By using the Lebesgue Theorem, we get

$$(\mathcal{T}\varphi)(\xi, x) = \int_{\mathcal{D}} A(\xi-\eta) \int_{\mathcal{E}} B(x, y)\varphi(\eta, y)\, dy\, d\eta = \int_{\mathcal{D}} A(\xi-\eta) \sum_{i=1}^{+\infty} \lambda_i \varphi_i(\eta) u_i(x)\, d\eta =$$

$$= \sum_{i=1}^{+\infty} (\lambda_i \int_{\mathcal{D}} A(\xi-\eta)\varphi_i(\eta)\, d\eta) u_i(x) = \sum_{i=1}^{+\infty} \varphi_i(\xi) u_i(x) = \varphi(\xi, x).$$

One can easily obtain the following result in the case of the space $C(\mathcal{D}, L^2(\mathcal{E}))$.

Proposition 3.3. *If* $f \in C(\mathcal{D}, L^2(\mathcal{E}))$ *and* $\sum_{i=1}^{+\infty} |\lambda_i| \, \|f_i\|_{C(\mathcal{D})} < +\infty$, *then* $\varphi(\xi, x)$ *given by* (3.16) *is a solution of* (*E*) *from* $C(\mathcal{D}, L^2(\mathcal{E}))$.

Proof. Using the relation

$$\|\varphi_i\|_{C(\mathcal{D})} \leq 2\|f_i\|_{C(\mathcal{D})}, \text{ for } i > k_0,$$

and the fact that φ_i is a solution of the *i*-equation of (Σ) we easily obtain the conclusion of Proposition 3.3.

Proposition 3.4. If $|\lambda_i| < \|A\|_1^{-1}$, $\forall i \in N$, *then the equation* (*E*) *has a unique solution in* $S^p(\Omega)$, $1 \leq p \leq +\infty$.

Proof. Using Lemma 3.1, we have

$$|\mathcal{T}\varphi|_x \leq \|\mathcal{B}\| \cdot \|A\|_1 |\varphi|_x.$$

But $\|\mathcal{B}\| = \sup\{|\lambda_i|;\ i \in N\} < \|A\|_1^{-1}$.

Therefore $\mathcal{T}$ is a contraction on $S^p(\Omega)$ and so for every $f \in L^p(\mathcal{D}, L^2(\mathcal{E}))$ $(C(\mathcal{D}, L^2(\mathcal{E})))$ we have a unique solution of (*E*) from $S^p(\Omega)$.

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A SIMPLICIAL PROOF OF THE KÜNNETH FORMULA FOR POLYHEDRA

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0. Introduction. Although the well-known Künneth exact sequence and the Künneth formula

$$H_n(X \times Y) \cong \bigoplus_{q+r=n} H_q(X) \otimes H_r(Y) \oplus \Big(\bigoplus_{q+r=n-1} H_q(X) * H_r(Y) \Big),$$

are essentially due to Herman Künneth [5], [6], in connection with the study of the Betti numbers and the torsion coefficients of a product of polyhedra, nevertheless in many books of algebraic topology these important relations are presented either in the framework of the singular homology, using the Eilenberg-Zilber theorem [7], [10], [11], or in the context of cellular homology (for X and Y CW-complexes) [8], or the background of the de Rham cohomology, using the Morse Theory or the cup-product (if X and Y are differentiable manifolds) [2], [4]. Even when X and Y are polyhedra, one begins by establishing the fomula for the singular homology and then one uses the isomorphism between the simplicial homology of a simplicial complex K and the singular homology of the space $|K|$, [3].

In this note, supposing X and Y being polyhedra, we give a direct simplicial proof of the Künneth formula, using the simplicial structure of the product $X \times Y$ and the calculus of the simplicial homology with a natural block dissection of the corresponding simplicial complex.

1. Some preliminaries regarding the triangulation of a product of polyhedra. For the data included in this paragraph we appeal essentially to [9, p. 109—111].

1.1. *The topological product of two simplexes* σ *and* τ *is a triagulated space.* If $\sigma = [a^0, a^1, \ldots, a^q] \subset \mathbf{R}^m$ and $\tau = [b^0, b^1, \ldots, b^q] \subset \mathbf{R}^n$, let $x_1 \in \sigma$, $x_1 = \Sigma \lambda_k(x_1) a^k$ and $x_2 \in \tau$, with $x_2 = \Sigma \mu_l(x_2) b^l$. One defines $\alpha_i(x_1) = \sum_{k=0}^{i} \lambda_k(x_1)$ and $\beta_j(x_2) = \sum_{l=0}^{j} \mu_l(x_2)$, $i = 0, 1, \ldots, q-1$; $j = 0, 1, \ldots r-1$.

We write these $q+r$ real numbers in a nonincreasing sequence $\gamma_1(x_1, x_2) \leq \ldots \leq \gamma_{q+r}(x_1, x_2)$. It is denoted $M_{qr} = \{\varphi | \varphi \subset \{1, 2, \ldots, q+r\}, |\varphi| = q\}$ and, for $\varphi \in M_{qr}$, one defines $s(\varphi) = \{(x_1, x_2) \in \sigma \times \tau \mid \forall p \in \varphi \rightarrow \gamma_p(x_1, x_2) \in \{\alpha_0(x_1), \ldots \alpha_{q-1}(x_1)\}\}$. It can be proved that $s(\varphi)$ is a simplex. Precisely, one shows that if $\varphi = \{j_1, \ldots, j_q\}$, with $j_1 < \ldots < j_q$, then considering the following $q+r+1$ points contained in $\mathbf{R}^{m+n}$ (the points of which are written in the form (x, y), with $x \in \mathbf{R}^m$, $y \in \mathbf{R}^n$)

$$
\begin{array}{ll}
 & (a^0, b^0), \ldots, (a^0, b^{j_1-1}) \\
 & (a^1, b^{j_1-1}), \ldots, (a^1, b^{j_2-2}) \\
(1) & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
 & (a^{q-1}, b^{j_{q-1}-(q-1)}), \ldots, (a^{q-1}, b^{j_q-q}) \\
 & (a^q, b^{j_q-q}), \ldots, (a^q, b^r)
\end{array}
$$

these points are independent and that simplex spanned by these vertices, denoted by $(\sigma \times \tau)(\varphi)$, agrees with the space $s(\varphi)$. In this way one obtains $\sigma \times \tau = \bigcup_{\varphi \in M_{qr}} (\sigma \times \tau)(\varphi)$ and it is proved that if $\varphi_1, \varphi_2 \in M_{qr}$, then $s(\varphi_1) \cap s(\varphi_2)$ agrees with the simplex spanned by the common vertices of the simplexes $s(\varphi_1)$ and $s(\varphi_2)$. It follows that simplexes $(\sigma \times \tau)(\varphi)$ together with their faces establish a triangulation of the space $\sigma \times \tau$.

1.2. *Remark.* If $\sigma' < \sigma$ and $\tau' < \tau$, then the simplexes of the form $(\sigma' \times \tau')(\psi)$ are simplexes of the above triangulation of the product $\sigma \times \tau$. But the converse is not generally true, as easily can be given simple counter examples.

1.3. *Let K and L be two simplicial complexes having the dimensions m and n respectively. Then the product $|K| \times |L|$ is a triangulated space of the dimension $m+n$.*

We have $|K| \times |L| = \bigcup_{\substack{\sigma \in K \\ \tau \in L}} \sigma \times \tau$. Applying 1.1, $\sigma \times \tau$ is a triangulated spaces and if $\sigma, \sigma' \in K$ and $\tau, \tau' \in L$, with $(\sigma \times \tau) \cap (\sigma' \times \tau') \neq \varnothing$, then $\sigma \cap \sigma' \neq \varnothing$ and $\tau \cap \tau' \neq \varnothing$, such that $(\sigma \times \tau) \cap (\sigma' \times \tau') = (\sigma \cap \sigma') \times (\tau \cap \tau')$ is a triangulated space. From this one deduces that the above decomposition of the product $|K| \times |L|$ defines a triangulation of this space. The dimension follows from 1.1. We will denote the corresponding simplicial complex by $K \times L$.

2. Block dissection of the simplicial complex $K \times L$ and some useful formulas on the incidence numbers of $K \times L$. Let $X = |K|$ and $Y = |L|$ be two simplicial complexes with $K \times L$ being the above considered simplicial complex, i.e. $X \times Y = |K \times L|$. We take a possible orientation for each of the complexes K and L. Let σ and τ be the simplexes considered in 1.1 and suppose that the fixed orientations of the complexes K and L are such that $+\sigma = \langle a^0, a^1, \ldots, a^q \rangle$ and $+\tau = \langle b^0, b^1, \ldots, b^r \rangle$ (here the notation $\langle , \rangle$ is for oriented simplexes). Now we choose for the simplicial complex $K \times L$ the orientation so that for every $\varphi \in M_{qr}$ the simplex $(\sigma \times \tau)(\varphi)$ and its faces have the orientations given by the order of the vertices in the table (1), reading left to right and top to bottom.

2.1. Lemma. *Let σ be an oriented q simplex of K and τ an oriented r-simplex of L. Let σ' be a $(q-1)$-face of σ, such that $(\sigma' \times \tau)(\varphi')$ is a face of the simplex $(\sigma \times \tau)(\varphi)$ and let τ' be a $(r-1)$-face of τ such that $(\sigma \times \tau')(\varphi'')$ is also a face of the same simplex $(\sigma \times \tau)(\varphi)$. Then the following relations hold :*

$$(-1)^{\varphi}[(\sigma \times \tau)(\varphi), (\sigma' \times \tau)(\varphi')] = (-1)^{\varphi'}[\sigma, \sigma'], \tag{2}$$

$$(-1)^{\varphi}[(\sigma \times \tau)(\varphi), (\sigma \times \tau')(\varphi'')] = (-1)^{q}(-1)^{\varphi''}[\tau, \tau'], \tag{3}$$

where the notation $(-1)^{\varphi}$, for $\varphi = \{j_1, \ldots, j_q\}$, means $(-1)^{j_1 + \ldots + j_q}$, with $(-1)^{\varphi} = -1$, if $\varphi = \varnothing$.

Proof. Let $+\sigma = \langle a^0, a^1, \ldots, a^q \rangle$, $+\sigma' = \langle a^0, a^1, \ldots, \hat{a}^h, \ldots, a^q \rangle$, $+\tau = \langle b^0, b^1, \ldots, b^r \rangle$, $+\tau' = \langle b^0, b^1, \ldots, \hat{b}^k, \ldots, b^r \rangle$. The cases $q=1$ for (2) and $q=0$ for (3) are evidently verified. That is why further on we suppose $q \geq 2$ for (2) and $q \geq 1$ for (3). Suppose that $\varphi' = \{j'_1, \ldots, j'_{q-1}\}$. Then, necessarily, $\varphi = \{j'_1, \ldots, j'_h - h, j'_{h+1}, \ldots, j'_{q-1}\}$ and we have $(-1)^{\varphi'}[\sigma, \sigma'] = (-1)^{h + j'_1 + j'_2 + \ldots + j'_{q-1}}$ and $(-1)^{\varphi}[(\sigma \times \tau)(\varphi), (\sigma' \times \tau)(\varphi')] = (-1)^{j'_1 + \ldots + j'_h + (j'_h - h) + j'_{h+1} + \ldots + j'_{q-1}} \cdot$

$$\cdot (-1)^{j'_1 + (j'_2 - j'_1) + \ldots + (j'_h - j'_{h-1})} = (-1)^{j'_1 + \ldots + j'_{q-1} + 2j'_h - h} = (-1)^{\varphi'}[\sigma, \sigma'].$$

Hence, the relation (2) is true. Now, let be $\varphi'' = \{j''_1, \ldots, j''_q\} \subset \{1, 2, \ldots, q+r-1\}$ and suppose that we have $j''_h - h \leq k \leq j''_{h+1} - (h+1)$. Then, $\varphi = \{j''_1, \ldots, j''_h, j''_{h+1} + 1, \ldots, j''_q + 1\}$ and we have: $(-1)^q \cdot (-1)^{\varphi''}[\tau, \tau'] = (-1)^{q + j''_1 + \ldots + j''_q + k}$ and

$$(-1)^{\varphi}[(\sigma \times \tau)(\varphi), (\sigma \times \tau')(\varphi'')] = (-1)^{j''_1 + \ldots + j''_h + (j''_{h+1} + 1) + \ldots + (j''_q + 1)} \cdot$$

$$\cdot (-1)^{j''_1 + (j''_2 - j''_1) + \ldots + (j''_h - j''_{h-1}) + (k - j''_h - h)} = (-1)^{j''_1 + \ldots + j''_q + (q-h)} \cdot (-1)^{k-h} =$$

$= (-1)^{j''_1 + \ldots + j''_q + q + k} = (-1)^q \cdot (-1)^{\varphi''}[\tau, \tau']$, so that the relation (3) is also verified.

2.2. Lemma. *If θ is a $(q+r-1)$-face of a simplex $(\sigma \times \tau)(\varphi)$, but not being either of the form $(\sigma' \times \tau)(\varphi'')$ or of the form $(\sigma \times \tau')(\psi'')$, then there exists $\psi \in M_{qr}$, $\psi \neq \varphi$, such that θ is also a face of the simplex $(\sigma \times \tau)(\psi)$ and moreover the following relation holds :*

$$(-1)^{\varphi}[(\sigma \times \tau)(\varphi), \theta] + (-1)^{\psi}[(\sigma \times \tau)(\psi), \theta] = 0. \tag{4}$$

Proof. It is clear that the simplex θ can be obtained from $(\sigma \times \tau)(\varphi)$, whose vertices we suppose to be those in the table (1), if and only if one omits a vertex of the form $(a^{h-1}, b^{j_h - h})$ or of the form $(a^h, b^{j_h - h})$, conditionated by the inequalities $j_h - j_{h-1} > 1$ and $j_{h+1} - j_h > 1$ respectively. Suppose that θ is obtained by omitting the vertex $(a^{h-1}, b^{j_h - h})$. Then, we can consider the set $\psi = \{j_1, \ldots, j_{h-1}, j_h - 1, j_{h+1}, \ldots, j_q\}$ and it is immediately deduced that the simplex θ is also a face of $(\sigma \times \tau)(\psi)$. Moreover $[(\sigma \times \tau)(\psi), \theta] = [(\sigma \times \tau)(\varphi), \theta)]$, while $(-1)^{\varphi} = -(-1)^{\psi}$. Analogously if θ is obtained by omitting the vertex $(a^h, b^{j_h - h})$ then $\psi = \{j_1, \ldots, j_{h-1}, j_h + 1, j_{h+1}, \ldots, j_q\}$ and the previous relations still hold.

2.3. Lemma. *If $\sigma \in K$ and $\tau \in L$ and we denote by $\overline{(\sigma \times \tau)}$ the boundary of the product $\sigma \times \tau$, with the induced triangulation, then the pair $(\sigma \times \tau, \overline{(\sigma \times \tau)})$ is a simplicial block of the complex $K \times L$. Moreover, when σ and τ cover the complexes K and L respectively, then one obtains a block dissection of the complex $K \times L$.*

Proof. It is clear that $\sigma \times \tau$ is a subcomplex of the simplicial complex $K \times L$ and $\overline{(\sigma \times \tau)}$ is a subcomplex of $\sigma \times \tau$. Then, applying [8, Th. 4.4.19] it follows that the pair $(\sigma \times \tau, \overline{(\sigma \times \tau)})$ is a $(q+r)$-block in the simplicial complex $K \times L$ (if $\dim \sigma = q$ and $\dim \tau = r$). Now let θ be a simplex of the complex $K \times L$. If $\theta = (\sigma \times \tau)(\varphi)$, then $\dim \theta = \dim \sigma + \dim \tau$ and $\theta \in (\sigma \times \tau) - \overline{(\sigma \times \tau)}$.

But if we suppose that 0 is not of this form, then according to Lemma 2.2, 0 is a common face of two simplexes $(\sigma\times\tau)(\varphi)$ and $(\sigma\times\tau)(\psi)$. Thus, every simplex of the complex $K\times L$ is in the interior of just one block. Then it is clear that the boundary $\overline{(\sigma\times\tau)}=\dot\sigma\times\tau\cup\sigma\times\dot\tau$ is in the union of the m blocks, for $m<\dim(\sigma\times\tau)$. In this way, all conditions of the definition of a block dissection [8, p. 134] are verified.

3. Proof of the Künneth formula. Let M_n be the block n-skeleton in the block dissection of the complex $K\times L$, given by Lemma 2.3. Hence $M_n=\bigcup_{\dim\sigma+\dim\tau\leqslant n}\sigma\times\tau$. We can associate to this block dissection of the complex $K\times L$ a chain complex (C^b, d^b), with $C^b=\bigoplus_n H_n(M_n, M_{n-1})$ and

$d^b=j_*\partial_*: H_n(M_n, M_{n-1})\xrightarrow{\partial_*}H_{n-1}(M_{n-1})\xrightarrow{j_*}H_{n-1}(M_{n-1}, M_{n-2})$ [8, p. 132].

3.1. Theorem. *The chain complexes* (C^b, d^b) *and* $(C(K)\otimes C(L), d'')$ *are chain isomorphic.*

Proof. Here, the second chain complex is the tensorial product of the simplicial chain complexes $C(K)$ and $C(L)$ [10, Ch. 5, § 3]. According to [8, Th. 4.4.20], $H_n(M_n, M_{n-1})$ is the free abelian group with generators in (1-1)-correspondence with the n-blocks $\sigma\times\tau$, for $\sigma\in K$ and $\tau\in L$. Then it is clear that we can define an isomorphism $\varphi_n: H_n(M_n, M_{n-1})\longrightarrow\bigoplus_{q+r=n}C_q(K)\otimes C_r(L)$, by $\varphi_n(\sigma\times\tau)=\sigma\otimes\tau$. We will show that $\bigoplus_n\varphi_n$ is a chain isomorphism. For this it is sufficient to prove the commutativity of the following diagram

$$\begin{array}{ccc} H_n(M_n, M_{n-1}) & \xrightarrow{\varphi_n} & \bigoplus_{q+r=n} C_q(K)\otimes C_r(L) \\ \Big\downarrow d^b & & \Big\downarrow d'' \\ H_{n-1}(M_{n-1}, M_{n-2}) & \xrightarrow{\varphi_{n-1}} & \bigoplus_{q'+r'=n-1} C_{q'}(K)\otimes C_{r'}(L) \end{array}$$

For $\sigma\in K$ and $\tau\in L$, with $\dim\sigma=q$, we have

(6) $$d''\varphi_n(\sigma\times\tau)=d''(\sigma\otimes\tau)=d\sigma\otimes\tau+(-1)^q\sigma\otimes d\tau.$$

On the other hand, according to the proof of the isomorphism

(7) $$H_n(C^b, d^b)\cong H_n(K\times L). \text{ [8, Th. 4.4.14]},$$

$H_n(M_n, M_{n-1})$ can be identified with the group of n-cycles $Z_n(C(M_n, M_{n-1}))\subset C_n(M_n, M_{n-1})\subset C_n(K\times L)$. If for σ, τ as above, we consider a chain $z=\sum_\varphi(-1)^\varphi(\sigma\times\tau)(\varphi)$, then by Lemma 2.2, this is a relative cycle in $Z_n(C(M_n, M_{n-1}))$, which can be identified with $\sigma\times\tau$. Also by the proof of the same isomorphism (7), we can write: $d^b(\sigma\times\tau)=dz=d(\sum_\varphi(-1)^\varphi(\sigma\times\tau)(\varphi))=\sum_\varphi(-1)^\varphi\{\sum_{\sigma',\varphi'}[(\sigma\times\tau)(\varphi), (\sigma'\times\tau)(\varphi')](\sigma'\times\tau)(\varphi')+\sum_{\tau',\varphi''}[(\sigma\times\tau)(\varphi), (\sigma\times\tau')(\varphi'')](\sigma\times\tau')(\varphi'')+\sum_0[(\sigma\times\tau)(\varphi), 0]0\}$,

where 0 is not of the form $(\sigma'\times\tau)(\varphi')$ neither of the form $(\sigma\times\tau')(\varphi'')$. Applying Lemmas 2.1 and 2.2, we deduce: $d^b(\sigma\times\tau)=\sum_{\sigma',\varphi'}(-1)^{\varphi'}[\sigma,\sigma'](\sigma'\times\tau)(\varphi')+$ $+\sum_{\tau',\varphi''}(-1)^q(-1)^{\varphi''}[\tau, \tau'](\sigma\times\tau')(\varphi'')=\sum_{\sigma'}[\sigma, \sigma'](\sigma'\times\sigma)+(-1)^q\sum_{\tau'}[\tau, \tau'](\sigma\times\tau')=$ $=(\sum_{\sigma'}[\sigma, \sigma']\sigma')\times\tau+(-1)^q\sigma\times(\sum_{\tau'}[\tau, \tau']\tau')=d\sigma\times\tau+(-1)^q\sigma\times d\tau.$

Therefore

(8) $$\varphi_{n-1}d^b(\sigma\times\tau)=d\sigma\times\tau+(-1)^q\sigma\times d\tau.$$

By comparing the relations (6) and (8) we deduce the commutativity of the diagram (5), which finishes the proof of the theorem.

3.2. Corollary. *Let* K *and* L *be two simplicial complexes. Then for every* $n\geqslant 0$ *the Künneth isomorphism holds:*

$$H_n(|K|\times|L|)\cong\bigoplus_{q+r=n}H_q(|K|)\otimes H_r(|L|)\oplus(\bigoplus_{q+r=n-1}H_q(|K|)*H_r(|L|)).$$

Proof. We have the following sequence of isomorphismes: $H_n(|K|\times|L|)\cong$ $\cong H_n(K\times L)\cong H_n(C^b, d^b)\cong H_n(C(K)\otimes C(L))$. Further on we apply the Künneth split exact sequence for abstract chain complexes of free abelian groups [10, Lemma 1, Ch. 4, §3] and the isomorphismes $H_*(K)\cong H_*(|K|)$ and $H_*(L)\cong$ $\cong H_*(|L|)$.

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