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SUMAR

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ON DECOMPOSITION OF GROUP-VALUED SUBMEASURES

BY

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In this paper we have established a decomposition theorem for exhaustive group-valued submeasures which generalizes the result of K h a n and R o w l a n d s in [2].

1. Notations and terminology. Throughout this paper we use the notations in [2]. Let G be a *commutative lattice group* (*l-group*), which is order complete (i.e., every bounded increasing net in G has a supremum). A set $U\subset G$ is said to be *solid* (see for ex. [3]) if $x\in U$ and $|y|\leqslant|x|$ implies $y\in U$. A solid $U\subset G$ is called a *l-band* if $A\subset U$ implies $\bigvee_{x\in A} x\in U$.

In what follows we assume that τ is a topology on G, in which *the lattice operations are continuous and τ possesses a base of l-band neighborhoods for the origin*, denoted by $\mathcal{B}(0)$. A G-valued function η defined on a ring $\mathcal{R}$ of subsets of a nonempty set X, is called a *submeasure* if

$$\eta(\Phi)=0,\ \eta(A\cup B)\leqslant\eta(A)+\eta(B),\ \textit{for all } A,\ B\in\mathcal{R},\ A\cap B=\Phi \textit{ and } \eta(A)\leqslant\eta(B),$$
$$\textit{if } A,\ B\in\mathcal{R},\ A\subseteq B.$$

A G-valued submeasure η is said to be *exhaustive* if, for any disjoint sequence $\{A_n\}$ in $\mathcal{R}$, $\lim\limits_{n\to\infty}\eta(A_n)=0$ in (G,τ).

Example. Let G be the additive group $\mathbf{R}\times\mathbf{R}$, equipped with the usual order (i.e. $(x, y)\leqslant(z, t)$ if $x\leqslant z$ and $y\leqslant t$). Then G is a complete order lattice.

Let $\mathcal{B}(0)$ be the collection of all subsets of the form $\{(x, y)\in\mathbf{R}\times\mathbf{R}/a\leqslant\leqslant x\leqslant b,\ c\leqslant y\leqslant d\}$, where $a<0<b$ and $c<0<d$. Then any element of $\mathcal{B}(0)$ is a l-band.

Define the function $\eta:\mathcal{R}\to\mathbf{R}\times\mathbf{R}$ by $\eta\ (A)=(\eta_1(A),\ \eta_2(A))$, for all $A\in\mathcal{R}$, where η_1, η_2 are real-valued exhaustive submeasures on $\mathcal{R}$. We deduce from the properties of η_1 and η_2, that η is an exhaustive submeasure.

For two G-valued submeasures λ and η on $\mathcal{R}$, we said that λ is *absolutely continuous* with respect to η (*η-continuous*) and write $\lambda\ll\eta$, if, for each $U\in\tau$, there exists $V\in\tau$ such that, $A\in\mathcal{R}$ and $\eta(A)\in V$, implies $\lambda(A)\in U$.

In the sequel we use the term of the additivity in the sense of D r e w n o w s k i ([1] or [2]). Let $\mathcal{D}$ denote a collection of pairwise disjoint sets in $\mathcal{R}$ and let Δ be the set of all such collections.

Given a set $\sigma\subseteq\mathcal{R}\times\Delta$ we write $\sigma(E)=\{\mathcal{D}\in\Delta/(E,\ \mathcal{D})\in\sigma\}$ for each $E\in\mathcal{R}$ and $\Delta_\sigma=\bigcup_{E\in\mathcal{R}}\sigma(E)$.

A set $\sigma \subseteq \mathcal{R} \times \Delta$ is said to be *an additivity* on $\mathcal{R}$ if it satisfies the conditions:

a) $\Delta f \subseteq \Delta_\sigma$, *where* Δf *consists of those collections* $\mathcal{D}$ *which have only finite number of elements*;

b) *if* $E \in \mathcal{R}$ *and* $\mathcal{D} \in \sigma(E)$, *then* $\cup \mathcal{D} = E$;

c) *if* $E \in \mathcal{R}$ *and* $\mathcal{D}_1, \mathcal{D}_2 \in \sigma(E)$, *then* $\mathcal{D}_1 \overset{0}{\cap} \mathcal{D}_2 = \{D_1 \cap D_2 / D_i \in \mathcal{D}_i,\ i=1, 2\} \in \sigma(E)$;

d) *if* $E_1, E_2 \in \mathcal{R}$, $E_1 \cap E_2 = \Phi$ *and* $\mathcal{D}_i \in \sigma(E_i)$ $(i=1, 2)$, *then* $\mathcal{D}_1 \overset{0}{\cup} \mathcal{D}_2 = \{D_1 \cup D_2 / D_i \in \mathcal{D}_i,\ i=1, 2\} \in \sigma(E_1 \cup E_2)$.

e) *if* $E, F \in \mathcal{R}$, $E \subseteq F$, *and* $\mathcal{D} \in \sigma(F)$, *then* $\mathcal{D} \overset{0}{\cap} E \in \sigma(E)$. Let σ be an additivity on $\mathcal{R}$. A G-valued submeasure η on $\mathcal{R}$ is said to be *σ-continuous* if, for each $E \in \mathcal{R}$, and $\mathcal{D} \in \sigma(E)$, $\lim\limits_{\mathcal{D}' \in f(\mathcal{D})} \eta(E - \cup \mathcal{D}') = 0$ in (G, τ), where $f(\mathcal{D})$ *denotes finite collections of members of* $\mathcal{D}$.

A G-valued submeasure η is said to be *σ-singular* if any σ-continuous G-valued submeasure λ on $\mathcal{R}$ such that $\lambda \ll \eta$ is identically zero.

2. The main results. Let μ be an exhaustive G-valued submeasure on $\mathcal{R}$ and suppose that σ is an additivity on $\mathcal{R}$. For each $E \in \mathcal{R}$, define

$$S_\mu(E) = \bigwedge_{\mathcal{D} \in \sigma(E)} \bigvee_{\mathcal{D}' \in f(\mathcal{D})} \mu(\cup \mathcal{D}')$$

$$S'_\mu(E) = \bigvee_{\mathcal{D} \in \sigma(E)} \bigwedge_{\mathcal{D}' \in f(\mathcal{D})} \mu(E - \cup D').$$

2.1. **Lemma.** *S_μ and S'_μ are G-valued exhaustive submeasures on $\mathcal{R}$.*

Proof. S_μ and S'_μ are G-valued submeasures on $\mathcal{R}$ as we can see from [2], Lemma 1. For exhaustivity let be any $E \in \mathcal{R}$. We have $S_\mu(E) \leqslant \mu(E)$ and $S'_\mu(E) \leqslant \mu(E)$ and so, if $U \in \mathcal{B}(0)$ such that $\mu(E) \in U$, implies $S_\mu(E) \in U$ and $S'_\mu(E) \in U$. Therefore both S_μ and S'_μ are exhaustive and μ-continuous.

2.2. **Lemma.** 1) *S_μ is σ-continuous*; 2) *S'_μ is σ-singular.*

Proof. 1) Suppose on the contrary, that there exist $U \in \mathcal{B}(0)$, $E \in \mathcal{R}$, and

(1) $\qquad \mathcal{D} \in \sigma(E)$ such that $S_\mu(E - \cup \mathcal{D}') \notin U$, for all $\mathcal{D}' \in f(\mathcal{D})$

Since S_μ is a submeasure and U is a solid we have $S_\mu(E) \notin U$ and $S_\mu(E) \leqslant \bigvee\limits_{\mathcal{D}' \in f(\mathcal{D})} \mu(\cup \mathcal{D}')$ implies $\bigvee\limits_{\mathcal{D}' \in f(\mathcal{D})} \mu(\cup \mathcal{D}') \notin U$. Since U is l-band there exists a $\mathcal{D}_1 \in f(\mathcal{D})$ such that $\mu(\cup \mathcal{D}_1) \notin U$. By property (5) of an additivity, $\mathcal{D} - \mathcal{D}_1 \in \sigma(E - \cup \mathcal{D}_1)$ and from (1), $S_\mu(E - \cup \mathcal{D}_1) \notin U$. Because $\bigvee\limits_{\mathcal{D}' \in f(\mathcal{D} - \mathcal{D}_1)} \mu(\cup \mathcal{D}') \geqslant S_\mu(E - \cup \mathcal{D}_1)$, it follows that $\bigvee\limits_{\mathcal{D}' \in f(\mathcal{D} - \mathcal{D}_1)} \mu(\cup \mathcal{D}') \notin U$ and so there exists a $\mathcal{D}_2 \in f(\mathcal{D} - \mathcal{D}_1)$ such that $\mu(\cup \mathcal{D}_2) \notin U$. Thus we construct by induction a disjoint sequence $\{\mathcal{D}_n;\ n=1, 2, \ldots\}$ such that $\mu(\cup \mathcal{D}_n) \notin U$. This contradicts the exhaustivity of μ and so S_μ is σ-continuous.

2) Suppose, on the contrary, that there exists a σ-continuous G-valued submeasure λ such that $\lambda \ll S'_\mu$ and $\lambda \neq 0$. Then there is a set $E \in \mathcal{R}$, and $U \in \mathcal{B}(0)$ such that $\lambda(E) \notin U + U + U + U$. Since $\lambda \ll S'_\mu$ there is $U' \in \mathcal{B}$ (0) such that

(2) $$(F \in \mathcal{R},\ S'_\mu(F) \in U') \Rightarrow (\lambda(F) \in U)$$

Thus $S'_\mu(E) \notin U'$; since U' is a l-band there exists a $\mathcal{D}_0 \in \sigma(E)$ such that $\mu(E - \cup \mathcal{D}') \notin U'$, for all $\mathcal{D}' \in f(\mathcal{D}_0)$. Because λ is σ-continuous, there exists $\mathcal{D}'_0 \in f(\mathcal{D}_0)$ such that $\lambda(E - \cup \mathcal{D}'_0) \in U$. Let $E_1 = \cup \mathcal{D}'_0$, $A_1 = E - E_1$. Then $\mu(A_1) \notin U'$, $\lambda(A_1) \in U$ and $\lambda(E_1) \notin U + U + U$. Hence $S'_\mu(E) \notin U'$ and so there exists a $\mathcal{D}_1 \in \sigma(E_1)$ such that $\mu(E_1 - \cup \mathcal{D}') \notin U'$, for all $\mathcal{D}' \in f(\mathcal{D}_1)$. Again σ-continuity of λ implies the existence of a $\mathcal{D}'_1 \in f(\mathcal{D}_1)$ such that $\lambda(E_1 - \cup \mathcal{D}'_1) \in V$, where $V \in \mathcal{B}(0)$ and $V + V \subseteq U$. Let $E_2 = \cup \mathcal{D}'_1$, $A_2 = E_1 - E_2$. Then $\mu(A_2) \notin U'$, $\lambda(A_2) \in V$ and $\lambda(E_2) \notin U + V$. It obtains a disjoint sequence $\{A_n,\ n=1, 2, \ldots\}$ in $\mathcal{R}$ such that $\mu(A_n) \notin U$, for $n=1, 2, \ldots$, which is a contradiction. Thus S'_μ is σ-singular.

2.3. **Lemma.** (1) *If λ is a σ-continuous G-valued submeasure on $\mathcal{R}$ such that $\lambda \ll \mu$, then $\lambda \ll S_\mu$.*

(2) *If ν is a σ-singular G-valued submeasure on $\mathcal{R}$ such that $\nu \ll \mu$, then $\nu \ll S'_\mu$.*

Proof. (1) Since $\lambda \ll \mu$, given $U \in \mathcal{B}(0)$, there exists $U' \in \tau$ such that

(3) $$(E \in \mathcal{R},\ \mu(E) \in U') \Rightarrow (\lambda(E) \in U),$$

Since U' is open, we can choose $U'' \in \mathcal{B}(0)$ such that $U'' \subset U'$. Let us prove that $S_\mu(E) \in U''$ implies $\lambda(E) \in U$. Suppose on the contrary that there exists an $E_0 \in \mathcal{R}$ such that $S_\mu(E_0) \in U''$ and $\lambda(E_0) \notin U + V$ for some $V \in \mathcal{B}(0)$. Going to a bigger $U'' \subset U'$ if it is necessary let us assume that there is a $\mathcal{D} \in \sigma(E_0)$ such that $\mu(\cup \mathcal{D}') \in U''$, for all $\mathcal{D}' \in f(D)$. Since λ is σ-continuous, there exists a $\mathcal{D}'_0 \in f(\mathcal{D})$ such that $\lambda(E_0 - \cup \mathcal{D}'_0) \in V$. It follows that $\lambda(\cup \mathcal{D}'_0) \notin U$. Thus $\mu(\cup \mathcal{D}'_0) \in U''$ and $\lambda(\cup \mathcal{D}'_0) \notin U$ (contradiction to (3)). So $\lambda \ll S_\mu$.

(2) Since $\nu \ll \mu$ given $U \in \mathcal{B}(0)$, there exists $U' \in \mathcal{B}(0)$ such that

(4) $$(E \in \mathcal{R}, \text{ and } \mu(E) \in U') \Rightarrow (\nu(E) \in U)$$

Suppose on the contrary there exists an $E_0 \in \mathcal{R}$ such that $S'_\mu(E_0) \in U'$ and $\nu(E_0) \notin U + V$, for some $V \in \mathcal{B}(0)$. Since U is solid, we have $\bigwedge\limits_{\mathcal{D}' \in f(\mathcal{D})} \mu(E - \cup \mathcal{D}') \in U'$, for all $\mathcal{D} \in \sigma(E_0)$.

Because $\nu \ll \mu$ and μ is exhaustive it follows that ν is exhaustive and so, by Lemma 2.2. 1), S_ν is σ-continuous. Moreover, $S_\nu \ll \nu$ and then, since ν is σ-singular it follows $S'_\nu = 0$. Thus there exists a $\mathcal{D}_0 \in \sigma(E_0)$ such that $\nu(\cup \mathcal{D}') \in V$, for all $\mathcal{D}' \in f(\mathcal{D}_0)$. Using the same arguments as in 1), we can choose $\mathcal{D}'_0 \in f \in (\mathcal{D}_0)$ so that $\mu(E_0 - \cup \mathcal{D}'_0) \in U'$. Let $F_0 = \cup \mathcal{D}'_0$, and it follows that $\nu(E_0 - F_0) \notin U$ which is a contradiction, and so $\nu \ll S'_\nu$.

In the following theorem $\lambda \sim \eta$ means that $\lambda \ll \eta$ and $\eta \ll \lambda$.

2.4. **Theorem.** *Let (G, τ) be a topological l-group which possesses a base of l-band neighborhoods for origin. Let μ be an exhaustive G-valued submeasure on $\mathcal{R}$ and σ an additivity on $\mathcal{R}$. Then $\mu \sim S_\mu + S'_\mu (\sim S_\mu \vee S'_\mu)$.*

If λ and ν are G-valued submeasures on $\mathcal{R}$, λ is σ-continuous and ν is σ-singular, respectively, such that $\mu \sim \lambda + \nu$, then $\lambda \sim S_\mu$ and $\nu \sim S'_\mu$.

Using the three lemmas presented previously, the proof follows as in the Theorem 2.1 in [2].

Particular case. Let G be an order complete l-group equipped with a quasi-norm q on G, which is a l-quasi-norm (i.e., $q(x) \leqslant q(y)$ for all $x, y \in G$, with $|x| \leqslant |y|$). Suppose that q is order continuous like in [2]. If we write $U_\delta = \{x \in G / q(x) \leqslant \delta\}$ for some $\delta > 0$, then $\mathcal{B}(0) = \{U_\delta\}\delta > 0$ is a base of neighborhoods for origin in (G, q).

Hence we deduce that Theorem 2.1 in [2] is a consequence of the preceding one.

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SOME SUFFICIENT CONDITIONS FOR CLOSE TO CONVEXITY AND STARLIKENESS OF MEROMORPHIC FUNCTIONS

BY

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1. Introduction. Let A denote the class of functions $f(z)$ regular in the open disc $D=\{z: |z|<1\}$ and normalized so that $f(0)=0=f'(0)-1$ and S denote the subclass of A consisting of univalent functions in D. Further C and S^* denote the subclasses of S whose members are close-to-convex and starlike in D respectively. Ram Singh and Sunder Singh [3] established a few sufficient conditions for close-to-convexity and starlikeness.

Let Σ denote the class of functions $f(z)=1/z+a_0+a_1z+a_2z^2+\ldots$ which are regular in the punctured disc $0<|z|<1$ with a simple pole at $z=0$ and Σ^*, Σ_c and Σ_s be the subclasses of Σ consisting of functions which are respectively starlike, close-to-convex and univalent. A function

$$g(z)=b_{-1}z^{-1}+b_0+b_1z+b_2z^2+\ldots \quad (|b_{-1}|=1), \tag{1}$$

which is regular in the punctures disc $0<|z|<1$ with a simple pole at $z=0$ is said to be starlike if $\operatorname{Re} zg'(z)/g(z)<0$, $z\in D$. A function f in Σ is said to be close-to-convex in the punctured disc $0<|z|<1$ if there exists a meromorphic, starlike, schlicht function $g(z)$ in $|z|<1$ with a simple pole at the origin, given by (1) such that $\operatorname{Re} zf'(z)/g(z)>0$ for $z\in D$. The concept of close-to-convexity to the case of meromorphic functions was introduced by R. J. Libera and M. S. Robertson [2]. It is well known that $\Sigma^*\subset\Sigma_s$ while $f\in\Sigma_c$ need not be univalent.

In this paper we obtain a few sufficient conditions for close-to-convexity and starlikeness of meromorphic functions. Methods are the same as could be found in [3].

2. We need the following result due to Jack [1].

Lemma. *Let $w(z)$ be regular in the unit disc D and such that $w(0)=0$. Then if $|w|$ attains its maximum value on the circle $|z|=r$ at the point z_0, we have $z_0w'(z_0)=kw(z_0)$, where $k\geqslant 1$.*

Theorem 1. *If $f\in\Sigma$ and*

$$|z^2f'(z)+1|^{1-\Gamma}|z^3f''(z)+2z^2f'(z)|^{\Gamma}<1, \quad z\in D, \tag{2}$$

for some $\Gamma\geqslant 0$, then f is meromorphically close-to-convex in $D-\{0\}$.

Proof. It suffices to show that (2) implies

$$|z^2f'(z)+1|<1, \quad z\in D. \tag{3}$$

Define $w(z)$ in D by

$$z^2f'(z)+1=w(z). \tag{4}$$