here, E is the expectation operator.

For a positive λ, we define

$$|||f|||=\sup\{\|f\|_t \cdot \exp(-\lambda(t+\tau));\ -\tau\leq t\leq T\}. \tag{2.2}$$

Since

$$\|f\|\cdot\exp(-\lambda(T+\tau))\leq|||f|||\leq\|f\|, \tag{2.3}$$

the relation (2.2) defines an equivalent norm in the space $L^2_{ad}(\Omega; C([-\tau, T]; H))$.

We shall denote by $L^2_{ad}(\Omega\times[0, T]; H)$ the space of stochastic processes $f\in L^2(\Omega\times[0, T], \mathcal{F}\times\mathcal{L}, P\times dt; H)=L^2(\Omega\times[0, T]; H)$ such that there is a function $\bar{f}$ belonging to the class f with the following property: $\omega\to\bar{f}(\omega, t):\Omega\to H$ is $(\mathcal{F}_t, \mathcal{B})$-measurable for almost all $t\in[0, T]$. (Throughout the paper $\mathcal{L}$ denotes the σ-algebra of Lebesgue measurable subsets of $[0, T]$ and with dt we denote the Lebesgue measure on $[0, T]$.) Also we note that every representative of f has the same property as $\bar{f}$ and $L^2_{ad}(\Omega\times[0, T]; H)$ is a closed linear subspace of the Hilbert space $L^2(\Omega\times[0, T]; H)$.

If $W(t)$, $t\geq 0$ is a H_0-valued Wiener process with respect to the stochastic base $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$, it is known that $W\in L^2_{ad}(\Omega; C([0, T]; H_0))$ and $W(\omega, t)=\sum_{i=1}^{\infty}\sqrt{\alpha_i}b_i(\omega, t)e_i$ where $\alpha_i\geq 0$, $\sum_{i=1}^{\infty}\alpha_i<+\infty$, $\{e_i\}$ is an orthonormal

basis in H_0 and b_i are independent real-valued standard Wiener processes with respect to the stochastic base $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$. We denote by $Q: H_0\to H_0$ the covariance operator of W, i.e. $Qh=\sum_{i=1}^{\infty}\alpha_i(h, e_i)e_i$ for all $h\in H_0$.

Finally, we shall denote by $\Lambda(W; H_0, H)$ the space of all stochastic processes $f:\Omega\times[0, T]\to L(H_0, H)$ which verify:

$$(2.4)\quad \begin{cases} \text{(i) for every } h\in H_0 \text{ the function } (\omega, t)\to f(\omega, t)h:\Omega\times[0, T]\to H \\ \quad\text{is } (\mathcal{F}\times\mathcal{L}, \mathcal{B})\text{-measurable (i.e. } f:\Omega\times[0, T]\to L(H_0, H) \text{ is pointwise measurable);} \\ \text{(ii) for any } h\in H_0 \text{ the function } \omega\to f(\omega, t)h:\Omega\to H \text{ is } (\mathcal{F}_t, \mathcal{B})- \\ \quad\text{measurable for a.a. } t\in[0, T]; \\ \text{(iii) } \|f\|_\Lambda^2 \stackrel{def}{=} E\int_0^T |f(t)|_Q^2\, dt<+\infty \text{ where } |f|_Q^2 \stackrel{def}{=} \operatorname{tr} fQf^* = \\ \quad=\operatorname{tr} f^*fQ=\sum_{i=1}^{\infty}\alpha_i|fe_i|^2. \end{cases}$$

Here, as usual, by $L(H_0, H)$ we have denoted the Banach space of linear continuous operators from H_0 into H.

If one identifies the elements $f, g\in\Lambda(W; H_0, H)$ which verify $\|f-g\|_\Lambda=0$ then $\Lambda(W; H_0, H)$ is a pre-Hilbert space and the stochastic integral $I(f)(\omega, t)=\int_0^t f(\omega, s)\,dW(\omega, s)$ is a linear continuous operator from $\Lambda(W; H_0, H)$ into $L^2_{ad}(\Omega; C([0, T]; H))$. Here we only mention that:

$$(2.5)\quad \begin{aligned} &\text{a) } E\, I(f)(t)=0; \\ &\text{b) } E|I(f)(t)|^2=E\int_0^t |f(s)|_Q^2\, ds; \\ &\text{c) } E\sup\{|I(f)(s)|^2;\ 0\leq 0\leq t\}\leq 4E\int_0^t |f(s)|_Q^2\, ds \end{aligned}$$

(see [4]).

Now we give an existence and uniqueness result for the following stochastic equation:

$$(2.6)\quad \begin{cases} x(\omega, s)=z(\omega, s), & -\tau\leq s\leq 0 \\ x(\omega, t)=z(\omega, t)+\int_0^t F(x)(\omega, s)\,ds+\int_0^t G(x)(\omega, s)\,dW(\omega, s), & 0\leq t\leq T, \end{cases}$$

where

$$(2.7)\quad \begin{cases} \text{(i) } F: L^2_{ad}(\Omega; C([-\tau, T]; H))\to L^2_{ad}(\Omega\times[0, T); H), \\ \quad G: L^2_{ad}(\Omega; C([-\tau, T]; H))\to\Lambda(W; H_0, H); \\ \text{(ii) there exists an } L>0 \text{ such that for all } t\in[0, T] \text{ and } x, y\in L^2_{ad}(\Omega; \\ \quad C([-\tau, T]; H)) \text{ one has } E\int_0^t |F(x)(s)-F(y)(s)|^2\, ds\leq \\ \quad L^2\int_0^t \|x-y\|_s^2\, ds \text{ and } E\int_0^t |G(x)-G(y)|_Q^2\, ds\leq L^2\int_0^t \|x-y\|_s^2\, ds; \\ \text{(iii) } z\in L^2_{ad}(\Omega; C([-\tau, T]; H)). \end{cases}$$

Theorem 2.1. *Under the hypotheses (2.7) the equation (2.6) has a unique solution* $x \in L^2_{ad}(\Omega; C([-\tau, T]; H))$. *Furthermore, there exists* $C_1 = C_1(L, T, \tau) > 0$ *such that for all* $x_0 \in L^2_{ad}(\Omega; C([-\tau, T]; H))$ *we have*

$$\|x\|^2 \leq C_1 (\|z\|^2 + \|x_0\|^2 + \|F(x_0)\|^2_{L^2(\Omega\times(0,T);H)} + \|G(x_0)\|^2_{\lambda}). \tag{2.8}$$

Proof. We note that the equation (2.6) may be written in the form $Sx = x$ where $S : L^2_{ad}(\Omega; C([-\tau, T]; H)) \to L^2_{ad}(\Omega; C([-\tau, T]; H))$ is defined by: $S(x)(\omega, t) = z(\omega, t)$ for $-\tau \leq t \leq 0$ and $S(x)(\omega, t) = z(\omega, t) + \int_0^t F(x)(\omega, s)\, ds + \int_0^t G(x)(\omega, s) dW(\omega, s)$ for $0 \leq t \leq T$. We show that there exists a $\lambda > 0$ such that the operator S is a contraction in $(L^2_{ad}(\Omega; C([-\tau, T]; H)), |||\cdot|||)$. Let $y_1, y_2 \in L^2_{ad}(\Omega; C([-\tau, T]; H))$ be given; for $-\tau \leq t \leq 0$ it holds $S(y_1)(t) - S(y_2)(t) = 0$ and, for $0 \leq t \leq T$, using the relations (2.5 – c) and (2.7 – ii) we obtain:

$$E \sup \{ | S(y_1)(s) - S(y_2)(s) |^2 ; 0 \leq s \leq t \} \leq 2E \sup_{s\in[0,t]} \left| \int_0^s (F(y_1) - F(y_2)) d\sigma \right|^2 +$$

$$+ 2E \sup_{s \in [0,t]} \left| \int_0^s (G(y_1) - G(y_2))\, dW(\sigma) \right|^2 \leq 2tE \int_0^t | F(y_1) - F(y_2) |^2 ds +$$

$$+ 8E \int_0^t | G(y_1) - G(y_2) |^2_Q \, ds \leq (2tL^2 + 8L^2) \int_0^t \| y_1 - y_2 \|^2_s \, ds \leq L^2(T+4)\,\cdot$$

$$\cdot\, |||y_1 - y_2|||^2 (e^{2\lambda(t+\tau)} - e^{2\lambda\tau})/\lambda \leq |||y_1 - y_2|||^2 \exp (2\lambda(t+\tau))/4.$$

whenever $\lambda > 4L^2(T+4)$. Hence, for every $t \in [-\tau, T]$ we have

$$E \sup_{s\in[-\tau,t]} | S(y_1)(s) - S(y_2)(s) |^2 \exp (-2\lambda(t+\tau)) \leq |||y_1 - y_2|||^2/4$$

and therefore

$$|||S(y_1) - S(y_2)||| \leq |||y_1 - y_2|||/2. \tag{2.9}$$

for all $y_1, y_2 \in L^2_{ad}(\Omega; C([-\tau, T]; H))$.

From (2.9), by the Banach fixed point theorem, it results that the equation (2.6) has a unique solution $x \in L^2_{ad}(\Omega; C([-\tau, T]; H))$ which may be obtained as the $L^2_{ad}(\Omega; C([-\tau, T]; H))$-limit of the following sequence:

$$x_0 \in L^2_{ad}(\Omega; C([-\tau, T]; H)); \; x_{n+1} = S(x_n), \; n = 0, 1, 2, \ldots \tag{2.10}$$

From the relations (2.9) and (2.10) we easily obtain

$$|||x_n - x_0||| \leq (1 + 2^{-1} + 2^{-2} + \ldots + 2^{-n+1})\, |||x_1 - x_0|||; \; n = 1, 2, \ldots$$

which, for $n \to \infty$, gives us

$$|||x||| \leq |||x_0||| + 2\, |||x_1 - x_0||| \leq \|x_0\| + 2\, \|x_1 - x_0\| \leq 3(\|x_0\| + \|x_1\|).$$

From this and the relation (2.3) we obtain

$$\|x\|^2 \leq 18 (\|x_0\|^2 + \|x_1\|^2) \exp (+2\lambda(T+\tau)). \tag{2.11}$$

Since $x_1 = S(x_0)$, using the definition of the operator S and the condition (2.5 – c), we have:

$$\|x_1\|^2 = E \sup_{t\in[-\tau,T]} | x_1(t) |^2 \leq 3\, \|z\|^2 + 3TE \int_0^T \| F(x_0)(s) \|^2 ds + 12\, E \int_0^T | G(x_0) |^2_Q \, ds. \tag{2.12}$$

From the relations (2.11) and (2.12) it may be easily obtained the relation (2.8) and Theorem 2.1 is completely proved.

2.2. Stochastic Differential Equations with Integral Contractors. Consider the following stochastic system:

$$\begin{cases} x(\omega, t) = z(\omega, t), & -\tau \leq t \leq 0, \\ x(\omega, t) = z(\omega, t) + \int_0^t f(s, x_s(\omega))\, ds + \int_0^t g(s, x_s(\omega))\, dW(\omega, s), & 0 \leq t \leq T. \end{cases} \tag{2.13}$$

Here, for $x \in L^2_{ad}(\Omega; C([-\tau, T]; H))$, $t \in [0, T]$ and $\omega \in \Omega$ the function $x_t(\omega) \in C([-\tau, 0]; H)$ is defined by:

$$x_t(\omega)(s) = x(\omega, t+s), \quad -\tau \leq s \leq 0. \tag{2.14}$$

The equation (2.13) will be studied under the following assumptions:

(2.15)

(i) $f : [0, T] \times C([-\tau, 0]; H) \to H$ *is measurable,*
$g : [0, T] \times C([-\tau, 0]; H) \to L(H_0, H)$ *is pointwise measurable;*

(ii) *there are positive constants* L *such that for all* $(t, \varphi), (t, \Psi) \in [0, T] \times C([-\tau, 0]; H)$ *one has:*
$| f(t, \varphi) - f(t, \Psi) | \leq L \sup \{ | \varphi(s) - \Psi(s) | ; -\tau \leq s \leq 0 \}$,
$| g(t, \varphi) - g(t, \Psi) |_Q \leq L \sup \{ | \varphi(s) - \Psi(s) | ; -\tau \leq s \leq 0 \}$;

(iii) $z \in L^2_{ad}(\Omega; C([-\tau, T]; H))$ *and* $C(z) = E \int_0^T | f(t, z_t) |^2 dt + E \int_0^T | g(t, z_t) |^2_Q \, dt < +\infty$.

Noticing that under the assumptions (2.15) the functions $F(x)(\omega, t) = f(t, x_t(\omega))$ and $G(x)(\omega, t) = g(t, x_t(\omega))$ verify all conditions of Theorem 2.1, we may state:

Proposition 2.1. *Under the assumptions (2.15) the equation (2.13) has a unique solution* $x \in L^2_{ad}(\Omega; C([-\tau, T]; H))$ *which satisfies the estimation (2.8).*

A natural extension of the notion of bounded stochastic integral contractor (given in [3] and [5]) for some functions $f : [0, T] \times C([-\tau, 0]; H) \to H$ and $g : [0, T] \times C([-\tau, 0]; H) \to L(H_0, H)$ is as follows:

Definition 2.1. *We say that the pair* (f, g) *has a bounded stochastic integral contractor* $\{I + \int_0^t \Gamma ds + \int_0^t \Phi dW(s)\}$ *whenever:*

(i) *$\Gamma : [-\tau, T] \times H \times H \to H$ is measurable.*
$\Phi : [-\tau, T] \times H \times H \to L(H_0, H)$ is pointwise measurable;

(ii) *there are positive constants M such that for all $(t, h_1) \in [-\tau, T] \times H$ and $h \in H$ we have: $|\Gamma(t, h_1, h)| \leq M |h|$, $|\Phi(t, h_1, h)|_Q \leq M |h|$ and, moreover, the mappings $\Gamma(t, h_1, .)$ and $\Phi(t, h_1, .)$ are linear;*

(iii) *there are positive constants K such that for all $t \in [0, T]$ and $x, y \in L^2_{ad}(\Omega; C([-\tau, T]; H))$ one has:*

$$(2.16)\quad \begin{cases} \text{a)} \ |f(t, x_t(\omega)+(A(x)\, y)_t(\omega))-f(t, x_t(\omega))-\Gamma(t, x(\omega, t), y(\omega, t))| \leq \\ \qquad \leq K \sup \{|y(\omega, s)|;\ -\tau \leq s \leq t\},\ \omega-\text{a.s.} \\ \text{b)} \ |g(t, x_t(\omega)+(A(x)\, y)_t(\omega))-g(t, x_t(\omega))-\Phi(t, x(\omega, t), y(\omega, t))|_Q \leq \\ \qquad \leq K \sup \{|y(\omega, s)|;\ -\tau \leq s \leq t\},\ \omega-\text{a.s.} \end{cases}$$

Here, the linear continuous operator $y \to A(x)\, y : L^2_{ad}(\Omega; C([-\tau, T]; H)) \to L^2_{ad}(\Omega; C([-\tau, T]; H))$ is given by:

$$(2.17)\quad (A(x)\, y)(\omega, t) = \begin{cases} y(\omega, t), \quad -\tau \leq t \leq 0 \\ y(\omega, t) + \int_0^t \Gamma(s, x(\omega, s), y(\omega, s))\, ds + \\ + \int_0^t \Phi(s, x(\omega, s), y(\omega, s))\, dW(\omega, s), \quad 0 \leq t \leq T \end{cases}$$

(see the definition (2.14) for the meaning of the function $(A(x)\, y)_t$).

Definition 2.2. *A bounded stochastic integral contractor $\{I + \int_0^t \Gamma ds + \int_0^t \Phi dW(s)\}$ is said to be regular if for any $x, z \in L^2_{ad}(\Omega; C([-\tau, T]; H))$ the equation*

$$(2.18)\qquad A(x)\, y = z$$

has at least one solution $y \in L^2_{ad}(\Omega; C([-\tau, T]; H))$.

Remark 2.1. We note that for $H_0 = H = R$, W a real standard Wiener process and $Q = I$ we have $\Lambda(W; R, R) = L^2_{ad}(\Omega \times [0, T]; R)$ and $|\cdot|_Q = |\cdot|$; in this particular case we easily may obtain the problems considered in the papers [3] and [5], namely: for the problem considered in [3] we only have to define $f(t, \varphi) = f_1(t, \varphi(0))$, $g(t, \varphi) = g_1(t, \varphi(0))$, where $f_1, g_1 : [0, T] \times \mathbf{R} \to \mathbf{R}$, and for the problem considered in [5] we may take $f(t, \varphi) = f_2(t, \varphi(0), \varphi(-\tau_1))$ and $g(t, \varphi) = g_2(t, \varphi(0), \varphi(-\tau_2))$, $0 \leq \tau_1 \leq \tau$, $0 \leq \tau_2 \leq \tau$, $f_2, g_2 : [0, T] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$. If we define $\Gamma(t, x, y) = \Gamma_1(t, x)\, y$ and $\Phi(t, x, y) = \Phi_1(t, x)\, y$ where $\Gamma_1, \Phi_1 : [-\tau, T] \times \mathbf{R} \to \mathbf{R}$ are bounded Borel measurable functions then from Definitions 2.1 and 2.2 we may obtain the definition of bounded stochastic integral contractors considered in [3] and [5].

Lemma 2.1. *Every bounded stochastic integral contractor is regular. Furthermore the equation (2.18) has a unique solution and there are positive constants $\tilde{L}$ such that:*

$$(2.19)\qquad \|y\| \leq \tilde{L}\, \|z\|.$$

Proof. Theorem 2.1 with the functions $F(y)(\omega, t) = -\Gamma(t, x(\omega, t), y(\omega, t))$, $G(y)(\omega, t) = -\Phi(t, x(\omega, t), y(\omega, t))$ and $x_0 = 0$ easily gives us the assertion of the lemma (recall that $x \in L^2_{ad}(\Omega; C([-\tau, T]; H))$ is a fixed stochastic process).

Theorem 2.2. *The pair (f, g) has a bounded stochastic integral contractor if and only if the Lipschitz condition (2.15 – ii) is verified.*

Proof. If the functions f and g verify the conditions (2.15 – ii) then $\{I + \int_0^t 0\, ds + \int_0^t 0\, dW(s)\}$ ($\Gamma = 0$ and $\Phi = 0$) is a bounded stochastic integral contractor.

Assume now that $\{I + \int_0^t \Gamma ds + \int_0^t \Phi\, dW(s)\}$ is a bounded stochastic integral contractor for the pair (f, g). Let us fix $x, z \in L^2_{ad}(\Omega; C([-\tau, T]; H))$ and let $y \in L^2_{ad}(\Omega; C([-\tau, T]; H))$ be the corresponding solution of the equation (2.18); the relations (2.16) become:

$$(2.20)\quad \begin{cases} \text{a)} \ |f(t, x_t(\omega)+z_t(\omega))-f(t, x_t(\omega))-\Gamma(t, x(\omega, t), y(\omega, t))| \leq \\ \qquad \leq K \sup \{|y(\omega, s)|;\ -\tau \leq s \leq t\}, \\ \text{b)} \ |g(t, x_t(\omega)+z_t(\omega))-g(t, x_t(\omega))-\Phi(t, x(\omega, t), y(\omega, t))|_Q \leq \\ \qquad \leq K \sup \{|y(\omega, s)|;\ -\tau \leq s \leq t\} \end{cases}$$

for all $t \in [0, T]$; ω – a.s.

Using the inequality (2.19), the relations (2.20) become:

$$(2.21)\quad \begin{aligned} &\text{a)}\ E\,|f(t, x_t+z_t)-f(t, x_t)|^2 \leq (K+M)^2 \tilde{L}^2\, E \sup \{|z(s)|^2;\ -\tau \leq s \leq T\}, \\ &\text{b)}\ E\,|g(t, x_t+z_t)-g(t, x_t)|_Q^2 \leq (K+M)^2 \tilde{L}^2\, E \sup \{|z(s)|^2;\ -\tau \leq s \leq T\}, \end{aligned}$$

for all $t \in [0, T]$; ω – a.s.

Now, let $t \in [0, T]$ and $\varphi, \Psi \in C([-\tau, 0]; H)$ be arbitrarily fixed. Let us define $x(\cdot\,; t, \Psi)$, $z(\cdot\,; t, \varphi, \Psi) \in L^2_{ad}(\Omega; C([-\tau, T]; H))$ as follows:

$$(2.22)\qquad x(s; t, \Psi) = \begin{cases} \Psi(-\tau), & s \leq t-\tau, \\ \Psi(s-t), & t-\tau \leq s \leq t, \\ \Psi(0), & s \geq t, \end{cases}$$

and

$$(2.23)\qquad z(s; t, \varphi, \Psi) = \begin{cases} \varphi(-\tau)-\Psi(-\tau), & s \leq t-\tau, \\ \varphi(s-t)-\Psi(s-t), & t-\tau \leq s \leq t, \\ \varphi(0)-\Psi(0), & s \geq t. \end{cases}$$

We notice that x and z are deterministic processes and

$$x_t(\cdot\,;\,t,\,\psi)=\psi,\quad z_t(\cdot\,;\,t,\,\varphi,\,\psi)=\varphi-\psi. \tag{2.24}$$

Let us consider the relations (2.21) where $x(\cdot\,;\,t,\,\psi)$ and $z(\cdot\,;\,t,\,\varphi,\,\psi)$ stand for x and z respectively. From (2.24) we obtain :

$$\text{(2.25)}\quad \begin{aligned} &\text{a)}\ |f(t,\,\varphi)-f(t,\,\psi)|^2\leqslant(K+M)^2\tilde{L}^2\sup\,\{|\varphi(s)-\psi(s)|^2:\ -\tau\leqslant s\leqslant 0\}.\\ &\text{b)}\ |g(t,\,\varphi)-g(t,\,\psi)|_0^2\leqslant(K+M)^2\tilde{L}^2\sup\,\{|\varphi(s)-\psi(s)|^2:\ -\tau\leqslant s\leqslant 0\}.\end{aligned}$$

Therefore the conditions (2.15—ii) are satisfied, and Theorem 2.2 is completely proved.

Remark 2.2. We have proved that the existence of a bounded stochastic integral contractor is equivalent with the Lipschitz condition (2.15—ii). But, in the proof of Proposition 2.1 the solution of Equation (2.13) is obtained as the limit in $L^2_{ad}(\Omega\,;\,C([-\tau,\ T]\,;\,H))$ of the successive approximations sequence :

$$x^{(0)}=z,\quad x^{(n+1)}=x^{(n)}-J(x^{(n)}),\quad y=0,\ 1,\ 2,\ \ldots \tag{2.26}$$

while, in a proof which used the integral contractor (see also [3] and [5]) the solution is obtained as the limit in $L^2_{ad}(\Omega\,;\,C([-\tau,\ T]\,;\,H))$ of the corrected successive approximations sequence :

$$x^{(0)}=z,\quad y^{(n)}=J(x^{(n)}),\ x^{(n+1)}=x^{(n)}-\Lambda(x^{(n)})\,y^{(n)},\quad y=0,\ 1,\ 2,\ \ldots \tag{2.27}$$

where

$$J(x)\,(\omega,\ t)=\begin{cases} x(\omega,\ t)-z(\omega,\ t), & -\tau\leqslant t\leqslant 0,\\ x(\omega,\ t)-z(\omega,\ t)-\int_0^t f(s,\ x_s(\omega))\,ds-\int_0^t g(s,\ x_s(\omega))\,dW(s), & 0\leqslant t\leqslant T\end{cases}$$

and $\Lambda(x)$ is defined by (2.17).

To prove the convergence in $L^2_{ad}(\Omega\,;\,C([-\tau,\ T]\,;\,H))$ of the sequence $x^{(n)}$ given by (2.27) we first remark that $J(x)\in L^2_{ad}(\Omega\,;\,C([-\tau,\ T)\,;\,H))$ for every $x\in L^2_{ad}(\Omega\,;\,C([-\tau,\ T]\,;\,H))$. Then, the main steps of the proof will be as follows :

a) $y^{(n+1)}=J(x^{(n)}-\Lambda(x^{(n)})\,y^{(n)})-J(x^{(n)})+y^{(n)},\ n=0,\ 1,\ 2,\ \ldots$

b) $\|\,J(x+\Lambda(x)\,y)-J(x)-y\,\|_t^2\leqslant K_1\int_0^t\|\,y\,\|_s^2\,ds,\ 0\leqslant t\leqslant T.$

c) $|||\,y^{(n+1)}\,|||\leqslant|||\,y^{(n)}\,|||/2$ (for $\lambda>2K_1$), $n=0,\ 1,\ 2,\ \ldots$

d) $|||\,x^{(n+1)}-x^{(n)}\,|||\leqslant M_1\,|||\,y^{(0)}\,|||/2^n,\ n=0,\ 1,\ 2,\ \ldots$.

We also note that the relations (2.27) may be rewriten in the form :

$$x^{(0)}=z,\quad x^{(n+1)}=x^{(n)}-\Lambda(x^{(n)})\,J(x^{(n)}),\quad n=0,\ 1,\ 2,\ \ldots \tag{2.28}$$

which, at least for $\Lambda\cong(J')^{-1}$, is a Newton-Kantorovich type method (see also the Remarks 1.2 and 1.3)

It is interesting to study, for the stochastic case, the speed of convergence of the method (2.28) with respect to the method (2.26) (recall that the relations (2.26) may be obtained from (2.28) for $\Gamma=0$ and $\Phi=0$), because it seems there are possibilities (see Remark 1.3) to choose integral contractors with better convergence results : but this problem is beyond of the scope of this paper.

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