$$c \leqslant g(y)/y \leqslant g'(y) \leqslant \alpha + g(y)/y \qquad (y \neq 0), \tag{35}$$

where $\alpha < \Delta d / 2ac^2$ and

$$d - 2ac^{-1}\Delta < h'(x) \leqslant d \ ; \ 0 < \beta \leqslant h(x)/x, \tag{36}$$

for all x, $x \neq 0$. Here, we require that $h(x)/x$ be bounded above by $d + \mu_2$ with μ_2 a positive constant satisfying the conditions of Lemma 1.

7. Conclusions. The results in this paper, have extended the results in [1], [2] and [4] to fourth order equations with two non-linear terms. Moreover, we have given another way by which the discussions in [1] can be extended to fourth-order differential equations of the forms (1) and (2). We have made use of all the parameters in Theorem 3, without necessarily putting them initially to zero.

While the major contribution of this paper is to prove in another way, results earlier studied by using a non frequency-domain technique, [3] and [5], we can claim to have by our approach, made some improvements on the range of the non-linear function $g(x)$ or $g_1(x)\dot{x}$ in equation (1) or (2) respectively. From Lemma 1, we require that

$$\max \{g(y)/y \ ; \ g'(y)\} \leqslant c + \mu_1 < A_1 a^{-2} + c.$$

We can conclude from (35) that

$$\max\{g(y)/y \ ; g'(y)\} = g'(y) < dA_1/2ac^2 + \varepsilon$$

where $0 \leqslant \varepsilon \leqslant (g(y)/y - c)$. So long as $g(y)/y$ is close enough to c, and $(2c - ad) > 0$, our hypothesis is an improvement on [5].

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Department of Mathematics
University of Ife,
Ile-Ife, Nigeria

An. şt. Univ. „Al. I. Cuza" Iaşi
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ON A CLASS OF LINEAR INTEGRAL EQUATIONS OF VOLTERRA-FREDHOLM TYPE

BY

N. LUCA and T. HAVÂRNEANU

1. Summary. In this paper we shall study some problems with respect to a class of integral equations in which the unknown function depends on several independent variables.

Using the Hilbert theory and the results from the spectral theory of linear operators, we have found some sufficient conditions which if they are fulfilled the study of a class of integral equations is equivalent with the study of a certain system (finite or not) of integral equations, for which we may apply the theory of M. G. Krein [2].

Now we shall present an example to show what kind of problems we treat in this paper. Let us consider the following equations

$$\int_0^{\xi} A(\xi - \tau) u(\tau, x) d\tau - \Delta u = f(\xi, x), \quad x \in \mathcal{E} \subset R^n, \ \xi \in [0; T], \tag{1.1}$$

and

$$\int_0^{T} A(\xi - \tau) u(\tau, x) d\tau - \Delta u = f(\xi, x), \quad x \in \mathcal{E} \subset R^n, \ \xi \in [0; T], \tag{1.2}$$

where $A \in L^1(R)$ and $\mathcal{E}$ is a smooth domain of R^n. It is well-known that using the Green function G for the operator Δ, we may write the equations (1.1) and (1.2) under the form

$$u(\xi, x) = \int_{\mathcal{E}} G(x, y) f(\xi, y) dy + \int_0^{\xi} \int_{\mathcal{E}} A(\xi - \tau) G(x, y) u(\tau, y) dy d\tau \tag{1.3}$$

and

$$u(\xi, x) = \int_{\mathcal{E}} G(x, y) f(\xi, y) dy + \int_0^{T} \int_{\mathcal{E}} A(\xi - \tau) G(x, y) u(\tau, y) dy d\tau \tag{1.4}$$

respectively. Applying the theory from the Section 3 we may obtain various existence and uniqueness results for the equations (1.3) and (1.4) under certain assumptions on f.

2. Preliminaries. We have denoted by R^m and R^n the Euclidian spaces of dimension m and n respectively, by $\mathcal{D}$, $\mathcal{E}$ and Ω some domains (bounded or not) from R^m, R^n and $R^x = R^m \times R^n$ respectively, and ξ, x and w are points from R^m, R^n and R^x respectively.

If $\xi \in \mathcal{D}$, $x \in \mathcal{E}$ we shall use the following spaces and norms

$$L^2(\mathcal{E}) = \{\varphi \ ; \ \varphi : \mathcal{E} \to C, \ \int_{\mathcal{E}} |\varphi(x)|^2 dx < +\infty\}, \tag{2.1}$$

(2.2) $$N^{(2)}(\varphi)=\left(\int_{\mathcal{E}}|\varphi(x)|^2dx\right)^{1/2},$$

(2.3) $$L^p(\mathcal{D})=\{\psi\ ;\ \psi:\mathcal{D}\to R,\ \int_{\mathcal{D}}|\psi(\xi)|^pd\xi<+\infty\},$$

(2.4) $$N^{(p)}(\psi)=\{\int_{\mathcal{D}}|\psi(\xi)|^pd\xi)^{1/p},\ \text{for}\ 1\leqslant p<+\infty\},$$

(2.5) $$L^\infty(\mathcal{D})=\{\psi\ ;\ \psi:\mathcal{D}\to R,\ N^{(\infty)}(\psi)=\operatorname*{ess\,sup}_{\xi\in\mathcal{D}}|\psi(\xi)|<+\infty\},$$

(2.6) $$C(\mathcal{D})=\{\psi\ ;\ \psi\in L^\infty(\mathcal{D}),\ \psi\ \text{is continuous on}\ \mathcal{D}\}.$$

If $\Omega=\mathcal{D}\times\mathcal{E}$, we define

(2.7) $$S^p(\Omega)=\{\chi\ ;\ \chi:\Omega\to\mathbf{C},\ N^{(2)}(\chi(\xi\ ;\ .))\in L^p(\mathcal{D}),\ 1\leqslant p<+\infty\},$$

$$S^\infty(\Omega)=\{\chi\ ;\ \chi:\Omega\to\mathbf{C},\ N^{(2)}(\chi(\xi,\ .))\in C(\mathcal{D})\},$$
$$|\chi|_\alpha=N^{(p)}N^{(2)}(\chi(\xi\ ;\ .))),\quad 1\leqslant p\leqslant+\infty,$$

where $\mathbf{C}$ is the complex field.

For every $\varphi_1,\varphi_2\in L^2(\mathcal{E})$, we shall denote by $<\varphi_1,\varphi_2>$ their inner product, i.e.

(2.8) $$<\varphi_1,\ \varphi_2>=\int_{\mathcal{E}}\varphi_1(x)\overline{\varphi_2(x)}dx.$$

Remark 2.1. The spaces $L^p(\mathcal{D})$, $1\leqslant p\leqslant+\infty$ are Banach spaces and the space $L^2(\mathcal{E})$ is a separable Hilbert space. Let $\{\varphi_l\}_{l=1}^{+\infty}$ be a sequence of elements from $L^2(\mathcal{E})$ ($L^p(\mathcal{D})$, $C(\mathcal{D})$). Then we shall denote by

(2.9) $$\varphi(x)=\sum_{l=1}^{+\infty}\varphi_l(x),\ x\in\mathcal{E},\ (\varphi(\xi)=\sum_{l=1}^{+\infty}\varphi_l(\xi),\ \xi\in\mathcal{D}),$$

if and only if

(2.10) $$N^{(2)}\left(\sum_{l=1}^{r}\varphi_l-\varphi\right)\quad\left(N^{(p)}\left(\sum_{l=1}^{r}\varphi_l-\varphi\right),\quad 1\leqslant p\leqslant+\infty\right)$$

tends to 0 as $r\to+\infty$.

In Section 3 we shall use the following generalization of the Minkowski inequality [5]

(2.11) $$\left(\int_{\mathcal{D}}\left|\int_{\mathcal{E}}\psi(\xi;\ x)dx\right|^pd\xi\right)^{1/p}\leqslant\int_{\mathcal{E}}\left(\int_{\mathcal{D}}|\psi(\xi\ ;\ x)|^pd\xi\right)^{1/p}dx,$$

i.e. if for almost every $x\in\mathcal{E}$, there exists $\int_{\mathcal{D}}|\psi(\xi\ ;\ x)|^pd\xi$ and this functions is integrable on $\mathcal{E}$, then the right hand of inequality (2.11) exists and, moreover, the left hand of (2.11) exists and this inequality is true.

Remark 2.2. The space $S^p(\Omega)$, $1\leqslant p\leqslant+\infty$ is in fact the space $L^p(\mathcal{D},\ L^2(\mathcal{E}))$ if $1\leqslant p<+\infty$ and $C(\mathcal{D},\ L^2(\mathcal{E}))$ if $p=+\infty$.

If $K\in L^2(\mathcal{E}\times\mathcal{E})$, then the operator

(2.12) $$(\mathcal{K}\varphi)\ (x)=\int_{\mathcal{E}}K(x,\ y)\varphi(y)dy$$

is a linear continuous operator from $L^2(\mathcal{E})$ in $L^2(\mathcal{E})$ and additionally

(2.13) $$N^{(2)}(\mathcal{K}\varphi)\leqslant M_1N^{(2)}(\varphi),\ \text{i.e.}\ \|\mathcal{K}\|\leqslant M_1\ \text{where}$$

(2.14) $$M_1=\left(\int_{\mathcal{E}}\int_{\mathcal{E}}|K(x,\ y)|^2dxdy\right)^{1/2}<+\infty.$$

It is well-known that if $|\lambda|<1/M_1$, then the equation

(2.15) $$\varphi(x)=\lambda\int_{\mathcal{E}}K(x,\ y)\varphi(y)dy+f(x)$$

has a unique solution $\varphi\in L^2(\mathcal{E})$, for every $f\in L^2(\mathcal{E})$, which is given by the Neumann series.

If the kernel K satisfies (2.14) and the Carleman condition (see [1])

(2.16) $$\int_{\mathcal{E}}|K^2(x,\ y)|dy\leqslant M_2=\text{const.}$$

then the Neumann series converges uniformly and absolutely to the solution of Eq. (2.15). We remark that in the hypothesis (2.13), all the theorems of Fredholm-Riesz theory relative to Eq. (2.15) work (see [4]).

Remark 2.3. If mes $\mathcal{E}=+\infty$, then the conditions (2.14) and (2.16) are independent. In the case that mes $\mathcal{E}<+\infty$, then the inequality (2.16) implies the inequality (2.14). If K satisfies (2.14) and

(2.17) $$K(x,\ y)=\overline{K(y,\ x)},$$

then the operator defined by (2.12) is called symmetrical. Therefore we may apply the Hilbert-Schmidt theory relative to Eq. (2.15) and the the corresponding homogeneous equation

(2.18) $$\varphi(x)=\lambda\int_{\mathcal{E}}K(x,\ y)\varphi(y)dy.$$

Finally we remark that, if the kernel K satisfies (2.14), (2.16) and (2.17) then the operator $\mathcal{K}$ defined by (2.12) is compact and self-adjoint and the Fourier series attached to a solution of Eq. (2.15) according to the system of eigenfunctions $\{\varphi_i\}_{i\in N}$ of $\mathcal{K}$, is convergent both in $L^2(\mathcal{E})$ and uniformly and absolutely.

3. The main results. Let us consider the following scalar liniar integral equation of Volterra-Fredholm type, with the unknown function φ

(E) $$\varphi(v)=f(v)+\int_{\Omega}T(v\ ;\ w)\varphi(w)dw,$$

where $v=(\xi,\ x)\in\Omega=\mathcal{D}\times\mathcal{E}$, $w=(\eta\ ;\ y)\in\Omega$, $dw=d\eta dy$ and f is a given function. Soppose that

(i) $$f\in S^p(\Omega),$$

(ii) $$T(v\ ;\ w)=A(\xi-\eta)B(x,\ y),\ \xi-\eta=(\xi^1-\eta^1,\ldots,\ \xi^m-\eta^m)\ ;$$

(iii) $$A\in L^1(R^m)\ ;$$

(iv) B satisfies (2.14), (2.16) and (2.17).

Let us consider the operators

(3.1) $$(\mathcal{A}\varphi)\ (v)=\int_{\mathcal{D}}A(\xi-\eta)\varphi(\eta,\ x)d\eta,$$

(3.2) $$(\mathcal{T}\varphi)(v)=\int_{\Omega} T(v, w)\varphi(w)dw.$$

Remark 3.1. If we take $\mathcal{D}=\mathcal{D}_\xi=(0\,;\,\xi)\subset R^1$, $\xi\in(0\,;\,T]$, $\Omega=\Omega_\xi=\mathcal{D}_\xi\times\mathcal{E}$ and $S^p(\Omega)=S^p(\Omega_\xi)$ we obtain from (E) a Volterra type equation (see the example (1.4)) and all the results of Section 3 remain true. That is why we called equation (E) of Volterra-Fredholm type.

Lemma 3.1. *In the hypotheses (ii)–(iv) the operators $\mathcal{A}$ and $\mathcal{T}$ map $S^p(\Omega)$ in $S^p(\Omega)$ and moreover satisfy*

$$|\mathcal{A}\varphi|_\alpha\leqslant\|A\|_1|\varphi|_\alpha \text{ where } \|A\|_1=\int_{R^m}|A(\xi)|d\xi,\quad \varphi\in S^p(\Omega),$$

$$|\mathcal{T}\varphi|_\alpha\leqslant\|A\|_1\cdot M_1|\varphi|_\alpha \text{ where } M_1=\Big(\int_{\mathcal{E}}\int_{\mathcal{E}}|B(x, y)|^2dxdy\Big)^{1/2},\quad \varphi\in S^p(\Omega).$$

Proof. Let $\varphi\in S^p(\Omega)$. Then, using (2.11), we have

$$|\mathcal{A}\varphi|_\alpha=\Big(\int_{\mathcal{D}}\Big(\int_{\mathcal{E}}\Big|\int_{\mathcal{D}}A(\xi-\eta)\varphi(\eta, x)d\eta\Big|^2dx\Big)^{p/2}d\xi\Big)^{1/p}\leqslant$$

$$\leqslant\Big(\int_{\mathcal{D}}\Big(\int_{\mathcal{D}}\Big(\int_{\mathcal{E}}|A(\xi-\eta)|^2\cdot|\varphi(\eta, x)|^2dx\Big)^{1/2}d\eta\Big)^p d\xi\Big)^{1/p}=$$

$$=\Big(\int_{\mathcal{D}}\Big(\int_{\mathcal{D}}|A(\xi-\eta)|\Big(\int_{\mathcal{E}}|\varphi(\eta, x)|^2dx\Big)^{1/2}d\eta\Big)^p d\xi\Big)^{1/p}.$$

But $\big(\int_{\mathcal{E}}|\varphi(\eta, x)|^2dx\big)^{1/2}=N^{(2)}(\varphi(\eta;\,.))\in L^p(\mathcal{D})$ and as $A\in L^1(R^m)$ it results that $\int_{\mathcal{D}}|A(\xi-\eta)|N^{(2)}(\varphi(\eta\,;\,.))d\eta\in L^p(\mathcal{D})$ and $\big(\int_{\mathcal{D}}\big(\int_{\mathcal{D}}|A(\xi-\eta)|N^{(2)}(\varphi(\eta, .))d\eta\big)^p\big)^{1/p}\leqslant\|A\|_1\cdot N^{(p)}(N^{(2)}(\varphi))$, where $\|A\|_1=\int_{R^m}|A(\xi)|d\xi$.

Therefore we have shown that $|\mathcal{A}\varphi|_\alpha\leqslant\|A\|_1\cdot|\varphi|_\alpha$.
To show that the operator $\mathcal{T}$ maps $S^p(\Omega)$ in $S^p(\Omega)$ let $\varphi\in S^p(\Omega)$ and using the hypotheses (ii)–(iv) we get

$$(\mathcal{T}\varphi)(v)=\int_{\mathcal{D}}A(\xi-\eta)\Big(\int_{\mathcal{E}}B(x, y)\varphi(\eta, y)dy\Big)d\eta.$$

Denoting by

(3.3) $$\psi(\eta, x)=\int_{\mathcal{E}}B(x, y)\varphi(\eta, y)dy,$$

and taking into account (iv) and the fact that $\varphi\in S^p(\Omega)$, we obtain

$$N^{(2)}(\psi(\eta\,;\,.))\leqslant M_1N^{(2)}(\varphi(\eta, .)),\quad M_1=\Big(\int_{\mathcal{E}}\int_{\mathcal{E}}|B(x, y)|^2dx\,dy\Big)^{1/2}$$

from where, using the definition of $S^p(\Omega)$, it results

(3.4) $$N^{(p)}(N^{(2)}(\psi))\leqslant M_1N^{(p)}(N^{(2)}(\varphi)).$$

Therefore $\psi\in S^p(\Omega)$. Now using the fact that $\mathcal{A}$ maps $S^p(\Omega)$ in $S^p(\Omega)$, we get

$$\int_{\mathcal{D}}A(\xi-\eta)\psi(\eta, x)d\eta\in S^p(\Omega),$$

i.e. $$\mathcal{T}\varphi\in S^p(\Omega) \text{ and } |\mathcal{T}\varphi|_\alpha\leqslant\|A\|_1M_1|\varphi|_\alpha.$$

Now, let us consider the Fredholm operator

(3.5) $$(\mathcal{B}u)(x)=\int_{\mathcal{E}}B(x, y)u(y)dy,$$

which, according to hypothesis (iv), maps $L^2(\mathcal{E})$ into itself and thus all the considerations of Section 2 may be applied for the operator $\mathcal{B}$.

Let $\{\lambda_n\}_{n\in N}$ and $\{u_n\}_{n\in N}$ be the system of eigenvalues and, respectively, eigenfunctions of the operator $\mathcal{B}$.

Lemma 3.2. *In the hypotheses (i)–(iv), if the equation (E) has solutions in $S^p(\Omega)$, then every solution has the form*

(3.6) $$\varphi(\xi, x)=\sum_{i=1}^{+\infty}\varphi_i(\xi)u_i(x),$$

where $\varphi_i(\xi)$ are the Fourier coefficients of the solution $\varphi(\xi, x)$ according to the system of eigenfunctions $\{u_i(x)\}_{i\in N}$ and these Fourier coefficients satisfy the following system of integral equations

(Σ) $$\varphi_i(\xi)=f_i(\xi)+\lambda_i\int_{\mathcal{D}}A(\xi-\eta)\varphi_i(\eta)d\eta,\quad i=1, 2, \ldots$$

Here $\{\lambda_i\}_{i\in N}$ and $\{f_i(\xi)\}_{i\in N}$ are the eigenvalues of $\mathcal{B}$ and the Fourier coefficients of $f(\xi, x)$ corresponding to the system $\{u_i(x)\}_{i\in N}$, respectively.

Proof. If $\varphi(\xi, x)\in S^p(\Omega)$ is a solution of (E), then multiplying both members of this equation by $u_i(x)$ and integrating on $\mathcal{E}$, we obtain (formaly)

(3.7) $$\int_{\mathcal{E}}\varphi(\xi, x)u_i(x)dx=\int_{\mathcal{E}}f(\xi, x)u_i(x)dx+$$
$$+\int_{\mathcal{E}}\Big(\int_{\mathcal{D}\times\mathcal{E}}A(\xi-\eta)B(x, y)\varphi(\eta, y)\,d\eta dy\Big)u_i(x)dx.$$

But

$$\int_{\mathcal{E}}\Big(\int_{\mathcal{D}\times\mathcal{E}}A(\xi-\eta)B(x, y)\varphi(\eta, y)\,d\eta\,dy\Big)\,u_i(x)dx=$$
$$=\int_{\mathcal{E}}\Big(\int_{\mathcal{D}}A(\xi-\eta)\Big(\int_{\mathcal{E}}B(x, y)\varphi(\eta, y)dy\Big)d\eta\Big)u_i(x)\,dx=$$
$$=\int_{\mathcal{D}}A(\xi-\eta)\Big(\int_{\mathcal{E}}\int_{\mathcal{E}}B(x, y)\varphi(\eta, y)u_i(x)\,dy\,dx\Big)\,d\eta=$$
$$=\int_{\mathcal{D}}A(\xi-\eta)\Big(\int_{\mathcal{E}}\varphi(\eta, y)\Big(\int_{\mathcal{E}}B(y, x)u_i(x)\,dx\Big)\,dy\Big)\,d\eta=$$
$$=\int_{\mathcal{D}}A(\xi-\eta)\Big(\int_{\mathcal{E}}\varphi(\eta, y)\lambda_iu_i(y)dyd\eta=\lambda_i\int_{\mathcal{E}}A(\xi-\eta)\varphi_i(\eta)d\eta.$$

Substituting this relation in the right hand of (3.7) and taking into account the semnifications of $f_i(\xi)$ and $\varphi_i(\xi)$, we get the relations (Σ).
The relations (Σ) may be regarded as a system of integral equations with the unknown functions $\varphi_i(\xi)$. From the above it results that a fundamental problem for the solubility of (E) is the solubility of the system (Σ) in $L^p(\mathcal{D})$.

Lemma 3.3. *We suppose that* (i)–(iv) *are satisfied. Then there exists a number n_0 such that the solubility of the system (Σ) is equivalent with the solubility of the first n_0 equations of this system.*

Proof. We distinguish two cases:
(a) $B(x, y)$ is a degenerate kernel,
(b) $B(x, y)$ is a nondegenerate kernel.

In the case (a), the system of the eigenvalues of $\mathcal{B}$ is finite and we shall denote by n_0 their number. Then (Σ) contains only n_0 equations and so in this case the assertion is true.

In the case (b) the system of eigenvalues is at most countable and the system (Σ) contains at most an infinity of integral equations. Because $\lambda_i \to 0$ as $i \to +\infty$, there exists a natural number n_0 such that $|\lambda_i| < \dfrac{1}{\|A\|_1}$, for $i > n_0$.

The operators

$$(U_i w)(\xi) = \lambda_i \int_{\mathcal{D}} A(\xi - \eta) w(\eta) d\eta$$

have the property that they map each of the spaces $L^p(\mathcal{D})$ and $C(\mathcal{D})$ into. As for i fixed, the norm of U_i in these spaces is

$$\|U_i\| = |\lambda_i| \int_{\mathcal{D}} |A(\xi)| d\xi$$

it results that

$$\|U_i\| < 1, \quad \forall i > n_0.$$

Therefore, for $i > n_0$, every U_i is a contraction on $L^p(\mathcal{D})$ and $C(\mathcal{D})$, respectively, and so the system (Σ) is uniquely solvable in each of these spaces. Thus the system (Σ) is solvable iff the system formed with the first n_0 equations of (Σ) is solvable.

Lemma 3.4. *In the hypotheses of Lemma 3.3, if* $|\lambda_i| < \|A\|_1^{-1}$, $i = 1, 2, \dots, n_0$, *then the system* (Σ) *is uniquely solvable in* $L^p(\mathcal{D})$ *and* $C(\mathcal{D})$ *respectively, for every* $f \in S^p(\Omega)$.

Proof. As in the proof of Lemma 3.3 it results that every U_i is a contraction on $L^p(\mathcal{D})$ or $C(\mathcal{D})$ for every $i \in N$ and so the system (Σ) is uniquely solvable in $L^p(\mathcal{D})$ or in $C(\mathcal{D})$.

Remark 3.2. The condition $|\lambda_i| < \|A\|_1^{-1}$, $i = 1, 2, \dots, n_0$ from Lemma 3.4 may be replaced by the condition that λ_i belongs to the resolvent set of the operator $\int_{\mathcal{D}} A(\xi - \eta)\varphi(\eta) d\eta$ for every $i \in N$.

Summarizing the results from Lemmas 3.2–3.4 the following theorem results:

Theorem 3.1. *Supposing that*

(C_1) *the conditions* (i)–(iv) *are fulfilled;*

(C_2) *the equation* (E) *has solutions in* $S^p(\Omega)$;

then

α) *the Fourier coefficients* $\varphi_i(\xi)$ *of every solution of the equation* (E) *from* $S^p(\Omega)$, *satisfy the system of integral equations* (Σ);

β) *the system* (Σ) *is solvable iff the system formed with the first* n_0 *equations of* (Σ) (n_0 *given by lemma 3.3*) *is solvable;*

γ) *if* λ_i *belongs to the resolvent set of the operator*

$$(\mathcal{A}\varphi)(\xi) = \int_{\mathcal{D}} A(\xi - \eta)\varphi(\eta) d\eta \tag{3.8}$$

for every $i \leq n_0$, *then the system* (Σ) *is uniquely solvable in* $L^p(\mathcal{D})$ $(C(\mathcal{D}))$ *for every* $f \in S^p(\Omega)$.

Let us consider the homogeneous equation corresponding to (E)

$$\psi(\xi; x) = \int_{\Omega} K(\xi, x; \eta, y)\psi(\eta, y) d\eta dy. \tag{E_0}$$

If $\psi(\xi; x)$ is a solution of (E_0) from $S^p(\Omega)$, then its Fourier coefficients $\psi_i(\xi)$ satisfy the homogeneous systems corresponding to (Σ)

$$\psi_i(\xi) = \lambda_i \int_{\mathcal{D}} A(\xi - \eta)\psi_i(\eta) d\eta, \quad i = 1, 2, 3, \dots \tag{Σ_0}$$

In this case we have obtained the following result.

Theorem 3.2. *If the conditions* (i)–(iv) *are satisfied, then the equation* (E_0) *is solvable iff the finite system*

$$\psi_i(\xi) = \lambda_i \int_{\mathcal{D}} A(\xi - \eta)\psi_i(\eta) d\eta, \quad i = 1, 2, \dots, n_0, \tag{3.9}$$

is solvable, where n_0 *is given by Lemma 3.3.*

Proof. „The only if" part. Let us suppose that (E_0) has a solution in $S^p(\Omega)$. Let $\psi(\xi, x)$ be such a solution. Then we may write

$$\psi(\xi, x) = \sum_{i=1}^{+\infty} \psi_i(\xi) u_i(x), \tag{3.10}$$

where $\{\psi_i(\xi)\}$ are Fourier coefficients according to the system $\{u_i(x)\}_{i \in N}$, and using Lemma 3.2, these coefficients satisfy the system (Σ) in which we have to take $f = 0$. Thus $\{\psi_i(\xi)\}$ satisfy the system (Σ_0). Let n_0 be the number given by Lemma 3.3. Then, if $B(x, y)$ is degenerate, the system (Σ_0) has only n_0 equations (see Lemma 3.3) and the „only if" part of Theorem 3.2 follows. If the kernel $B(x, y)$ is nondegenerate, then for $i > n_0$ (n_0 given by Lemma 3.3), using the fact that every equation of (Σ) for $i > n_0$ has a unique solution and the Fredholm alternative theorem, it results that every equation of (Σ_0) for $i > n_0$ has only the trivial solution, the other equations of (Σ_0) being solvable because $\psi(\xi, x)$ is a solution of (E_0). Therefore every solution from $S^p(\Omega)$ has the form

$$\psi(\xi, x) = \sum_{i=1}^{n_0} \psi_i(\xi) u_i(x), \tag{3.11}$$

where $\psi_i(\xi)$ is a solution of the i-equation from (Σ_0), $1 \leq i \leq n_0$.

The „if part". If $\psi_i(\xi)$ is a solution of the i-equation from (Σ_0), $1 \leq i \leq n_0$, then the function

$$\psi(\xi, x) = \psi_i(\xi) u_i(x), \tag{3.12}$$

is a solution of (E_0), because

$$\int_{\Omega} K(\xi, x; \eta, y)\psi_i(\eta) u_i(y) d\eta dy = \int_{\mathcal{D} \times \mathcal{E}} A(\xi - \eta) B(x, y) \psi_i(\eta) u_i(y) d\eta dy =$$

$$= \int_{\mathcal{D}} \left(\int_{\mathcal{E}} B(x, y) u_i(y) dy \right) A(\xi - \eta)\psi_i(\eta) d\eta = \lambda_i \int_{\mathcal{D}} u_i(x) A(\xi - \eta)\psi_i(\eta) d\eta = \lambda_i \psi_i(\xi) u_i(x).$$

Then, the linearly independent solutions of (E_0) are formed with linearly independent solution of the system (3.9). Indeed, let $\psi(\xi, x)$ and $\theta(\xi, x)$ be two linearly independent solutions of (E_0). Then, evidently, these have the form

$$\psi(\xi, x) = \sum_{i=1}^{n_0} \psi_i(\xi) u_i(x), \quad \theta(\xi, x) = \sum_{i=1}^{n_0} \theta_i(\xi) u_i(x), \tag{3.13}$$

where $\psi_i(\xi)$, $\theta_i(\xi)$ satisfy the system (3,9). Because $\psi(\xi, x)$ and $\theta(\xi, x)$ are linearly independent, it results

$$\alpha\psi(\xi, x)+\beta\theta(\xi, x)=\sum_{i=1}^{n_0}(\alpha\psi_i(\xi)+\beta\theta_i(\xi))u_i(x)=0 \tag{3.14}$$

$$\alpha,\ \beta\in R(C)\to\alpha=\beta=0.$$

But $\{u_i(x)\}_{i\in N}$ is an orthonormal system and so, from relation (3.14) we obtain

$$\alpha\psi(\xi, x)+\beta\theta(\xi, x)=0\Leftrightarrow\alpha\psi_i(\xi)+\beta\theta_i(\xi)=0,\quad i=1, 2,\dots, n_0.$$

Therefore $\alpha\psi_i(\xi)+\beta\theta_i(\xi)=0\Rightarrow\alpha=\beta=0$, i.e. $\psi_i(\xi)$ and $\theta_i(\xi)$ are linearly independent solutions of (Σ_0). Then, the solutions (3.12) obtained with the help of the functions $\psi_i(\xi)$ for different λ_i are linearly independent because if we should have

$$\alpha\psi_i(\xi)u_i(x)+\beta\psi_j(\xi)u_j(x)=0\ \text{ with }\ |\alpha|^2+|\beta|^2\neq 0, \tag{3.15}$$

then it would results that $u_i(x)$ and $u_j(x)$ would be linearly dependent, that is impossible because $\{u_i\}_{i\in N}$ are eigenfunctions. This completes the proof.

Now we shall give some sufficient conditions to make the function

$$\varphi(\xi, x)=\sum_{i=1}^{+\infty}\varphi_i(\xi)u_i(x), \tag{3.16}$$

belong to $S^p(\Omega)$, where $\varphi_i(\xi)$ are the solutions of (Σ) in $L^p(\mathcal{D})$ $(C(\mathcal{D}))$.

Proposition 3.1. *If $f\in S^p(\Omega)$ has the properties*

$$\sum_{i=1}^{+\infty}\|f_i\|^2_{L^p}|\lambda_i|^2<+\infty, \tag{a_1}$$

(a_2) $1\leqslant p<+\infty$ *and* mes $\mathcal{D}<+\infty$ *or* $p=2$ *and* mes $\mathcal{D}\leqslant+\infty$, *then* $\varphi\in S^p(\Omega)$.

Proof. We have to prove that $\varphi(\xi, x)$ given by (3.16) belongs to $L^p(\mathcal{D}, L^2(\mathcal{E}))$ $(C(\mathcal{D}; L^2(\mathcal{E}))$.

From the i-equation of the system (Σ) it results

$$\begin{aligned}|\varphi_i(\xi)|^2&\leqslant 2(|f_i(\xi)|^2+|\lambda_i|^2|\int_{\mathcal{D}}A(\xi-n)\varphi_i(\eta)d\eta|^2)\leqslant 2(|f_i(\xi)|^2+\\&+|\lambda_i|^2\cdot\|A\|_1^2\cdot\|\varphi_i\|^2_{L^p}.\end{aligned} \tag{3.17}$$

But

$$\varphi_i(\xi)=f_i(\xi)+\lambda_i\int_{\mathcal{D}}A(\xi-\eta)\varphi_i(\eta)d\eta,\ \text{i.e.}$$

$$\varphi_i(\xi)=f_i(\xi)+\lambda_i(\mathcal{A}\varphi_i)(\xi)$$

with

$$\|\mathcal{A}\|=\|A\|_1=\int_{\mathcal{D}}|A(\xi)|d\xi.$$

Thus for $|\lambda_i|<\dfrac{1}{\|A\|_1}$ there exists $(I-\lambda_i\mathcal{A})^{-1}: S^p(\Omega)\to S^p(\Omega)$ and $(I-\lambda_i\mathcal{A})^{-1}\in\in\mathcal{L}(S^p(\Omega))$ from where it follows

$$\|\varphi_i\|_{L^p}\leqslant\|(I-\lambda_i\mathcal{A})^{-1}\|\cdot\|f_i\|_{L^p},\quad i>n_0$$

(n_0 given in Lemma 3.3), and so there exists $k_0>n_0$ such that

$$\|\varphi_i\|_{L^p}\leqslant 2\cdot\|f_i\|_{L^p}\ \text{ for }\ i>k_0. \tag{3.18}$$

From (3.17) and (3.18) we obtain

$$\begin{aligned}\sum_{i=1}^{+\infty}|\varphi_i(\xi)|^2&\leqslant 2(\sum_{i=1}^{+\infty}|f_i(\xi)|^2+\|A\|_1\sum_{i=1}^{k_0}|\lambda_i|^2\|\varphi_i\|^2_{L^p}+\\&+4\|A\|_1^2\sum_{i=k_0+1}^{+\infty}|\lambda_i|^2\|f_i\|^2_{L^p})<+\infty\ \text{a.e.}\ \xi\in\mathcal{D}.\end{aligned} \tag{3.19}$$

In conclusion, (3.19) shows that $\varphi(\xi,.)\in L^2(\mathcal{E})$ and

$$N^{(2)}(\varphi(\xi,.))=(\sum_{i=1}^{+\infty}|\varphi_i(\xi)|^2)^{1/2}.$$

1) If mes $\mathcal{D}<+\infty$, then from (3.19) it results

$$\int_{\mathcal{D}}(\sum_{i=1}^{+\infty}|\varphi_i(\xi)|^2)^{p/2}d\xi\leqslant 2^{\frac{p-1}{2}}(\int_{\mathcal{D}}(\sum_{i=1}^{+\infty}|f_i(\xi)|^2)^{p/2}d\xi+C\ \text{mes}\ \mathcal{D}).$$

That is

$$|\varphi|_\alpha\leqslant C_1(|f|_\alpha+C_2),$$

thus $\varphi\in S^p(\Omega)$.

2) If $p=2$, we have immediately from (3.18)

$$\int_{\mathcal{D}}(\sum_{i=1}^{+\infty}|\varphi_i(\xi)|^2)d\xi\leqslant 2^2\cdot\int_{\mathcal{D}}(\sum_{i=1}^{+\infty}|f_i(\xi)|^2)d\xi,\ \text{i.e.}$$

$$|\varphi|_\alpha\leqslant 2|f|_\alpha\ \text{and so}\ \varphi\in S^2(\Omega)\ \text{if}\ f\in S^2(\Omega).$$

Proposition 3.2. *If $f\in S^p(\Omega)$ and satisfies the condition (a_2) from Proposition 3.1, and*

$$\sum_{1=i}^{+\infty}|\lambda_i|^2\|f_i\|_{L^p}<+\infty, \tag{a}$$

$$\sum_{i=1}^{+\infty}|\lambda_i\ |f_i(\xi)|\ \text{is measurable}, \tag{b}$$

then $\varphi(\xi, x)$ given by (3.16) is a solution of (E).

Proof. Because

$$\varphi_i(\xi)=f_i(\xi)+\lambda_i\int_{\mathcal{D}}A(\xi-\eta)\varphi_i(\eta)d\eta,$$

we have

$$|\varphi_i(\xi)|\leqslant|f_i(\xi)|+|\lambda_i|\cdot\|A\|_1\|\varphi_i\|_{L^p},$$

from where, using (3.18), we obtain

$$\sum_{i=1}^{+\infty}|\lambda_i|\,|\varphi_i(\xi)|\leqslant\sum_{i=1}^{+\infty}|\lambda_i|\cdot|f_i(\xi)|+\sum_{i=1}^{k_0}|\lambda_i|^2\|A\|_1\|\varphi_i\|_{L^p}+2\|A\|_1\sum_{k_0+1}^{+\infty}|\lambda_i|^2\|f_i\|_{L^p}.$$

By using the Lebesgue Theorem, we get

$$(\mathcal{T}\varphi)(\xi, x) = \int_{\mathcal{D}} A(\xi-\eta) \int_{\mathcal{E}} B(x, y)\varphi(\eta, y)\, dy\, d\eta = \int_{\mathcal{D}} A(\xi-\eta) \sum_{i=1}^{+\infty} \lambda_i \varphi_i(\eta) u_i(x)\, d\eta =$$

$$= \sum_{i=1}^{+\infty} (\lambda_i \int_{\mathcal{D}} A(\xi-\eta)\varphi_i(\eta)\, d\eta) u_i(x) = \sum_{i=1}^{+\infty} \varphi_i(\xi) u_i(x) = \varphi(\xi, x).$$

One can easily obtain the following result in the case of the space $C(\mathcal{D}, L^2(\mathcal{E}))$.

Proposition 3.3. *If* $f \in C(\mathcal{D}, L^2(\mathcal{E}))$ *and* $\sum_{i=1}^{+\infty} |\lambda_i| \, \|f_i\|_{C(\mathcal{D})} < +\infty$, *then* $\varphi(\xi, x)$ *given by* (3.16) *is a solution of* (E) *from* $C(\mathcal{D}, L^2(\mathcal{E}))$.

Proof. Using the relation

$$\|\varphi_i\|_{C(\mathcal{D})} \leqslant 2\|f_i\|_{C(\mathcal{D})}, \text{ for } i > k_0,$$

and the fact that φ_i is a solution of the i-equation of (Σ) we easily obtain the conclusion of Proposition 3.3.

Proposition 3.4. If $|\lambda_i| < \|A\|_1^{-1}$, $\forall i \in N$, *then the equation* (E) *has a unique solution in* $S^p(\Omega)$, $1 \leqslant p \leqslant +\infty$.

Proof. Using Lemma 3.1, we have

$$|\mathcal{T}\varphi|_x \leqslant \|\mathcal{B}\| \cdot \|A\|_1 |\varphi|_x.$$

But $\|\mathcal{B}\| = \sup\{|\lambda_i|;\ i \in N\} < \|A\|_1^{-1}$.

Therefore $\mathcal{T}$ is a contraction on $S^p(\Omega)$ and so for every $f \in L^p(\mathcal{D}, L^2(\mathcal{E}))$ $(C(\mathcal{D}, L^2(\mathcal{E})))$ we have a unique solution of (E) from $S^p(\Omega)$.

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Faculty of Mathematics
University of Iaşi,
Iaşi, Romania

An. şt. Univ. „Al. I. Cuza" Iaşi
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A SIMPLICIAL PROOF OF THE KÜNNETH FORMULA FOR POLYHEDRA

BY

I. POP

0. Introduction. Although the well-known Künneth exact sequence and the Künneth formula

$$H_n(X \times Y) \cong \bigoplus_{q+r=n} H_q(X) \otimes H_r(Y) \oplus \left(\bigoplus_{q+r=n-1} H_q(X) * H_r(Y)\right),$$

are essentially due to Herman Künneth [5], [6], in connection with the study of the Betti numbers and the torsion coefficients of a product of polyhedra, nevertheless in many books of algebraic topology these important relations are presented either in the framework of the singular homology, using the Eilenberg-Zilber theorem [7], [10], [11], or in the context of cellular homology (for X and Y CW-complexes) [8], or the background of the de Rham cohomology, using the Morse Theory or the cup-product (if X and Y are differentiable manifolds) [2], [4]. Even when X and Y are polyhedra, one begins by establishing the fomula for the singular homology and then one uses the isomorphism between the simplicial homology of a simplicial complex K and the singular homology of the space $|K|$, [3].

In this note, supposing X and Y being polyhedra, we give a direct simplicial proof of the Künneth formula, using the simplicial structure of the product $X \times Y$ and the calculus of the simplicial homology with a natural block dissection of the corresponding simplicial complex.

1. Some preliminaries regarding the triangulation of a product of polyhedra. For the data included in this paragraph we appeal essentially to [9, p. 109–111].

1.1. The topological product of two simplexes σ *and* τ *is a triagulated space.* If $\sigma = [a^0, a^1, \ldots, a^q] \subset \mathbf{R}^m$ and $\tau = [b^0, b^1, \ldots, b^q] \subset \mathbf{R}^n$, let $x_1 \in \sigma$, $x_1 = \Sigma\lambda_k(x_1)a^k$ and $x_2 \in \tau$, with $x_2 = \Sigma \mu_l(x_2)b^l$. One defines $\alpha_i(x_1) = \sum_{k=0}^{i} \lambda_k(x_1)$ and $\beta_j(x_2) = \sum_{l=0}^{j} \mu_l(x_2)$, $i = 0, 1, \ldots, q-1$; $j = 0, 1, \ldots r-1$.

We write these $q+r$ real numbers in a nonincreasing sequence $\gamma_1(x_1, x_2) \leqslant \ldots \leqslant \gamma_{q+r}(x_1, x_2)$. It is denoted $M_{qr} = \{\varphi | \varphi \subset \{1, 2, \ldots, q+r\}, |\varphi| = q\}$ and, for $\varphi \in M_{qr}$, one defines $s(\varphi) = \{(x_1, x_2) \in \sigma \times \tau | \forall p \in \varphi \rightarrow \gamma_p(x_1, x_2) \in \{\alpha_0(x_1), \ldots \alpha_{q-1}(x_1)\}\}$. It can be proved that $s(\varphi)$ is a simplex. Precisely, one shows that if $\varphi = \{j_1, \ldots, j_q\}$, with $j_1 < \ldots < j_q$, then considering the following $q+r+1$ points contained in $\mathbf{R}^{m+n}$ (the points of which are written in the form (x, y), with $x \in \mathbf{R}^m$, $y \in \mathbf{R}^n$)