Corollary 5. *If* $f \in \Sigma$, *and*

(18) $$\mathrm{Re}\,(1+zf''(z)/f'(z)) < -1/2,\ z \in D$$

then f *is meromorphically close-to-convex.*

Our next theorem strengthens the results of Corollary 5.

Theorem 6. *If* $f \in \Sigma$ *satisfies (18) then*

(19) $$-z^2 f'(z) \prec 1-z,\quad z \in D.$$

Moreover f *is close-to-convex.*

Proof. It is easy to see that $g(z)=1-z$ is univalent in D. Define a regular function $w(z)$ in D, $w(0)=0$ such that

(20) $$-z^2 f'(z) = 1 - w(z)$$

Differentiating (20) we obtain $zf''(z)/f'(z)+1=-1-zw'(z)/(1-w(z))$.

We claim that $|w(z)|<1$, otherwise there exists $z_0 \in D$ such that

$$\max_{|z|\leqslant|z_0|} |w(z)| = |w(z_0)| = 1.$$

Then applying Jack's lemma to w at z_0 we have

$$z_0 w'(z_0) = k w(z_0) \text{ where } k \geqslant 1, \text{ so that}$$

$$\mathrm{Re}\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right) = -1 - \mathrm{Re}\,\frac{k w(z_0)}{1-w(z_0)} = -1+\frac{k}{2} \geqslant -\frac{1}{2},$$

which contradicts (18). This proves that $|w(z)|<1$ in D and hence (19) holds, which in it turn implies

$$\mathrm{Re}\,(-z^2 f'(z)) = \mathrm{Re}\,(1-w(z)) > 0 \text{ for } |z|<1.$$

Hence f is close-to-convex.

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Department of Mathematics
Karnatak University
Dharwad-580 003
India
Department of Mathematics
P. C. Jabin Science College
Hubli-580 031
India

An. şt. Univ. „Al. I. Cuza" Iaşi
Matematica, 35 (1989)

UNIFORM DISSIPATIVITY OF SOME FOURTH-ORDER NON-LINEAR DIFFERENTIAL EQUATIONS

BY

ANTHONY UYI AFUWAPE

1. Introduction. We shall be concerned in this paper with differential equations of the form

(1) $$x^{\mathrm{IV}} + a\dddot{x} + b\ddot{x} + g(\dot{x}) + h(x) = q(t, x, \dot{x}, \ddot{x}, \dddot{x}),$$

and

(2) $$x^{\mathrm{IV}} + a\dddot{x} + b\ddot{x} + g_1(x)\dot{x} + h(x) = q(t, x, \dot{x}, \ddot{x}, \dddot{x}).$$

under the basic assumptions that

(3) a, b are positive constants and functions g, g_1, h and q are real-valued and continuous in their respective arguments;

(4) there is a positive constant ρ_0 such that for all $t, x, \dot{x}, \ddot{x}, \dddot{x}$,

$$|q(t, x, \dot{x}, \ddot{x}, \dddot{x})| \leqslant \rho_0.$$

In an earlier investigation, Barbalat and Halanay [2] and Halanay [4] used the frequency-domain technique to study uniform dissipativity for equations of the forms (1) and (2), but for which at most one of the functions g, g_1 or h is non-linear. Also, the author in [1] studied the equivalent equations to (1) and (2), but for third-order non-linear equations.

Our objective in this paper is to employ the frequency-domain technique in proving uniform dissipativity for equations (1) and (2). We shall thus extend earlier results in [1], [2] and [4] to fourth-order differential equations with two non-linear terms.

2. Statement of results. Two theorems will be proved in this paper. Our first result is concerned with equation (1).

Theorem 1. *Suppose, in addition to the basic assumptions (3) and (4), there exist positive constants* c, d, μ_1 *and* μ_2 *such that*

(5) $$ab > c,\ (ab-c)c - a^2 d > 0,$$

and for all $|z| \geqslant \lambda_0$, λ_0 *sufficiently large,*

(6) $$c \leqslant g(z)/z \leqslant c+\mu_1,\quad c \leqslant g'(z) \leqslant c+\mu_1,$$

and

(7) $$d \leqslant h(z)/z \leqslant d+\mu_2,\ d \leqslant h'(z) \leqslant d+\mu_2.$$

Then, if

(8) $$\lim_{|z|\to\infty} (1/z^2)\{\int_0^z g(s)ds - zg(z)/2\} \geq 0,$$

and

(9) $$\lim_{|z|\to\infty} (1/az^2)(ab-2c)\{\int_0^z h(s)ds - zh(z)/2\} \geq 0,$$

equation (1) *is uniformly dissipative.*

We note that if

(i) $g(\dot{x}) = c\dot{x}$, $c>0$, in equation (1), then inequalities (6) and (8) are trivially satisfied with μ_1 identically equal to zero, a situation investigated in [2];
(ii) $h(x) = dx$, $d>0$, in equation (1), then inequalities (7) and (9) are trivially satisfied with μ_2 identically equal to zero, a situation investigated in [2] and [4].

Our second result is concerned with equations of the form (2).

Theorem 2. *Suppose, in addition to the basic assumptions* (3) *and* (4), *there exist positive constants* c, d, μ_1 *and* μ_2 *such that inequalities* (5) *and* (7) *are satisfied as well as*

(10) $$c \leq G_1(x)/x \leq c+\mu_1,$$

for all $|x| \geq \lambda_0$, λ_0 *sufficiently large, where* $G_1(x) = \int_0^x g_1(s)ds$.
Then, if inequality (9) *holds, equation* (2) *is uniformly dissipative.*

We remark that if in equation (2), $h(x) = dx$, $d>0$, we obtain

$$x^{\mathrm{IV}} + a\dddot{x} + b\ddot{x} + g_1(x)\dot{x} + dx = q(t, x, \dot{x}, \ddot{x}, \dddot{x}).$$

Uniform dissipativity for this equation was studied in [2].

3. Theorem of Yacubovich. The proofs of our theorems, using the frequency-domain technique are based on the following result of Y a c u b o v i c h [7], and generalized by B a r b a l a t and H a l a n a y [2] to allow for systems of the form

(11) $$\dot{X} = AX - B\Phi(\sigma) + P(t, X),\ \sigma = C^*X.$$

Theorem 3. *Suppose that in system* (11), A *is an* $n\times n$ *stable matrix,* B *and* C *are* $n\times m$ *matrices with* C^* *as the complex conjugate of* C,
$\Phi(\sigma) = \mathrm{col}\,(\Phi_j(\sigma_j))$, $j=1, 2,\ldots, m$; *and for all* $|\alpha| \geq \lambda_0$,

$$0 \leq \alpha\Phi_j(\alpha) \leq \mu_0^j\alpha^2, \quad -\alpha_1^j \leq \Phi_j'(\alpha) \leq \alpha_2^j,$$

$$\mu_0^j \leq \alpha_2^j,\ \alpha_1^j \geq 0; \quad \textit{and } |P(t, X)| < \rho_0 \textit{ for all } (t, X).$$

If there exist diagonal matrices $D_1>0$, D_2, $D_3 \geq 0$ *such that*

(12) $$\{\{D_1 + Re\{[D_1 \mathrm{diag}\ \mu_0^j + i\omega D_2]G(i\omega)\} + \omega^2 Re[D_3(I + \mathrm{diag}\,(\alpha_2^j - \alpha_1^j)G(i\omega))] - G^*(-i\omega)D_3\ \mathrm{diag}\ \alpha_1^j\alpha_2^j)G(i\omega)\} > 0$$

for all ω *in* $\overline{\mathbf{R}}$, *where* $G(iw) = C^*(i\omega I - A)^{-1}B$, *then, if further, we have*

(13) $$\lim_{|\lambda|\to\infty} D_2(1/\lambda^2)\ \mathrm{diag}\ \{\int_0^\lambda \Phi_j(\alpha)d\alpha - (1/2)\lambda\Phi(\lambda)\} \geq 0,$$

system (11) *is uniformly dissipative.*

We note that if $\alpha_1^j = 0$, the last term in (12) vanishes and inequality (12) reduces to

(14) $$\{D_1 + Re\{[D_1\ \mathrm{diag}\ \mu_0^j + i\omega D_2]G(i\omega)\} + \omega^2 Re\{D_3[I + (\mathrm{diag}\ \alpha_2^j)G(i\omega)]\}\} > 0,$$

for all ω in $\overline{\mathbf{R}}$.

Furthermore, if $\Phi_j(\alpha)$, $(j = 1, 2,\ldots, m)$, is linear in α, the j-th component of (13) is zero.

In the same line with the frequency-domain technique, we introduce the following definition:
The system

(15) $$\dot{Y} = A_1Y - B_1\Phi_1(\sigma_1) + P_1(t, Y),\ \sigma_1 = C_1^*Y,$$

will be referred to as the *dual* of system (11), if matrices A_1, B_1, C_1 are respectively equal or equivalent to matrices A^*, C and B.

With this definition, the frequency-domain conditions (12) for system (11) are equivalent to those for system (15).

4. The Proof of Theorem 1. *4.1. Some Preliminary Results.* Let $g(\dot{x}) = c\dot{x} + \hat{g}(\dot{x})$ and $h(x) = dx + \hat{h}(x)$, where c, d are positive constants and g, h are differentiable real-valued continuous functions. Thus, inequalities (6) to (9) become

(16) $$0 \leq \hat{g}(z)/z \quad 0 \leq \hat{g}'(z) \leq \mu_1'$$

(17) $$0 \leq \hat{h}(z)/z \leq \mu_2;\ 0 \leq \hat{h}'(z) \leq \mu_2,$$

for all $|z| \geq \lambda_0$:

(18) $$\lim_{|z|\to\infty} (1/z^2)\{\int_0^z \hat{g}(s)ds - (z/2)\hat{g}(z)\} \geq 0,$$

and

(19) $$\lim_{|z|\to\infty} (1/az^2)(ab-2c)\{\int_0^z \hat{h}(s)ds - (z/2)\hat{h}(z)\} \geq 0,$$

respectively.

For convenience, we set $x_1 = x$, $\dot{x}_1 = x_2$, $\dot{x}_2 = x_3$, $\dot{x}_3 = x_4$ to obtain for equation (1), an equivalent system

(20) $$\dot{X} = AX - B\Phi(\sigma) + P(t, X),\ \sigma = C^*X,$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -d & -c & -b & -a \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and } P(t, X) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \hat{q}(t, x_1, x_2, x_3, x_4) \end{pmatrix},$$

with $\hat{q}(t, x_1, x_2, x_3, x_4) = q(t, x_1, \dot{x}_1, \ddot{x}_1, \dddot{x}_1)$.

In that case,

$$\sigma = C^{*}X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad \Phi(\sigma) = \begin{pmatrix} \hat{h}(x_1) \\ \hat{g}(x_2) \end{pmatrix}.$$

Observe that, since $(ab-c)>0$ and $(ab-c)c-a^2d>0$ with $c>0$ and $d>0$, the characteristic roots of matrix A have negative real parts. Thus matrix A is stable. Further, for all real values of ω, $(i\omega I - A)^{-1}$ exists.

A simple calculation shows that

$$G(i\omega) = C^{*}(i\omega I - A)^{-1}B = (1/\Delta(i\omega)) \begin{pmatrix} 1 & 1 \\ i\omega & i\omega \end{pmatrix},$$

where $\Delta(i\omega) = (\omega^4 - b\omega^2 + d) + i\omega(c - a\omega^2)$. We shall denote $\Delta(i\omega)$ simply by Δ, and its conjugate $\Delta(-i\omega)$ by $\bar{\Delta}$. Thus, $|\Delta|^2 = \Delta\bar{\Delta} = (\omega^4 - b\omega^2 + d)^2 + \omega^2(c - a\omega^2)^2$. Clearly, for all ω in $\bar{\mathbf{R}}$, $|\Delta|^2 > 0$.

For us to use the generalized theorem of Yacubovich, we choose

$$\text{(21)} \qquad D_1 = \begin{pmatrix} \tau_1/\mu_1 & 0 \\ 0 & \tau_2/\mu_2 \end{pmatrix} \qquad D_2 = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}$$

$$D_3 = \begin{pmatrix} \delta_1/\mu_1 & 0 \\ 0 & \delta_2/\mu_2 \end{pmatrix} \quad \text{and diag}\ \mu_0^j = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}.$$

with $\tau_j \geqslant 0$, $\delta_j \geqslant 0$ and θ_j $(j=1, 2)$ as parameters to be chosen appropriately later on. By inequalities (16) and (17), μ_0^j, μ_1^j, μ_2^j of Theorem 3 are respectively μ_j, 0 and μ_j $(j=1, 2)$. With these, we are left the evaluation of inequality (14) for all ω in $\bar{\mathbf{R}}$.

In view of (21), the frequency-domain condition (14) becomes

$$\text{(22)} \qquad \pi(\omega) = \begin{pmatrix} \pi_1(\omega) & \pi_2(\omega) \\ \pi_3(\omega) & \pi_4(\omega) \end{pmatrix} > 0$$

where $\pi_1(\omega)$, $\pi_2(\omega)$, $\pi_3(\omega)$ and $\pi_4(\omega)$ are defined by

$$\pi_1(\omega) = \tau_1\{(1/\mu_1) + [(\omega^4 - b\omega^2 + d) + \omega^2\theta_1(c - a\omega^2)/(\tau_1 + \omega^2\delta_1)]/|\Delta|^2\}$$

$$\pi_2(\omega) = \{[(\tau_1 + \omega^2(\delta_1 - \theta_2))(\omega^4 - b\omega^2 + d) + \omega^2(\tau_2 + \omega^2\delta_2 + \theta_1)(c - a\omega^2)] -$$

$$- i\omega[(\tau_1 + \omega^2(\delta_1 + \theta_2))(c - a\omega^2) + (\tau_2 + \omega^2\delta_2 - \theta_1)(\omega^4 - b\omega^2 + d)]\}/2|\Delta|^2$$

$$\pi_3(\omega) = \bar{\pi}_2(\omega)$$

and

$$\pi_4(\omega) = \tau_2\{(1/\mu_2) + \omega^2[(c - a\omega^2) - \theta_2(\omega^4 - b\omega^2 + d)/(\tau_2 + \omega^2\delta_2)]/|\Delta|^2\}.$$

4.2. Verification of (22). Matrix $\pi(\omega)$ will be positive definite for all real ω, if either of the following two conditions hold:

$$\text{(23)} \qquad \pi_1(\omega) > 0 \text{ and } \pi_1(\omega)\pi_4(\omega) - |\pi_2(\omega)|^2 > 0,$$

or

$$\text{(24)} \qquad \pi_4(\omega) > 0 \text{ and } \pi_1(\omega)\pi_4(\omega) - |\pi_2(\omega)|^2 > 0$$

for all real ω.

Conditions (24) are obtained by a non singular transformation of $\pi(\omega)$. To achieve these, the following results will be useful.

Lemma 1. *Let* $\theta_1 = (ab-2c)(\tau_1 + \delta_1 c/a)/ac$ *and* $\theta_2 = c(\tau_2 + \omega_1^2\delta_2)/d(b^2-4d)^{1/2}$. *Then for all real* ω, $\pi_1(\omega) > 0$, *and* $\pi_4(\omega) > 0$, *provided* $\mu_1 < [(ab-c)c - a^2d]/a^2$ *and* $\mu_2 < (a\omega_1^2 - c)$, *where* $2\omega_1^2 = [b + (b^2-4d)^{1/2}]$, ω_1^2 *being the larger root of* $\omega^4 - b\omega^2 + d = 0$.

Proof of Lemma 1. It is clear from the definitions of $\pi_1(\omega)$, $\pi_4(\omega)$ that $\pi_1(\omega) > 0$ and $\pi_4(\omega) > 0$, if for all ω in $\bar{\mathbf{R}}$

$$(1/\mu_1) > \max\{P_1(\omega^2)/|\Delta|^2\}, \text{ and } (1/\mu_2) > \max\{P_2(\omega^2)/|\Delta|^2\}$$

where

$$P_1(\omega^2) = \{(a\omega^2 - c)\omega^2\theta_1 - (\tau_1 + \omega^2\delta_1)(\omega^4 - b\omega^2 + d)\}/(\tau_1 + \omega^2\delta_1)$$

and

$$P_2(\omega^2) = \omega^2\{(\omega^4 - b\omega^2 + d)\theta_2 + (\tau_2 + \omega^2\delta_2)(a\omega^2 - c)\}/(\tau_2 + \omega^2\delta_2).$$

If the maximum of $P_1(\omega^2)/|\Delta|^2$ is attained at $\omega^2 = c/a$, then it has the value a^2/A_4 with $A_4 = (ab-c)c - a^2d$. Thus $(1/\mu_1) > a^2/A_4$. On the otherhand, if the maximum of $P_2(\omega^2)|/\Delta|^2$ is attained at $\omega^2 = \omega_1^2$, we have

$$P_2(\omega_1^2)/\omega_1^2(c - a\omega_1^2)^2 = 1/(a\omega_1^2 - c).$$

That is, $1/\mu_2 > 1/(a\omega_1^2 - c)$. We observe that

$$\text{(25)} \qquad \frac{a^2}{A_4} \geqslant \frac{\{\omega^2(a\omega^2 - c)(ab-2c)(a+tc)/a^2c\} - (1+t\omega^2)(\omega^4 - b\omega^2 + d)}{(1+t\omega^2)|\Delta|^2}$$

for all real values of ω, where $t = \delta_1/\tau_1 > 0$.

In a similar way, we note that for all real values of ω,

$$\text{(26)} \quad (a\omega_1^2 - c)^{-1} \geqslant \frac{c(\omega^2 - \omega_1^2)(\omega^2 - \omega_0^2)(1 + \omega_1^2 s) + d(a\omega^2 - c)(1 + \omega^2 s)(\omega_1^2 - \omega_0^2)}{d(1 + \omega^2 s)(\omega_1^2 - \omega_0^2)|\Delta|^2}$$

where $s = \delta_2/\tau_2$ and ω_0 is the smaller root of $\omega^4 - b\omega^2 + d = 0$, given by $2\omega_0^2 = b - (b^2-4d)^{1/2}$.

Further, we note that equality occur in (25) and (26) if we respectively have $\omega^2 = c/a$ and $\omega^2 = \omega_1^2$. This completes the proof of Lemma 1.

Lemma 2. *For all real values of* ω, *and all positive choices of*

$$t = \delta_1/\tau_1 \text{ and } s = \delta_2/\tau_2, \quad \pi_1(\omega)\pi_4(\omega) - |\pi_2(\omega)|^2 > 0.$$

Proof of Lemma 2. From the definitions of $\pi_1(\omega)$, $\pi_2(\omega)$ and $\pi_4(\omega)$, we have that

$$\pi_1(\omega)\pi_1(\omega)-|\pi_2(\omega)|^2=$$

$$\tau_1\tau_2\left\{\mu_1^{-1}+|\Delta|^{-2}\left[(\omega^4-b\omega^2+d)+\frac{\omega^2(ab-2c)\,(a+lc)(c-a\omega^2)}{a^2c(1+\omega^2 l)}\right\}\times\right.$$

(27) $$\times\left\{\mu_2^{-1}+\frac{\omega^2}{|\Delta|^2}\left[(c-a\omega^2)-\frac{c(1+\omega_1 s)\,(\omega^4-b\omega^2+d)}{d(\omega_1^2-\omega_0^2)(1+\omega^2 s)}\right]\right\}-$$

$$-(4|\Delta|^4)^{-1}\{[(\tau_1+\omega^2(\delta_1-\theta_2))(\omega^4-b\omega^2+d)+\omega^2(\tau_2+\omega^2\delta_2+\theta_1)(c-a\omega^2)]^2+$$

$$+\omega^2[(\tau_1+\omega^2(\delta_1+\theta_2))\,(c-a\omega^2)+(\tau_2+\omega^2\delta_2-\theta_1)\,(\omega^4-b\omega^2+d)]^2\}.$$

Since $|\Delta|^2$ is a polynomial of order eight in ω, or a polynomial of order four in ω^2, for all $\omega^2>\omega_1^2$, the last expression in brace bracket is always very small. We can therefore choose μ_1 and μ_2 such that $\pi_1(\omega)\pi_1(\omega)-|\pi_2(\omega)|^2>0$. In particular, if $\omega^2=c/a$, then (27) becomes

$$\tau_1\tau_2(\mu_2^{-1}-a^2A_4^{-1})\left\{\mu_1^{-1}+\frac{a^2c^2(1+\omega_1^2 s)}{A_1(\omega_1^2-\omega_0^2)\,(a+sc)d}\right]-$$

$$-(a^2/4A_4)\left\{\left[\tau_1+\frac{c}{a}(\delta_1-\theta_2)\right]^2+\frac{c}{a}\left(\tau_2+\frac{c}{a}\delta_2-\theta_1\right)^2\right\}.$$

Thus μ_1, μ_2, can be chosen as in Lemma 1 such that this expression is positive. This completes the proof of Lemma 2.

4.3. Supplementary Conditions. The supplementary conditions from (13) in relation to our system (20), and our choice of matrix D_2 will be

(28) $$\lim_{|z|\to\infty}\frac{1}{z^2}\begin{pmatrix}\theta_1 & 0\\ 0 & \theta_2\end{pmatrix}\begin{pmatrix}\left\{\int_0^z\hat h(s)ds-\frac{z}{2}\hat h(z)\right\} & 0\\ 0 & \left\{\int_0^z\hat g(s)ds-\frac{z}{2}\hat g(z)\right\}\end{pmatrix}\geqslant 0$$

A simplification of this shows that (28) is valid if and only if

$$\lim_{|z|\to\infty}\frac{1}{az^2}(ab-2c)\left\{\int_0^z\hat h(s)ds-\frac{z}{2}\hat h(z)\right\}\geqslant 0,\quad\text{and}\quad \lim_{|z|\to\infty}\frac{1}{z^2}\left\{\int_0^z\hat g(s)ds-\frac{z}{2}\hat g(z)\right\}\geqslant 0.$$

since θ_2 is positive.

4.4. Completion of the Proof of Theorem 1. The choice of matrices D_1, D_2 and D_3 with the conditions imposed on them by Lemmas 1 and 2 make the frequency-domain conditions valid for system (20). Thus all the conditions of Theorem 3 are fulfilled for this system. This completes the proof of Theorem 1.

5. Proof of Theorem 2. In equation (2), let $\int_0^x g_1(s)ds=cx+\hat g_1(x)$ and $h(x)=dx+\hat h(x)$, where c, d are positive constants and $\hat g_1(x)$ and $\hat h(x)$ are continuous differentiable functions satisfying $\hat g_1(0)=0=\hat h(0)$. An equivalent system for equation (2), is given by

(29) $$\dot x_1=-dx_4-\hat h(x_4)+\hat q(t, x_1, x_2, x_3, x_4),$$
$$\dot x_2=x_1-cx_4-\hat g_1(x_4),\quad \dot x_3=x_2-bx_4,\quad \dot x_4=x_3-ax_4,$$

where

$$\hat q(t, x_1, x_2, x_3, x_4)=q(t, x_4, \dot x_4, \ddot x_4, \dddot x_4).$$

As before, system (29) can be written in the vector form

$$\dot X=A_1X-B_1\Phi(\bar\sigma)+P_1(t, X),\quad \sigma=C_1^*X,$$

with

$$A_1=\begin{pmatrix}0&0&0&-d\\1&0&0&-c\\0&1&0&-b\\0&0&1&-a\end{pmatrix},\qquad B_1=\begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix},\qquad C_1=\begin{pmatrix}1&1\\0&0\\0&0\\0&0\end{pmatrix}.$$

Thus, we have a system dual to system (20) The frequency-domain conditions for both systems are the same. Hence, all the conclusions for system (20) are valid for system (29). This completes the proof of Theorem 2.

6. Remarks. We note that some work has been done by Ezeilo and Tejumola [3], Harrow [5] and others [6], on the ultimate boundedness of some fourth-order equations which can be reduced to the forms (1) and (2). They have all employed Lyapunov's second method in their proofs.

Of particular interest is the work of Harrow [5]. The equation he discussed reduces to equation (1), if we let $f(\ddot x)=b\ddot x$ in [5]. Then, the Lyapunov function employed reduces to

(30) $$2V(x, y, z, w)=(b\delta-d/a)y^2+(a-\delta+a^{-1}b)z^2+a^{-1}w^2+$$
$$+2\{wz+\delta wy+ayz+h(x)y+a^{-1}h(x)z+a^{-1}g(y)z+\textstyle\int_0^y g(s)ds+\delta\int_0^x h(s)ds\}.$$

This function has the following properties

(31) $$V(0, 0, 0, 0)=0,$$

(32) $$V(x, y, z, w)\geqslant D_1(x^2+y^2+z^2+w^2),$$

and

(33) $$\dot V\leqslant -D_2(y^2+z^2),$$

where D_j, $(j=1, 2)$ are some positive constants depending on a, b, c, d and δ. The frequency-domain technique uses a special Lyapunov's function (cf. [2]), satisfying in addition to conditions (31) and (32), the following condition

(34) $$\dot V\leqslant -D(x^2+y^2+z^2+w^2),$$

with D as a positive constant depending on a, b, c, and d. This makes a difference, since, with this condition, the full requirement of Lyapunov's Theorem can be exploited to advantage. Furthermore, this gives the frequency-domain technique an advantage, especially with the appearance of the x and w parameters in (34), over the Lyapunov second method as used by Harrow in [5]. The results also obtained improve considerably the results of Harrow in this direction.

We further remark that, the conditions on the non-linear functions $g(y)$ and $h(x)$ in [5] and the conditions here are the same. In fact, in [5], it was required that

$$c \leqslant g(y)/y \leqslant g'(y) \leqslant \alpha + g(y)/y \qquad (y \neq 0), \tag{35}$$

where $\alpha < \Delta d/2ac^2$ and

$$d - 2ac^{-1}\Delta < h'(x) \leqslant d\ ;\ 0 < \beta \leqslant h(x)/x, \tag{36}$$

for all x, $x \neq 0$. Here, we require that $h(x)/x$ be bounded above by $d + \mu_2$ with μ_2 a positive constant satisfying the conditions of Lemma 1.

7. Conclusions. The results in this paper, have extended the results in [1], [2] and [4] to fourth order equations with two non-linear terms. Moreover, we have given another way by which the discussions in [1] can be extended to fourth-order differential equations of the forms (1) and (2). We have made use of all the parameters in Theorem 3, without necessarily putting them initially to zero.

While the major contribution of this paper is to prove in another way, results earlier studied by using a non frequency-domain technique, [3] and [5], we can claim to have by our approach, made some improvements on the range of the non-linear function $g(x)$ or $g_1(x)\dot{x}$ in equation (1) or (2) respectively. From Lemma 1, we require that

$$\max\,\{g(y)/y\ ;\ g'(y)\} \leqslant c + \mu_1 < A_4 a^{-2} + c.$$

We can conclude from (35) that

$$\max\{g(y)/y\ ; g'(y)\} = g'(y) < dA_4/2ac^2 + \varepsilon$$

where $0 \leqslant \varepsilon \leqslant (g(y)/y - c)$. So long as $g(y)/y$ is close enough to c, and $(2c - ad) > 0$, our hypothesis is an improvement on [5].

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Department of Mathematics
University of Ife,
Ile-Ife, Nigeria

An. şt. Univ. „Al. I. Cuza" Iaşi
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ON A CLASS OF LINEAR INTEGRAL EQUATIONS OF VOLTERRA-FREDHOLM TYPE

BY

N. LUCA and T. HAVÂRNEANU

1. Summary. In this paper we shall study some problems with respect to a class of integral equations in which the unknown function depends on several independent variables.

Using the Hilbert theory and the results from the spectral theory of linear operators, we have found some sufficient conditions which if they are fulfilled the study of a class of integral equations is equivalent with the study of a certain system (finite or not) of integral equations, for which we may apply the theory of M. G. Krein [2].

Now we shall present an example to show what kind of problems we treat in this paper. Let us consider the following equations

$$\int_0^\xi A(\xi - \tau)u(\tau, x)d\tau - \Delta u = f(\xi, x),\ x \in \mathcal{E} \subset R^n,\ \xi \in [0;\ T], \tag{1.1}$$

and

$$\int_0^T A(\xi - \tau)u(\tau, x)d\tau - \Delta u = f(\xi, x),\ x \in \mathcal{E} \subset R^n,\ \xi \in [0;\ T], \tag{1.2}$$

where $A \in L^1(R)$ and $\mathcal{E}$ is a smooth domain of R^n. It is well-known that using the Green function G for the operator Δ, we may write the equations (1.1) and (1.2) under the form

$$u(\xi, x) = \int_{\mathcal{E}} G(x, y)f(\xi, y)dy + \int_0^\xi \int_{\mathcal{E}} A(\xi - \tau)G(x, y)u(\tau, y)dyd\tau \tag{1.3}$$

and

$$u(\xi, x) = \int_{\mathcal{E}} G(x, y)f(\xi, y)dy + \int_0^T \int_{\mathcal{E}} A(\xi - \tau)G(x, y)u(\tau, y)dyd\tau \tag{1.4}$$

respectively. Applying the theory from the Section 3 we may obtain various existence and uniqueness results for the equations (1.3) and (1.4) under certain assumptions on f.

2. Preliminaries. We have denoted by R^m and R^n the Euclidian spaces of dimension m and n respectively, by $\mathcal{D}$, $\mathcal{E}$ and Ω some domains (bounded or not) from R^m, R^n and $R^x = R^m \times R^n$ respectively, and ξ, x and w are points from R^m, R^n and R^x respectively.

If $\xi \in \mathcal{D}$, $x \in \mathcal{E}$ we shall use the following spaces and norms

$$L^2(\mathcal{E}) = \{\varphi\ ;\ \varphi : \mathcal{E} \to C,\ \int_{\mathcal{E}} |\varphi(x)|^2 dx < +\infty\}, \tag{2.1}$$