Hence we deduce that Theorem 2.1 in [2] is a consequence of the preceding one.

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SOME SUFFICIENT CONDITIONS FOR CLOSE TO CONVEXITY AND STARLIKENESS OF MEROMORPHIC FUNCTIONS

BY

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1. Introduction. Let A denote the class of functions $f(z)$ regular in the open disc $D=\{z:|z|<1\}$ and normalized so that $f(0)=0=f'(0)-1$ and S denote the subclass of A consisting of univalent functions in D. Further C and S^* denote the subclasses of S whose members are close-to-convex and starlike in D respectively. Ram Singh and Sunder Singh [3] established a few sufficient conditions for close-to-convexity and starlikeness.

Let Σ denote the class of functions $f(z)=1/z+a_0+a_1z+a_2z^2+\ldots$ which are regular in the punctured disc $0<|z|<1$ with a simple pole at $z=0$ and Σ^*, Σ_c and Σ_s be the subclasses of Σ consisting of functions which are respectively starlike, close-to-convex and univalent. A function

$$g(z)=b_{-1}z^{-1}+b_0+b_1z+b_2z^2+\ldots \quad (|b_{-1}|=1), \tag{1}$$

which is regular in the punctures disc $0<|z|<1$ with a simple pole at $z=0$ is said to be starlike if $\operatorname{Re} zg'(z)/g(z)<0$, $z\in D$. A function f in Σ is said to be close-to-convex in the punctured disc $0<|z|<1$ if there exists a meromorphic, starlike, schlicht function $g(z)$ in $|z|<1$ with a simple pole at the origin, given by (1) such that $\operatorname{Re} zf'(z)/g(z)>0$ for $z\in D$. The concept of close-to-convexity to the case of meromorphic functions was introduced by R. J. Libera and M. S. Robertson [2]. It is well known that $\Sigma^*\subset\Sigma_s$ while $f\in\Sigma_c$ need not be univalent.

In this paper we obtain a few sufficient conditions for close-to-convexity and starlikeness of meromorphic functions. Methods are the same as could be found in [3].

2. We need the following result due to Jack [1].

Lemma. *Let $w(z)$ be regular in the unit disc D and such that $w(0)=0$. Then if $|w|$ attains its maximum value on the circle $|z|=r$ at the point z_0, we have $z_0w'(z_0)=kw(z_0)$, where $k\geq 1$.*

Theorem 1. *If $f\in\Sigma$ and*

$$|z^2f'(z)+1|^{1-\Gamma}|z^3f''(z)+2z^2f'(z)|^{\Gamma}<1, \quad z\in D, \tag{2}$$

for some $\Gamma\geq 0$, then f is meromorphically close-to-convex in $D-\{0\}$.

Proof. It suffices to show that (2) implies

$$|z^2f'(z)+1|<1, \quad z\in D. \tag{3}$$

Define $w(z)$ in D by

$$z^2f'(z)+1=w(z). \tag{4}$$

Then clearly $w(0)=0$ and $w(z)$ is regular in D. We claim that $|w(z)|<1$ in D. Differentiating (4) we obtain

$$z^3f''(z)+2z^2f'(z)=zw'(z).$$

Hence

(5) $$|z^2f'(z)+1|^{1-\Gamma}|z^3f''(z)+2z^2f'(z)|^{\Gamma}=|w(z)|^{1-\Gamma}|zw'(z)|^{\Gamma}.$$

Suppose there exists a point z_0 in D such that

$$\max_{|z|\le|z_0|}|w(z)|=|w(z_0)|=1.$$

By applying of Jack's lemma to w at z_0 we have $z_0w'(z_0)=kw(z_0)$, where $k\ge 1$, so that (5) yields

$$|z_0^2f'(z_0)+1|^{1-\Gamma}|z_0^3f''(z_0)+2z_0^2f'(z_0)|^{\Gamma}=k^{\Gamma}\ge 1,$$

which contradicts (2). Therefore $|w(z)|<1$ in D so that $|z^2f'(z)+1|<1$ in D which shows that $f(z)$ is meromorphically close-to-convex in $D-\{0\}$. Putting $\Gamma=1$ in Theorem 1 we obtain

Corollary 1. *If $f\in\Sigma$ and*

(6) $$|z^3f''(z)+2z^2f'(z)|<1,\quad z\in D,$$

then f is meromorphically close-to-convex in $D-\{0\}$.

Theorem 2. *If $f\in\Sigma$ satisfies*

(7) $$|z^2f'(z)+1|^{1-\Gamma}|3+zf''(z)/f'(z)|^{\Gamma}<(3/2)^{\Gamma},\quad z\in D,$$

for some $\Gamma\ge 0$, then f is meromorphically close-to-convex in $D-\{0\}$.

The proof is similar to that of Theorem 1. Taking $\Gamma=1$ in Theorem 2, we obtain

Corollary 2. *If $f\in\Sigma$ satisfies*

(8) $$|3+zf''(z)/f'(z)|<3/2,\quad z\in D,$$

then f is meromorphically close-to-convex in $D-\{0\}$.

Theorem 3. *If $f\in\Sigma$ satisfies for $z\in D$ the condition*

(9) $$|zf'(z)/f(z)+1|^{1-\Gamma}|zf''(z)/f'(z)-2zf'(z)/f(z)|^{\Gamma}<(3/2)^{\Gamma}$$

for some $\Gamma>0$, then $f\in\Sigma^*$.

Proof. It suffices to prove that (9) implies Re $zf'(z)/f(z)<0$, $z\in D$.

Define $w(z)$ in D by

(10) $$zf'(z)/f(z)=-(1+w(z))/(1-w(z)).$$

Then clearly $w(0)=0$ and $w(z)$ is regular in D. We claim that $|w(z)|<1$ in D. Differentiating (10) logarithmically and simplifying we obtain

(11) $$|zf'(z)/f(z)+1|^{1-\Gamma}|zf''(z)/f'(z)-2zf'(z)/f(z)|^{\Gamma}= |2w(z)/(1-w(z))|\,|1+zw'(z)/[w(z)(1+w(z))]|^{\Gamma}.$$

By applying Jack's lemma to w at z_0 we get $z_0w'(z_0)=kw(z_0)$, where $k\ge 1$, so that (11) yields

$$|z_0f'(z_0)/f(z_0)+1|^{1-\Gamma}|z_0f''(z_0)/f'(z_0)-2z_0f'(z_0)/f(z_0)|^{\Gamma}\ge(1+k/2)^{\Gamma}\ge(1+1/2)^{\Gamma},$$

which contradicts (9). Hence $|w(z)|<1$ in D and from (10) it follows that $f\in\Sigma^*$.

Taking $\Gamma=1$ in Theorem 3, we obtain

Corollary 3. *If $f\in\Sigma$ satisfies*

$$|zf''(z)/f'(z)-2zf'(z)/f(z)|<3/2,\quad z\in D$$

then f is meromorphically starlike in D.

Theorem 4. *If $f\in\Sigma$ satisfies*

(12) $$|\alpha(zf'(z)/f(z)+1)+(1-\alpha)(2+4zf'(z)/f(z)+z^2f''(z)/f(z))|<1,\quad z\in D$$

for $0\le\alpha\le 1$, then f is meromorphically starlike in D.

Proof. It suffices to prove that (12) implies the inequality

$$|zf'(z)/f(z)+1|<1,\quad z\in D.$$

We define $w(z)$ in D by

(13) $$w(z)=zf'(z)/f(z)+1.$$

Then clearly $w(0)=0$ and $w(z)$ is regular in D. Logarithmic differentiation of (13) yields $2+zf''(z)/f'(z)=w(z)+zw'(z)/(w(z)-1)$

and hence

(14) $$|\alpha(zf'(z)/f(z)+1)+(1-\alpha)(2+4zf'(z)/f(z)+z^2f''(z)/f(z))| = |w(z)|\,|1+(1-\alpha)(w(z)+zw'(z)/w(z))|$$

We show that $|w(z)|<1$, $z\in D$. Suppose z_0 is a point in D such that

$$\max_{|z|\le|z_0|}|w(z)|=|w(z_0)|=1.$$

Applying Jack's lemma to w at the point z_0, we get $z_0w'(z_0)=kw(z_0)$, where $k\ge 1$ and $w(z_0)=e^{i\theta}$, so that we obtain from (14)

$$|\alpha(z_0f'(z_0)/f(z_0)+1)+(1-\alpha)(2+4z_0f'(z_0)/f(z_0)+z_0^2f''(z_0)/f(z_0))| = |1+(1-\alpha)(e^{i\theta}+k)|\ge 1,$$

which contradicts (12). This proves that $|w(z)|<1$ in D, and hence $|zf'(z)/f(z)+1|<1$ in D, from which it follows that f is meromorphically starlike in D. Taking $\alpha=0$ in Theorem 4, we obtain

Corollary 4. *If $f\in\Sigma$ satisfies*

(15) $$|2+4zf'(z)/f(z)+z^2f''(z)/f(z)|<1,\quad z\in D$$

then $f\in\Sigma^*$.

Theorem 5. *If $f\in\Sigma$, $\alpha\ge 0$, and*

$$\mathrm{Re}\,\{\alpha(1+zf''(z)/f'(z))+(1-\alpha)(z^2f'(z))^{-1}\}<-1/2,\quad z\in D$$

then f is meromorphically close-to-convex.

Proof. It suffices to prove that

(16) $$|z^2f'(z)+1|<1 \text{ for } z\in D.$$

Define $w(z)$ in D by

(17) $$z^2f'(z)+1=w(z).$$

Then clearly $w(0)=0$ and $w(z)$ is regular in D. We claim that $|w(z)|<1$ in D. The remaining proof is similar to that of Theorem 1. Putting $\alpha=1$ in Theorem 5 we obtain

Corollary 5. *If $f \in \Sigma$, and*

(18) $$\mathrm{Re}\,(1 + zf''(z)/f'(z)) < -1/2,\ z \in D$$

then f is meromorphically close-to-convex.

Our next theorem strengthens the results of Corollary 5.

Theorem 6. *If $f \in \Sigma$ satisfies (18) then*

(19) $$-z^2 f'(z) \prec 1 - z,\quad z \in D.$$

Moreover f is close-to-convex.

Proof. It is easy to see that $g(z) = 1 - z$ is univalent in D. Define a regular function $w(z)$ in D, $w(0) = 0$ such that

(20) $$-z^2 f'(z) = 1 - w(z)$$

Differentiating (20) we obtain $zf''(z)/f'(z) + 1 = -1 - zw'(z)/(1 - w(z))$.

We claim that $|w(z)| < 1$, otherwise there exists $z_0 \in D$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then applying Jack's lemma to w at z_0 we have

$$z_0 w'(z_0) = k w(z_0) \text{ where } k \geq 1, \text{ so that}$$

$$\mathrm{Re}\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) = -1 - \mathrm{Re}\,\frac{k w(z_0)}{1 - w(z_0)} = -1 + \frac{k}{2} \geq -\frac{1}{2},$$

which contradicts (18). This proves that $|w(z)| < 1$ in D and hence (19) holds, which in it turn implies

$$\mathrm{Re}\,(-z^2 f'(z)) = \mathrm{Re}\,(1 - w(z)) > 0 \text{ for } |z| < 1.$$

Hence f is close-to-convex.

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UNIFORM DISSIPATIVITY OF SOME FOURTH-ORDER NON-LINEAR DIFFERENTIAL EQUATIONS

BY

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1. Introduction. We shall be concerned in this paper with differential equations of the form

(1) $$x^{\mathrm{IV}} + a\dddot{x} + b\ddot{x} + g(\dot{x}) + h(x) = q(t, x, \dot{x}, \ddot{x}, \dddot{x}),$$

and

(2) $$x^{\mathrm{IV}} + a\dddot{x} + b\ddot{x} + g_1(x)\dot{x} + h(x) = q(t, x, \dot{x}, \ddot{x}, \dddot{x}),$$

under the basic assumptions that

(3) a, b are positive constants and functions g, g_1, h and q are real-valued and continuous in their respective arguments;

(4) there is a positive constant ρ_0 such that for all $t, x, \dot{x}, \ddot{x}, \dddot{x}$,

$$|q(t, x, \dot{x}, \ddot{x}, \dddot{x})| < \rho_0.$$

In an earlier investigation, B a r b a l a t and H a l a n a y [2] and H a l a n a y [4] used the frequency-domain technique to study uniform dissipativity for equations of the forms (1) and (2), but for which at most one of the functions g, g_1 or h is non-linear. Also, the author in [1] studied the equivalent equations to (1) and (2), but for third-order non-linear equations.

Our objective in this paper is to employ the frequency-domain technique in proving uniform dissipativity for equations (1) and (2). We shall thus extend earlier results in [1], [2] and [4] to fourth-order differential equations with two non-linear terms.

2. Statement of results. Two theorems will be proved in this paper. Our first result is concerned with equation (1).

Theorem 1. *Suppose, in addition to the basic assumptions* (3) *and* (4), *there exist positive constants c, d, μ_1 and μ_2 such that*

(5) $$ab > c,\ (ab - c)c - a^2 d > 0,$$

and for all $|z| \geq \lambda_0$, λ_0 sufficiently large,

(6) $$c \leq g(z)/z \leq c + \mu_1,\ c \leq g'(z) \leq c + \mu_1,$$

and

(7) $$d \leq h(z)/z \leq d + \mu_2,\ d \leq h'(z) \leq d + \mu_2.$$