An. şt. Univ. „Al. I. Cuza" Iaşi
Matematica, 35 (1989)

HAMILTON GEOMETRY

BY

RADU MIRON

A Lagrange space $L^n=(M, L)$ is a C^∞-real n-dimensional manifold M endowed with a Lagrangian $L: TM \to R$ for which the d-tensor field $g_{ij}(x, y)= \frac{1}{2}\partial^2 L/\partial y^i \partial y^j$, $(i, j, k,..=1,..., n)$, is nondegenerate. One obtains a natural extension (see [11, 12, 13]) by defining the generalized Lagrange space as a pair $M^n=(M, g_{ij}(x, y))$. As the Riemann space V^n and the Finsler spaces F^n are particularly M^n-spaces, we have the inclusions: $\{V^n\}\subset\{F^n\}\subset\{L^n\}\subset\{M^n\}$.

The geometry of the Lagrange spaces plays an important role in the construction of a Lagrangian model of general relativity (see [12]), as well as in solving of some problems in variational calculus (see [17]). The following problem arises naturally: what corresponds to the notion of Lagrange space in the duality between tangent cotangent bundles ?

This paper gives an answer to this question: this is the Hamilton space. A Hamilton space is a pair $H^n=(M, H)$, where $H: T^*M \to R$ is a Hamilton function for which the d-tensor field $g^{ij}(x, p)= \frac{1}{2}\partial^2 H/\partial p_i \partial p_j$ is nondegenerate. Here T^*M is the total space of the cotangent bundle (T^*M, π^*, M). If the Hamiltonian $H(x, p)$ is 2-homogeneous with respect to p_i, then H^n is named a Cartan space $\mathcal{C}^n$ (see [3, 5, 9, 19]). More general, a pair $M^{*n}=(M, g^{ij}(x, p))$, where $g^{ij}(x, p)$ is a d-tensor field symmetric and nondegenerate, will be called a generalized Hamilton space. We shall prove that the following inclusions

$$\{V^n\}\subset\{\mathcal{C}^n\}\subset\{H^n\}\subset\{M^{*n}\}$$

are true.

It is obvious that the geometry of the Hamilton spaces is interesting for its applications in Theoretical Physics and in Variational Calculus. By our knowledge, the spaces H^n and M^{*n} were not defined until now. In the theory of the Cartan spaces some important facts were discovered by Cartan, [5]. But our point of view on these spaces leads to results which are completly new. We are treating the geometry of the Hamilton spaces by using methods of the geometry of the Lagrange spaces and of the geometry of the total space of a vector bundle (see [11, 14]).

The paper is organized as follows. In the first part we define the notion of nonlinear connection N on T^*M, we introduce the almost symplectic structure θ associated to N as well as the canonical Liouville 1-form $\tilde{p}$, and we prove that $d\tilde{p}=\theta+\eta$. If N is a symmetric nonlinear connection it follows $\eta=0$ and then θ is a symplectic structure. Then, we define the d-connections $H\Gamma(N)$, the h- and v-covariant derivatives associated to $H\Gamma(N)$ and we esta-

blish the Ricci formulas and the Bianchi identities. Finally, we study the paralle-
lism of vectors and the transformations of d-connections. The structure of
the group of transformations of d-connections is studied too. In the second
part we introduce the notion of the generalized Hamilton space, of metrical
d-connections and that of d-connections of Berwald, Rund or Hashiguchi
type. The duality between the spaces L^n and H^n allows us to find the canonical
nonlinear connection $\overset{c}{N}$ of the Hamilton space H^n and the invariant
form of the system of Hamilton equations. For the Cartan spaces a canonical
connection which is dual to the Cartan connection from Finsler spaces is de-
termined. By using the canonical connection $C\Gamma$, some remarkable properties of
the Cartan spaces $\mathcal{C}^n$ are established. Then the spaces M^{*n} which are weakly
regular or regular or locally Minkowski type are investigated. In order to
illustrate the efficiency of the these geometrical models the examples of Pyra-
gas and Alexandrov [15, 16] from the Hamiltonian theory of physical fields
are revisited. The third part contains the almost Hermitian model H^{2n} of the
Hamilton space H^n or of the generalized Hamilton space M^{*n}. We prove that
H^{2n} is an almost Kählerian space K^{2n} for the Hamilton or Cartan spaces.

Some special results from the theory of the Hamilton and Cartan spaces will be published in the Journals Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences (Paris) and Progress of Mathematics (Varanasi).

Part I. REMARKABLE GEOMETRICAL OBJECTS ON T*M.

The fields of the distinguished geometrical objects on a differentiable manifold give a natural framework for the geometry of Hamilton spaces. These allow us to define the efficient operators of covariant derivation and to introduce the fundamental concepts of this geometry.

§ 1. Nonlinear connection on T*M. Let M be a C^∞-differentiable real n-dimensional manifold and $\pi^*: T^*M \to M$ its cotangent bundle. If (U, φ) is a local chart on M then the coordinates of a point $u \in \pi^{*-1}(U) \subset T^*M$ will be denoted by (x^i, p_i), $(i, j, k \ldots = 1, \ldots, n)$.

A local coordinate transformation on T^*M is given by

$$\tilde{x}^i = \tilde{x}^i(x^1, \ldots, x^n), \ \det \|\partial\tilde{x}^i/\partial x^j\| \neq 0, \qquad (1.1)$$
$$\bar{p}_i = \partial x^j/\partial\bar{x}^i p_j.$$

Denoting by $\partial_i = \partial/\partial x^i$, $\dot\partial^i = \partial/\partial p_i$ the local natural basis of the module of vector fields $\mathcal{X}(T^*M)$, by using (1.1) one obtains:

$$\bar{\partial}_i = \partial x^j/\partial\bar{x}^i \partial_j - p_k \partial^2 x^k/\partial\bar{x}^i \partial\bar{x}^j \partial\bar{x}^j/\partial\bar{x}^h \dot\partial^h, \qquad \bar{\dot\partial}^i = \partial\bar{x}^i/\partial x^j \dot\partial^j, \qquad (1.2)$$

$$d\bar{x}^i = \partial\bar{x}^i/\partial x^j dx^j, \quad d\bar{p}_i = p_k \partial^2 x^k/\partial\bar{x}^i \partial\bar{x}^j \partial\bar{x}^j/\partial x^h dx^h + \partial x^j/\partial\bar{x}^i dp_j. \qquad (1.2)'$$

From (1.1) and (1.2)' it follows:

Proposition 1.1. *The field of 1-forms*

$$\tilde{p} = p_i dx^i \qquad (1.3)$$

is global defined on T^*M.

$\tilde{p}$ is called the fundamental 1-form.

The kernel of the differential of the projection $\pi^*: T^*M \to M$ will be called the vertical subbundle VT^*M of the tangent bundle TT^*M. Associating to each point $u \in T^*M$ the fiber V_u of VT^*M we obtain a distribution of the class C^∞, $V: u \in T^*M \to V_u \subset T_u T^*M$ whose local dimension is n. The *vertical* distribution V is integrable, $\{\dot\partial^i\}$ being a local basis of it.

Definition 1.1. *A nonlinear connection on T^*M is a distribution of the class C^∞, $N: u \in T^*M \to N_u \subset T_u T^*M$, which is supplementary to the vertical distribution V, i.e. $T_u T^*M = N_u \oplus V_u$, $\forall u \in T^*M$.*

In the following the point $u \in T^*M$ for which $x = \pi^*(u)$ will be denoted by (x, p) and its local coordinates by (x^i, p_i).

A nonlinear connection N on T^*M determines a subbundle HT^*M of TT^*M, such that we have the Whitney sum

$$TT^*M = HT^*M \oplus VT^*M$$

Sometimes N will be called a horizontal distribution on T^*M.

Let $\delta/\delta x^i = \delta_i$ be given by

$$\delta_i = \partial_i + N_{ij}(x, p)\dot\partial^j, \quad (i = \overline{1, n}) \qquad (1.4)$$

the adapted local basis of the horizontal distribution N. The functions $N_{ij}(x, p)$ are well determined on $\pi^{*-1}(U)$. They are called the „coefficients" of the nonlinear connection N.

Proposition 1.2. *With respect to the transformation* (1.1) *the coefficients $N_{ij}(x, p)$ of a nonlinear connection N have the following rule of transformation*

$$\overline{N}_{ij}(\bar{x}, \bar{p}) = \frac{\partial x^h}{\partial\bar{x}^i} \frac{\partial x^k}{\partial\bar{x}^j} N_{hk}(x, p) + p_k \frac{\partial^2 x^k}{\partial\bar{x}^i \partial\bar{x}^j}. \qquad (1.5)$$

Conversely, a system of the function $N_{ij}(x, p)$, defined on each domain of local chart from T^*M, which verifies (1.5) *with respect to* (1.1), *determines a nonlinear connection N on T^*M.*

The proof is immediate, if we remark that (1.4) and (1.2) imply (1.5).

Now, we note that $(\delta_1, \ldots, \delta_n, \dot\partial^1, \ldots, \dot\partial^n)$ is a local basis adapted to the supplementary distributions N and V. By virtue of (1.1) it follows

$$\bar{\delta}_i = \partial x^h/\partial\bar{x}^i \delta_h, \quad \bar{\dot\partial}^i = \partial\bar{x}^i/\partial x^h \dot\partial^h. \qquad (1.6)$$

The dual basis $(dx^i, \delta p_i)$, where

$$\delta p_i = dp_i - N_{ji}(x, p)dx^j, \qquad (1.4)'$$

satisfies

$$d\bar{x}^i = \partial\bar{x}^i/\partial x^j dx^j, \quad \delta\bar{p}_i = \partial x^j/\partial\bar{x}^i \delta p_j. \qquad (1.6)'$$

By applying the general results from [11, 14] one obtains:

Proposition 1.3. *If M is a paracompact differentiable manifold, then there exist nonlinear connections on T^*M.*

§ 2. Distinguished geometrical objects on M. A distinguished geometrical object field on M, shortly a „d-field", is given by its components $\Omega^{\wedge}(x, p)$, on each domain of local chart of T^*M, such that with respect to (1.1) we have

$$\overline{\Omega}^{\wedge}(x, p)=\mathcal{F}^{\wedge}(\Omega^{\Sigma}(x, p)\ ;\ \bar{x}^i, \partial_j\bar{x}^i,\ldots,\partial_{j_1}\ldots_{j_k}\bar{x}^i)$$

in which $\mathcal{F}^{\wedge}$ has the known classical properties.

Note that in $\mathcal{F}^{\wedge}$ do not enter explicitly p_i and $\bar{p}_i$. For example, a d-tensor field K of the type (1,1) on M is given by its components $K^i_j(x, p)$, $\forall(x, p)\in\pi^{*-1}(U)$, having the property:

$$\overline{K}^i_j(\bar{x}, \bar{p})=\frac{\partial\bar{x}^i}{\partial x^h}\frac{\partial x^k}{\partial\bar{x}^j}K^h_k(x, p). \tag{2.1}$$

The coefficients $N_{ij}(x, p)$ of a nonlinear connection N do not define a d-geometrical object field on M. But the functions

$$\tau_{ij}=N_{ij}-N_{ji} \tag{2.2}$$

define a d-tensor field of the type (0,2) on M. If $\tau_{ij}=0$ then N will be called a symmetric nonlinear connection. Also, let us consider

$$R_{ijk}=\delta_i N_{jk}-\delta_j N_{ik}. \tag{2.3}$$

A direct calculus leads to:

Proposition 2.1. 1°. *$\tau_{ij}(x, p)$ is an antisymmetric d-tensor field on M.*

2° *R_{ijk} is a d-tensor field on M of the type (0,3), antisymmetric with respect to the first two indices.*

3° *We have the brackets*

$$[\delta_i, \delta_j]=R_{ijk}\dot{\partial}^k,\ [\delta_i, \dot{\partial}^j]=-\dot{\partial}^j N_{ih}\dot{\partial}^h,\ [\dot{\partial}^i, \dot{\partial}^j]=0. \tag{2.4}$$

Theorem 2.1. *The horizontal distribution N on T^*M is integrable if and only if $R_{ijk}=0$.*

For a nonlinear connection N we consider the fields of 2-forms:

$$\eta=\frac{1}{2}\tau_{ij}dx^i\wedge dx^j, \tag{2.5}$$

$$\theta=\delta p_i\wedge dx^i. \tag{2.6}$$

Theorem 2.2. 1°. *The fields of 2-forms θ and η are global defined on T^*M.*

2°. *θ is an almost symplectic structure on T^*M.*

Proof. The tensorial character of τ_{ij} and (1.6)′ lead us to 1°. 2° is a consequence of the:

Proposition 2.2. *The associated tensor field of the 2-form θ is $\tilde{\theta}$, expressed in the adapted basis $(\delta_i, \dot{\partial}^i)$ by*

$$\tilde{\theta}(\delta_i, \delta_j)=0,\ \ \tilde{\theta}(\delta_i, \dot{\partial}^j)=-\delta^j_i,\ \ \tilde{\theta}(\dot{\partial}^j, \delta_i)=\delta^j_i,\ \ \tilde{\theta}(\dot{\partial}^i, \dot{\partial}^j)=0. \tag{2.6'}$$

θ will be called the associate almost symplectic structure of the nonlinear connection N.

To find a characterization of the integrability of θ we prove first:

Proposition 2.3. *The exterior differentials of the forms $\bar{p}$, η and θ verify the relations*

$$d\bar{p}=\theta+\eta. \tag{2.7}$$

$$d\theta=-d\eta=-1/6(R_{ijk}+R_{jki}+R_{kij})\,dx^i\wedge dx^j\wedge dx^k-\dot{\partial}^k\tau_{ij}\delta p_k\wedge dx^i\wedge dx^j. \tag{2.8}$$

Proof. We note that, for a function $f: T^*M\to R$, we have $df=\delta_k f dx^k+\dot{\partial}^k f\delta p_k$. Therefore (2.5), (2.6) and $\bar{p}=p_i dx^i$ give us (2.7) and (2.8).

Easy consequences of this theorem are the following two theorems:

Theorem 2.3. *If the nonlinear connection N is symmetric, then 1°. θ is integrable;*

2°. $R_{ijk}+R_{jki}+R_{kij}=0$.

Theorem 2.4. *The almost symplectic structure θ, associated to the nonlinear connection N, is integrable if and only if $\dot{\partial}^k\tau_{ij}=0$ and $R_{ijk}+R_{jki}+R_{kij}=0$.*

§ 3. d-Connections. We define a d-connection on the manifold base M as a pair of d-geometrical special objects on M having remarkable geometrical meanings as we shall show in the last part of the paper.

Let N be a fixed nonlinear connection on T^*M.

Definition 3.1. *It is called a d-connection on M a pair of d-geometrical object fields on M, $H\Gamma(N)=(H^i_{jk}(x, p), C_i^{jk}(x, p))$, where C_i^{jk} is a d-tensor field of the type (2, 1) and H^i_{jk} is a d-geometric object field with the law of transformation:*

$$\overline{H}^i_{jk}(\bar{x}, \bar{p})=\frac{\partial\bar{x}^i}{\partial x^h}\frac{\partial x^r}{\partial\bar{x}^j}\frac{\partial x^s}{\partial x^k}H^h_{rs}(x, p)+\frac{\partial\bar{x}^i}{\partial x^h}\frac{\partial^2 x^h}{\partial\bar{x}^j\partial\bar{x}^k}. \tag{3.1}$$

By using the proposition 1.3, the formulae (1.5) and (1.3) we get:

Proposition 3.1. *If $H\Gamma(N)=(H^i_{jk}, C_i^{jk})$ is a d-connection on M, then $N_{kj}=p_i H^i_{kj}$ are the coefficients of a nonlinear connection on T^*M.*

Proposition 3.2. *If $N(N_{jk})$ is a nonlinear connection on T^*M, then $B\Gamma(N)=(\dot{\partial}^i N_{kj}, 0)$ is a d-connection on M.*

Therefore, we have:

Theorem 3.1. *If M is a paracompact differentiable manifold, then there exist a d-connection on M.*

Now, we shall define h- and v-covariant derivatives of d-tensor fields on M with respect to a d-connection $H\Gamma(N)$, by

$$K^i_{j|h}=\delta_h K^i_j+K^r_j H^i_{rh}-K^i_r H^r_{jh}, \tag{3.2}$$

$$K^i_j|^h=\dot{\partial}^h K^i_j+K^r_j C_r^{ih}-K^i_r C_j^{rh} \tag{3.3}$$

and similarly for every d-tensor field on M.

From (1.6), (2.1) and (3.1), it follows that $K^i_{j|h}$ is a d-tensor field of the type (1,2) and $K^i_j|^h$ is a d-tensor field of the type (2,1) on M. Also it results hat h- and v-covariant derivatives determine each of them an algorithm of covariant derivative in the algebra of the d-tensor fields on M.

Now, we can define the covariant differential of $K^i_j(x, p)$ as :

$$\nabla K^i_j = K^i_{j|h} dx^h + K^i_j|^h \delta p_h. \tag{3.4}$$

Obviously, ∇ preserves the type of the d-tensor fields and it has all properties of a covariant differential operator.

Denoting

$$\omega^i_j = H^i_{jh} dx^h + C_j{}^{ih} \delta p_h, \tag{3.4)'$$

it follows :

$$\nabla K^i_j = dK^i_j + K^r_j \omega^i_r - K^i_r \omega^r_j. \tag{3.4)''$$

As an application, we shall consider the *d-fundamental covector field* p_i. Its h- and v-covariant derivatives are given by

$$p_{i|h} = N_{hi} - H^0_{ih}, \quad p_i|^h = \delta^h_i - C_i{}^{0h}, \tag{3.5}$$

and ∇p_i is

$$\nabla p_i = p_{i|h} dx^h + p_i|^h \delta p_h.$$

Throughout this paper we denote the contraction with p_i by the index „0", as in (3.5).

The d-tensor field

$$D_{ih} = p_{i|h} = N_{hi} - H^0_{ih}, \tag{3.5)'$$

is called the deflexion tensor field of the d-connection $H\Gamma(N)$.

The d-connection $H\Gamma(N)$ is called of Cartan type if $p_{i|h} = 0$, $p_i|^h = \delta^h_i$.

Proposition 3.3. 1°. *$H\Gamma(N)$ is of Cartan type if and only if*

$$N_{hi} = H^0_{ih}, \quad C_j{}^{0h} = 0. \tag{3.5)''$$

2°. *If $H\Gamma(N)$ is of Cartan type, then the covariant differential ∇p_i is*

$$\nabla p_i = \delta p_i. \tag{3.5)'''$$

Proof. 1°. $p_{i|h} = 0$ and $p_i|^h = \delta^h_i$ are equivalent to (3.5)''. 2°. In these conditions $\nabla p_i = \delta^h_i \delta p_h = \delta p_i$.

The formulae of commutation of h- and v-second covariant derivatives introduce d-torsion tensor fields of the d-connection $H\Gamma(N)$:

$$T_{jk}{}^i = H^i_{jk} - H^i_{kj}, \quad S_i{}^{jk} = -(C_i{}^{jk} - C_i{}^{kj}), \quad P_{jk}{}^i = -(\dot{\partial}^i N_{kj} - H^i_{jk}), \quad C_i{}^{jk}, \quad R_{ijk}, \tag{3.6}$$

as well as d-curvature tensor fields of d-connection $H\Gamma(N)$:

$$\begin{aligned} R_j{}^i{}_{hl} &= \delta_l H^i_{jh} - \delta_h H^i_{jl} + H^r_{jh} H^i_{rl} - H^r_{jl} H^i_{rh} + C_j{}^{ir} R_{hlr}, \\ P_j{}^i{}_h{}^l &= \dot{\partial}^l H^i_{jh} - C_j{}^{il}|_h + C_j^{ir} P_{rh}{}^l, \\ S_j{}^{ihl} &= \dot{\partial}^l C_j{}^{ih} - \dot{\partial}^h C_j{}^{il} + C_j{}^{rh} C_r{}^{il} - C_j{}^{rl} C_r{}^{ih}. \end{aligned} \tag{3.7}$$

Theorem 3.1. *The h- and v-covariant derivatives with respect to $H\Gamma(N)$ verify the Ricci formulae :*

$$\begin{aligned} K^i_{j|h|l} - K^i_{j|l|h} &= K^r_j R_r{}^i{}_{hl} - K^i_r R_j{}^r{}_{hl} - T_{hl}{}^r K^i_{j|r} - R_{hlr} K^i_j|^r, \\ K^i_{j|h}|^l - K^i_j|^l{}_{|h} &= K^r_j P_r{}^i{}_h{}^l - K^i_r P_j{}^r{}_h{}^l - C_h{}^{rl} K^i_{j|r} - P_{rh}{}^l K^i_j|^r, \\ K^i_j|^h|^l - K^i_j|^l|^h &= K^r_j S_r{}^{ihl} - K^i_r S_j{}^{rhl} - S_r{}^{hl} K^i_j|^r. \end{aligned} \tag{3.8}$$

Proof. Using (2.4), (3.2), (3.3) and the expressions (3.6), (3.7) of d-tensors of torsion and curvature, by a straightforward calculus we get (3.8).

As an application we have :

Proposition 3.4. *If $H\Gamma(N)$ is a d-connection of Cartan type then*

$$R_{hli} + R_i{}^0{}_{hl} = 0, \quad P_{ih}{}^l + P_i{}^0{}_h{}^l = 0, \quad S_i{}^{hl} + S_i{}^{0hl} = 0. \tag{3.9}$$

Indeed, applying the Ricci formulae to the d-fundamental covector field p_i and using the equations $p_{i|k} = 0$, $p_i|^k = \delta^k_i$, we obtain (3.9).

Proposition 3.5. *If $H\Gamma(N)$ has the properties*

$$p_{i|j} = D_{ij}, \quad p_i|^j = \delta^j_i,$$

where the deflexion tensor D_{ij} is given, then we have

1° $\nabla p_i = D_{ij} dx^j + \delta p_i$;

2° $$\begin{aligned} D_{jh|l} - D_{jl|h} &= -(R_j{}^0{}_{hl} + R_{hlj} + T_{hl}{}^r D_{jr}), \\ D_{jh}|^l &= -(P_j{}^0{}_h{}^l + P_{jh}{}^l + C_h{}^{rl} D_{jr}), \\ 0 &= S_j{}^{0kl} + S_j{}^{kl}. \end{aligned} \tag{3.10}$$

For the proof it is sufficient to apply again (3.4) and (3.8).

Remark. We will use the previous formulae (3.8) in the theory of physical fields, where the deflection tensor D_{ij} will be expressed by the electromagnetic field.

§ 4. Bianchi identities. Each d-connection $H\Gamma(N)$ satisfies a number of eleven Bianchi identities. To deduce these important identities we consider the connexion forms (3.4)' and the *structure equations* of $H\Gamma(N)$

$$d(dx^i) - dx^h \wedge \omega^i_h = -\Omega^i, \quad d(\delta p_i) + \delta p_h \wedge \omega^h_i = -\Omega_i, \quad d\omega^i_j - \omega^h_j \wedge \omega^i_h = -\Omega^i_j. \tag{4.1}$$

The exterior differential of 1-forms δp_i is given by

$$d(\delta p_i) = -\frac{1}{2} R_{hki} dx^h \wedge dx^k - \dot{\partial}^h N_{kj} \delta p_h \wedge dx^k. \tag{4.2}$$

Therefore, (3.4)', (3.6) and (3.7) imply

$$\begin{aligned} \Omega^i &= 1/2\, T_{jk}{}^i dx^j \wedge dx^k + C_j{}^{ik} dx^j \wedge \delta p_k, \\ \Omega_i &= 1/2\, R_{jki} dx^j \wedge dx^k + P_{ij}{}^k dx^j \wedge \delta p_k + 1/2\, S_i^{jk} \delta p_j \wedge \delta p_k, \\ \Omega^i_j &= 1/2\, R_j{}^i{}_{hk} dx^h \wedge dx^k + P_j{}^i{}_h{}^k dx^h \wedge \delta p_k + 1/2\, S_j{}^{ihk} \delta p_h \wedge \delta p_k. \end{aligned} \tag{4.3}$$

Using the symbol $\underset{(i,j,k)}{\mathcal{S}}$ for the cyclic sum and $\underset{(i,j)}{Q}$ for the alternate sum and differentiating (4.1) and (4.3), we get :

Theorem 4.1. *Each d-connection $H\Gamma(N)=(H^i_{jk}, C_i^{jk})$ verifies the Bianchi identities*

$$\underset{(h,r,s)}{\mathcal{S}}\{T_{hr|s}^{i}+T_{hl}^{\ i}T_{rs}^{\ l}-R_{h\ rs}^{\ i}+R_{hrl}C_s^{\ il}\}=0,$$

$$\text{(4.4)}\qquad \underset{(h,r,s)}{\mathcal{S}}\{S_k^{\ hr}|^s+S_k^{\ hl}S_l^{\ rs}+S_k^{\ hrs}\}=0,$$

$$\underset{(h,r,s)}{\mathcal{S}}\{R_{hrj|s}+T_{hr}^{\ l}R_{slj}+R_{hrl}P_{j\ s}^{\ l}\}=0,$$

$$\underset{(h,r,s)}{\mathcal{S}}\{R_{j\ hr|s}^{\ i}+R_{j\ hl}^{\ i}T_{rs}^{\ l}+P_{j\ h}^{\ i\ l}R_{rsl}\}=0,$$

$$\text{(4.5)}\qquad \underset{(h,r,s)}{\mathcal{S}}\{S_j^{\ ihr}|^s+S_j^{\ ihl}S_l^{\ rs}\}=0,$$

$$\underset{(r,s)}{\mathcal{Q}}\{P_{h\ s}^{\ i\ r}+C_h^{\ lr}T_{sl}^{\ i}+C_h^{\ ir}{}_{|s}-C_h^{\ il}P_{ls}^{\ r}\}-T_{hs}^{\ i}|^r+T_{hs}^{\ l}C_l^{\ ir}=0,$$

$$\text{(4.6)}\qquad S_h^{\ irs}-\underset{(r,s)}{\mathcal{Q}}\{\dot\partial^s C_h^{\ ir}+C_h^{\ lr}C_l^{\ is}\}=0.$$

$$\text{(4.7)}\qquad R_{k\ rs}^{\ h}+R_{rsk}|^h-S_k^{\ lr}R_{rsl}-P_{kl}^{\ h}T_{rs}^{\ l}+\underset{(r,s)}{\mathcal{Q}}\{R_{lsk}C_r^{\ lh}-P_{kr\ |s}^{\ h}-P_{lr}^{\ h}P_{ks}^{\ l}\}=0,$$

$$\underset{(h,r)}{\mathcal{Q}}\{P_{k\ s}^{\ h\ r}-P_{ks}^{\ h}|^r-P_{kl}^{\ h}C_s^{\ lr}-S_k^{\ lh}P_{ls}^{\ r}\}+P_{ks}^{\ l}S_l^{\ hr}-S_k^{\ hr}{}_{|s}=0.$$

$$R_{j\ rs}^{\ i}|^k-P_{j\ l}^{\ i\ k}T_{rs}^{\ l}+S_j^{\ ikl}R_{rsl}+\underset{(r,s)}{\mathcal{Q}}\{R_{j\ lr}^{\ i}C_r^{\ lk}+P_{j\ s\ |r}^{\ i\ k}+P_{j\ rl}^{\ i}P_{ls}^{\ k}\}=0,$$

$$\text{(4.8)}\qquad S_j^{\ isk}{}_{|r}+P_{j\ r}^{\ i\ l}S_l^{\ sk}+\underset{(s,k)}{\mathcal{Q}}\{P_{j\ r}^{\ i\ s}|^k+P_{j\ l}^{\ i\ s}C_r^{\ lk}+S_j^{\ ilr}P_{lr}^{\ k}\}=0.$$

The previous identities become simpler in the following cases

a. $H\Gamma(N)$ is h- and v-symmetric, i.e. $T_{jk}^{\ i}=0$, $S_i^{\ jk}=0$,

b. $H\Gamma(N)$ is h-symmetric and h-connection, i.e. $T_{jk}^{\ i}=0$, $C_i^{\ jk}=0$,

c. $H(\Gamma N)$ is of the Berwald type i.e. $H^i_{jk}=\dot\partial^i N_{kj}$, $C_i^{\ jk}=0$, $(\Rightarrow P^i_{kj}=0)$.

For example, in the last case, if moreover $\tau_{ij}=0$, we have the following Bianchi identities:

$$\underset{(h,r,s)}{\mathcal{S}} R_{h\ rs}^{\ i}=0,\qquad \underset{(h,r,s)}{\mathcal{S}} R_{hrj|s}=0,$$

$$\underset{(h,r,s)}{\mathcal{S}}\{R_{j\ h|s}^{\ i}+P_{j\ h}^{\ i\ l}R_{rsl}\}=0,$$

$$\underset{(h,s)}{\mathcal{Q}} P_{h\ s}^{\ i\ r}=0,\qquad R_{k\ rs}^{\ h}+R_{rsk}|^h-\underset{(r,s)}{\mathcal{Q}} P_{kr\ |s}^{\ h}=0,$$

$$\underset{(h,r)}{\mathcal{Q}} P_{k\ s}^{\ h\ r}-=0,\qquad R_{j\ ks}^{\ i}|^k+\underset{(r,s)}{\mathcal{Q}} P_{j\ s\ |r}^{\ i\ k}=0,$$

$$\underset{(s,k)}{\mathcal{Q}} P_{j\ r}^{\ i\ s}|^k=0.$$

§ 5. Parallelism. The existence of the horizontal distribution N and the vertical distribution V allows us to consider several types of the auto-parallel curves.

A point $(x, p)\in T^*M$ is called a support element in the point $x\in M$. An m-parameters family of the support elements is given by:

$$\text{(5.1)}\qquad x^i=x^i(u^1,\dots,u^m),\ \operatorname{rank}\left\|\frac{\partial x^i}{\partial u^\alpha}\right\|=m,\ (\alpha,\beta=1,\dots,m),\ 1\leq m<n.$$
$$p_i=p_i(u^1,\dots,u^m),\ p(u^\alpha)\neq 0.$$

Definition 5.1. *An m-parameter family is called geodesic if its support elements are horizontal.*

The horizontality condition for the family (5.1) is given by

$$\delta p_i=\left[\frac{\partial p_i}{\partial u^\alpha}-N_{ji}(x(u),\ p(u))\frac{\partial x^j}{\partial u^\alpha}\right]du^\alpha=0.$$

Hence, we have:

Proposition 5.1. *The m-parameter family $\mathcal{F}$ is geodesic if and only if*

$$\text{(5.2)}\qquad \frac{\partial p_i}{\partial u^\alpha}-N_{ji}(x(u),\ p(u))\frac{\partial x^j}{\partial u^\alpha}=0.$$

Proposition 5.2. *The system formed by the first equations (5.1) and (5.2) is completely integrable if and only if the equations*

$$R_{ijk}\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta}=0,$$

hold.

This system is identically verified for $m=1$. Therefore we obtain:

Theorem 5.1. *For every differentiable curve: $x^i=x^i(t)$, $t\in(a,b)$ in M there exists, locally, a geodesic 1-parameter family, on the given curve, of the support elements.*

Let $\mathcal{F}$ be an 1-parameter family of support elements, given by

$$\text{(5.1)}'\qquad \begin{cases} x^i=x^i(t), & t\in(a,b),\quad \operatorname{rank}\left\|\dfrac{dx^i}{dt}\right\|=1,\\ p_i=p_i(t), & p(t)\neq 0.\end{cases}$$

Definition 5.2. 1° *The d-vectorial field $X^i(x,p)$ is said to be parallel on $\mathcal{F}$ with respect to $H\Gamma(N)$ if $\dfrac{\nabla X^i}{dt}=0$ along of $\mathcal{F}$.*

2°. *$H\Gamma(N)$ is called with absolute parallelism of d-vectors if $\dfrac{\nabla X^i}{dt}=0$ on every family $\mathcal{F}$(i.e. $X^i_{|k}=0$, $X^i|^k=0$, is a completely integrable system).*

Since (3.4) implies

$$\text{(5.3)}\qquad \frac{\nabla X^i}{dt}=\frac{dX^i}{dt}+\frac{\omega^i_j}{dt}X^j=X^i_{\ |k}\frac{dx^k}{dt}+X^i|^k\frac{\delta p_k}{dt},$$

we obtain:

Theorem 5.2. *Let $\mathcal{F}$ be a family (5.1)′ and X_0 be a given d-vector in the point $t=t_0$ of $\mathcal{F}$. Then there exists, locally on $\mathcal{F}$, a unique d-vector field X^i, parallel with respect to $H\Gamma(N)$, which for $t=t_0$ coincides to X_0^i.*

Theorem 5.3. *The d-connection $H\Gamma(N)$ is with absolute parallelism of d-vectors if and only if its d-curvature tensors vanish.*

Indeed, from (5.3), it follows that $\nabla X^i/dt=0$ is independent of the family $\mathcal{F}$ if and only if

$$X^i{}_{|k}=0, \qquad X^i|^k=0.$$

But this system is completely integrable if and only if the d-curvature tensor fields of $H\Gamma(N)$ vanish.

Definition 5.3. *An h-path, with respect to $H\Gamma(N)$, is a geodesic family $\mathcal{F}$ which has the vectors dx^i/dt parallel on $\mathcal{F}$.*

Theorem 5.4. *The family $\mathcal{F}$ from (5.1)′ is an h-path if and only if it verifies the system of differential equations:*

$$\frac{d^2x^i}{dt^2}+H^i_{jk}(x(t),p(t))\frac{dx^j}{dt}\frac{dx^k}{dt}=0, \qquad \frac{dp_i}{dt}-N_{ji}(x(t),p(t))\frac{dx^j}{dt}=0. \tag{5.4}$$

This property follows by virtue of the equations (5.3) and (3.4)′.

The previous theorem assures the local existence and uniqueness of the h-paths in the apriori given initial data.

A family $\mathcal{F}$ of the support elements given by

$$x^i=x^i_0\ (=\text{const}),\quad p_i=p_i(t),\ t\in(a,b),\ p(t)\neq 0,$$

will be called vertical.

Definition 5.4. *A vertical family $\mathcal{F}$ is a v-path if its d-vectors $\delta p_i/dt$ are parallel on $\mathcal{F}$ with respect to $H\Gamma(N)$.*

Theorem 5.5. *A v-path, with respect to $H\Gamma(N)$ is characterized by the differential equations:*

$$x^i=x^i_0,\quad \frac{d^2p_i}{dt^2}-C_i{}^{jk}(x_0,p(t))\frac{dp_j}{dt}\frac{dp_k}{dt}=0. \tag{5.5}$$

This property results from (5.3) and (3.4).

The local existence of v-paths is immediate by virtue of (5.5).

§ 6. Transformations of d-connections. We are interested in the effect of the change of the nonlinear connection N or of the d-connection $H\Gamma(N)$.

A transformation of nonlinear connection $N(N_{ij})\to\bar{N}(\bar{N}_{ij})$ is given by

$$\bar{N}_{ij}=N_{ij}+A_{ij}, \tag{6.1}$$

where $A_{ij}(x,p)$ is an arbitrary d-tensor field of the type (0,2).

The adapted cobasis $(dx^i,\delta p_i)$ to the distribution N and V is transformed in the cobasis $(dx^i,\bar{\delta}p_i)$ adapted to the distributions $\bar{N}$ and V, where

$$\bar{\delta}p_i=\delta p_i-A_{ji}dx^j \tag{6.2}$$

Proposition 6.1. *The transformation (6.1) carries the d-connection $H\Gamma(N)=(H,C)$ in the d-connection $H\bar{\Gamma}(\bar{N})=(\bar{H},\bar{C})$ such that $\nabla X^i=\bar{\nabla}X^i$, for every $X^i(x,p)$ if and only if*

$$\bar{H}^i_{jk}=H^i_{jk}+C_i{}^{jl}A_{kl},\quad \bar{C}_i{}^{jk}=C_i{}^{jk}. \tag{6.3}$$

Indeed, from (3.4)′ we get $\bar{\omega}^i_j=\omega^i_j$ if and only if (6.2) holds. But (6.2) is equivalent to (6.3), q.e.d.

By applying to (6.3) an usual transformation of d-connection: $\bar{\omega}^i_j=\omega^i_j+\eta^i_j$ we obtain:

Theorem 6.1. *The most general transformation of d-connections: $t(A,B,D)$: $H\Gamma(N)\to H\bar{\Gamma}(\bar{N})$ is given by the equations:*

$$\bar{N}_{ij}=N_{ij}+A_{ij},\quad \bar{H}^i_{jk}=H^i_{jk}+C_i{}^{jl}A_{kl}+B^i_{jk},\quad \bar{C}_i{}^{jk}=C_i{}^{jk}+D_i{}^{jk}, \tag{6.4}$$

where $(A,B,D)=(A_{ij},B^i_{jk},D_i{}^{jk})$ are arbitrary d-tensor fields of the type (0,2), (1.2) and (2,1) respectively.

We briefly denote the transformation (6.4) by

$$t(A,B,D):(N,H,C)\to(N+A,H+CA+B,C+D). \tag{6.4′}$$

Now we can easily prove:

Theorem 6.2. *The set $\mathcal{T}$ of the transformations (6.4) together with the product of maps is a group, wich acts transitively on the set of d-connections on M.*

The subset $\mathcal{T}_1\subset\mathcal{T}$ of the transformations of the form:

$$t(A,0,0):(N,H,C)\to(N+A,H+CA,C)$$

and the product of maps is a normal subgroup in $\mathcal{T}$. Also

$$\mathcal{T}_2=\{t(0,B,0)\in\mathcal{T}\}\ \text{and}\ \mathcal{T}_3=\{t(0,0,D)\in\mathcal{T}\}$$

are subgroups in $\mathcal{T}$.

Theorem 6.3. *The group $\mathcal{T}$ of the transformations of d-connections on M is the semidirect product of the subgroup $\mathcal{T}_1$ by the direct product of the subgroups $\mathcal{T}_2$ and $\mathcal{T}_3$:*

$$\mathcal{T}=\mathcal{T}_1\dot{\times}(\mathcal{T}_2\times\mathcal{T}_3).$$

Theorem 6.4. *The following diagram is commutative*

$$\begin{tikzcd}
 & & (N+A,H+CA+B,C) \arrow[dr, "t(0,0,D)"] & \\
(N,H,C) \arrow[r, "t(A,0,0)"] & (N+A,H+CA,C) \arrow[ur, "t(0,B,0)"] \arrow[rr, "t(0,B,D)"] \arrow[dr, "t(0,0,D)"'] & & (N+A,H+CA+B,C+D) \\
 & & (N+A,H+CA,C+D) \arrow[ur, "t(0,B,0)"'] &
\end{tikzcd}$$

Part II. HAMILTON SPACES

The theory of d-geometrical objects on the differentiable manifold M has an initial application in the study of Hamilton spaces. We begin with the definition of generalized Hamilton spaces, then we shall treat the Hamilton spaces, Cartan spaces and also the special classes of weakly regular, regular and local Minkowski spaces.

§ 7. Generalized Hamilton spaces. Definition 7.1. *A generalized Hamilton space is a pair $M^{*n}=(M, g^{ij}(x, p))$ in which $g^{ij}(x, p)$ is a d-tensor field symmetric nondegenerate of the type* (2, 0).

$g^{ij}(x, p)$ is called the metric Hamilton structure or the fundamental tensor of the space M^{*n}.

Proposition 7.1. *If M is a paracompact differentiable manifold, then there exists on M an Hamilton metrical structure.*

Proof. M being paracompact, T^*M has the same property. Let G be a Riemann metric on T^*M. Then $g^{ij}(x, p)=G(\dot{\partial}^i, \dot{\partial}^j)$, $\forall (x, p)\in T^*M$ is a generalized metric structure on M. q.e.d.

Let us consider the d-tensor field $g_{ij}(x, p)$ uniquely determined by

$$g_{ik}(x, p)\, g^{kj}(x, p)=\delta_i^j. \tag{7.1}$$

A d-connection $H\Gamma(N)=(H^i_{jk}, C_i^{jk})$ will be called compatible with the Hamilton metrical structure $g^{ij}(x, p)$ if

$$g^{ij}{}_{|k}=0, \quad g^{ij}|^k=0. \tag{7.2}$$

Supposing $g^{ij}(x, p)$ positively defined, we can consider the square of the length of a d-covector field $\omega_i(x, p)$ given by

$$\|\omega\|^2=g^{ij}(x, p)\omega_i\omega_j, \quad \forall (x, p)\in T^*M. \tag{7.3}$$

Denoting by $S=(x^i(t), p_i(t), t\in(a, b))$ an 1-parameter family of support elements we have:

Theorem 7.1. *A d-connection $H\Gamma(N)$ has the property $d/dt\|\omega\|^2=0$, on every family S, for an arbitrary d-covector field ω_i, parallel on S with respect to $H\Gamma(N)$ if and only if $H\Gamma(N)$ is compatible with the Hamilton metrical structure $g^{ij}(x, p)$.*

Proof. From (7.3) and $\dfrac{\nabla\omega_i}{dt}=0$ on S it results $\dfrac{d}{dt}(\|\omega\|^2)=\dfrac{\nabla}{dt}(g^{ij}\omega_i\omega_j)=$ $=\left(g^{ij}{}_{|k}\dfrac{dx^k}{kdt}+g^{ij}|^k\dfrac{\delta p_k}{dt}\right)\omega_i\omega_j=0$. But ω_i and S being arbitrary it follows $g^{ij}|_k=0$, $g^{ij}|^k=0$.

Conversely, if $H\Gamma(N)$ has the property (7.2) and $\nabla\omega_i/dt=0$ on S then we have $d/dt(\|\omega\|^2)=0$. q.e.d.

A d-connection compatible with g^{ij} will be called a d-metrical connection. By virtue of (7.1) it results that (7.2) is equivalent to:

$$g_{ij}|_k=0, \quad g_{ij}|^k=0. \tag{7.2'}$$

We suppose that a nonlinear connection N on T^*M was fixed. Using the operators $\delta_k=\delta/\delta x^k=\partial_k+N_{kj}\dot{\partial}^j$, we can construct the „Christoffel symbols"

$$\begin{aligned} H^i_{jk}&=1/2\, g^{ih}(\delta_k g_{hj}+\delta_j g_{hk}-\delta_h g_{jk}),\\ C_i^{jk}&=-1/2\, g_{ih}(\dot{\partial}^k g^{hj}+\dot{\partial}^j g^{hk}-\dot{\partial}^h g^{jk}). \end{aligned} \tag{7.4}$$

Theorem 7.2. 1°. *$(H^i_{jk}(x, p), C_i^{jk}(x, p))$ from (7.4) give us a d-connection on M.*

2°. *The d-connection (7.4) is a d-metrical connection.*

Proof. 1°. Observing (1.6) and the law of transformation of $g^{ij}(x, p)$ we obtain, for H^i_{jk} from (7.4), the law of transformation (3.1). Also, for C_i^{jk} we get a tensorial law of transformation. 2°. By a straightforward calculus we prove that $H\Gamma(N)$ of (7.4) verifies the equations (7.2). q.e.d.

Remark. The d-connection (7.4) depends on the fundamental d-tensor g^{ij} and on the nonlinear connection $N(N_{ij})$, only. It will be called the N-canonical connection of the space M^{*n} and denoted by $C\Gamma(N)$.

In the following we denote

$$C^{ijk}=g^{ih}C_h^{jk}. \tag{7.5}$$

Theorem 7.3. *The N-canonical connection $C\Gamma(N)$ has the properties:*

1°. $T_{jk}{}^i=0$, $S_i^{jk}=0$.

2°. $S^{ijkl}=g^{ih}S_h{}^{jkl}=\dot{\partial}^l C^{ijk}-\dot{\partial}^k C^{ijl}+C_r{}^{jk}C^{ril}-C_r{}^{jl}C^{rik}$.

Proof. The expressions (7.4) clearly show 1°. Using (3.7), the formulae 2° follow. q.e.d.

Let us consider the Obata's operators

$$\Omega^{ij}_{hk}=1/2(\delta^i_h\delta^j_k-g_{hk}g^{ij}), \quad \Omega^{*ij}_{hk}=1/2\,(\delta^i_h\delta^j_k+g_{hk}g^{ij}). \tag{7.6}$$

We have:

Proposition 7.2. 1°. *The Obata's operators are absolute parallel with respect to every d-metrical connection $H\Gamma(N)$.*

2°. *Obata's operators are supplementary projectors on the module of the d-tensor field of the type (r, s), $r\geq 1$, $s\geq 1$.*

Theorem 7.4. *If N is a fixed nonlinear connection on T^*M, then the set of all d-metrical connection $H\overline{\Gamma}(N)=(\overline{H}^i_{jk}, \overline{C}_i^{jk})$ is given by*

$$\overline{H}^i_{jk}=H^i_{jk}+\Omega^{is}_{rj}X^r_{sk}, \qquad \overline{C}_i^{jk}=C_i^{jk}+\Omega^{js}_{ri}Y^{rk}_s, \tag{7.7}$$

where $C\Gamma(N)=(H^i_{jk}, C_i^{jk})$ is the N-canonical connection and X^i_{jk}, Y_i^{jk} are d-arbitrary tensor fields of the type (1, 2) *and* (2, 1), *respectively.*

Proof. Taking into account the transformation $C\Gamma(N)\to H\overline{\Gamma}(N)$:

$$\overline{H}=H+B, \qquad \overline{C}=C+B, \tag{a}$$

we see that the d-connection $H\overline{\Gamma}(N)$ is a metrical one if and only if

$$\Omega^{*is}_{rj}B^r_{sk}=0, \qquad \Omega^{*is}_{rj}D^{rk}_s=0. \tag{b}$$

Then the general solution of (b) inserted in (a), give us (7.7).

The following theorem holds.

Theorem 7.5. 1°. *There exists a unique d-metrical connection* $H\overline{\Gamma}(N)$ *with the property* $T_{jk}{}^{i}=0$, $S_{i}{}^{jk}=0$. *This is* $C\Gamma(N)$.

2°. *There exists a unique d-metrical connection* $H\overline{\Gamma}(N)$ *whose d-torsion tensor fields* $T_{jk}{}^{i}$, $S_{i}{}^{jk}$ *are given a priori. This is*

$$\overline{H}^{i}_{jk}=H^{i}_{jk}+\frac{1}{2}g^{ir}(g_{rh}T_{jk}{}^{h}-g_{jh}T_{rk}{}^{h}+g_{kh}T_{jr}{}^{h}),$$

(7.8)

$$\overline{C}^{jk}_{i}=C_{i}{}^{jk}-\frac{1}{2}g_{ir}(g^{rh}S_{h}{}^{jk}-g^{jh}S_{h}{}^{rk}+g^{kh}S_{h}{}^{jr}).$$

Theorem 7.6. *Every d-metrical connection* $H\Gamma(N)$ *has the property.*

(7.9) $$R^{ij}{}_{hk}+R^{ji}{}_{hk}=0,\quad P^{ij}{}_{h}{}^{l}+P^{ji}{}_{h}{}^{l}=0,\quad S^{ijhk}+S^{jihk}=0,$$

where $R^{ij}{}_{hk}=g^{il}R_{l}{}^{j}{}_{hk}$, etc.

Indeed, by applying the Ricci formulae (3.8) to $g^{ij}(x,p)$ and taking into account (7.2), we arrive to the announced result.

It is clear that the transformations (7.7) of d-metrical connections determine a subgroup of the group $\mathcal{T}$. Some particular transformations of $\mathcal{T}$, associated to $C\Gamma(N)$ are useful in applications. If $C\Gamma(N)=(H^{i}_{jk}, C_{i}{}^{jk})$ is the N-canonical connection, we associate to it the d-connections of the type: Berwald $B\Gamma(N)=(\dot{\partial}^{i}N_{kj}, 0)$; Rund $R\Gamma(N)=(H^{i}_{jk}, 0)$ and Hashiguchi $\mathcal{H}\Gamma(N)=(\dot{\partial}^{i}N_{kj}, C_{i}{}^{jk})$ [10, 11].

Theorem 7.7. *The following diagram is commutative*

$$\begin{tikzcd} & R\Gamma(N) \arrow[dr, "t_2"] & \\ C\Gamma(N) \arrow[ur, "t_1"] \arrow[rr] \arrow[dr, "t_2"'] & & B\Gamma(N) \\ & \mathcal{H}\Gamma(N) \arrow[ur, "t_1"'] & \end{tikzcd}$$

where $t_1=t(0,\ 0,\ -C_{i}{}^{jk})$, $t_2=t(0, -P_{jk}{}^{i}, 0)$, $t_3=t(0,\ -P_{jk}{}^{i},\ -C_{i}{}^{jk})$.

Indeed, taking in the diagram in the theorem 6.4: $A=0$, $B=-C$, $D=-P$ and looking at the definitions of the d-connections of the type Berwald, Rund and Hashiguchi, we obtain the above commutative diagram. These d-connections have some important properties:

Proposition 7.3. *The d-connection of the Berwald type* $B\Gamma(N)$ *has the properties:*

1°. *It is not a d-metrical connection;*

2°. *It has d-torsion tensors*

$$T_{jk}{}^{i}=\dot{\partial}^{i}\tau_{jk},\quad C_{i}{}^{jk}=0,\quad S_{i}{}^{jk}=0,\quad P_{jk}{}^{i}=0,$$

and, generally, $R_{ijk}\neq 0$;

3°. *It has the d-curvature tensors* $R_{j}{}^{i}{}_{kh}$ *in* (3.7) *with* $H^{i}_{jk}=\dot{\partial}^{i}N_{kj}$, $C_{i}{}^{jk}=0$; $P_{j}{}^{i}{}_{k}{}^{l}=\dot{\partial}^{l}\dot{\partial}^{i}N_{kj}$; $S_{j}^{ihk}=0$;

4°. *It has v-paths:* $x^{i}=x^{i}_{0}$, $\dfrac{d^{2}p_{i}}{dt^{2}}=0$;

5°. *Its h-paths are given by* $\dfrac{\nabla}{dt}\left(\dfrac{dx^{i}}{dt}\right)=0$, $\dfrac{\delta p_{i}}{dt}=0$.

Proposition 7.4. *The d-connection of Rund type,* $R\Gamma(N)$, *has the properties*

1°. *It is h-metrical:* $g^{ij}{}_{|k}=0$, $g^{ij}|^{k}\neq 0$;

2°. *It is metrical if and only if* (M, g^{ij}) *is a Riemann space;*

3°. *Its d-torsion tensors* $T_{jk}{}^{i}$, $S_{i}{}^{jk}$, $C_{i}{}^{jk}$ *vanish;*

4°. *Its d-curvature tensors are*

$$R_{j}{}^{i}{}_{hl}=\underset{(k,l)}{Q}\{\delta_{l}H^{i}_{jk}+H^{r}_{jk}H^{i}{}_{rl}\},\quad P_{j}{}^{i}{}_{h}{}^{l}=\dot{\partial}^{l}H^{i}_{jh},\quad S_{i}{}^{jkl}=0.$$

5°. *Its v-paths are given by* $x^{i}=x^{i}_{0}$, $\dfrac{d^{2}p_{i}}{dt^{2}}=0$;

6°. *Its h-paths are given by* $\dfrac{\nabla}{dt}\left(\dfrac{dx^{i}}{dt}\right)=0$, $\dfrac{\delta p_{i}}{dt}=0$.

Similar properties has $\mathcal{H}\Gamma(N)$. Some particular classes of the M^{*n} spaces will be studied after the introduction of the notion of Hamilton space.

§ 8. The duality between the Lagrange and Hamilton spaces. The most important problem in the study of generalized Hamilton spaces M^{*n} or Hamilton spaces H^{n} consists of the determination of the nonlinear connections on $T^{*}M$ which are intrinsically related to these spaces. To this aim it is necessary to extend here the Legendre transformation from the analytical mechanics. It leads us to the concept of duality between the Lagrange and Hamilton spaces.

Definition 8.1. *A Hamilton space is a pair* $H^{n}=(M, H)$, *where* $H: T^{*}M\to R$ *is a* C^{∞}*-function on* $T^{*}M\setminus\{0\}$, *continous for* $p=0$ *and with the property that the Hessian*

(8.1) $$g^{ij}(x,p)=\frac{1}{2}\frac{\partial^{2}H}{\partial p_{i}\partial p_{j}},\qquad \forall (x,p)\in T^{*}M,\ p\neq 0,$$

is nondegenerate.

$H(x,p)$ is called the Hamilton function or fundamental function of H^{n} and $g^{ij}(x,p)$ the metrical structure or fundamental tensor of the Hamilton space H^{n}. Obviously, every space H^{n} is a generalized Hamilton space, M^{*n}, but not conversely.

Proposition 8.1. *If* M *is a* C^{∞}*-differentiable paracompact manifold then there exists a function* $H(x,p)$ *on* $T^{*}M$ *such that* (M, H) *is a Hamilton space.*

Indeed, M being paracompact, let $g^{ij}(x)$ be a Riemann structure on M. Then

$$H(x,p)=(mc)^{-1}g^{ij}(x)p_{i}p_{j},$$

where m and c are constants, is a Hamilton function on M. Therefore $(M, H(x,p))$ is a Hamilton space.

Remark. The function $\mathcal{H}=1/2\,H=(2mc)^{-1}g^{ij}(x)p_{i}p_{j}$ determines the gravitational field on M, which we shall consider in § 13.

The theory developed in the previous paragraphs can be applied to the spaces H^{n} taking into account the equations (8.1) for determining of the fundamental tensor field $g^{ij}(x,p)$.

But firstly we shall study the existence of a nonlinear connection N on $T^{*}M$, determined by the Hamilton function H, only.

As it is well-known, [11, 12], a Lagrange space is a pair $L^n = (M, L(x, y))$, where $L : TM \to R$ is a C^∞-function on $TM/\{0\}$, continous for $y=0$ and with the property that d-tensor field

$$a_{ij}(x, y) = 1/2\, \partial^2 L/\partial y^i \partial y^j, \ \forall (x, y) \in TM,\ y \neq 0, \tag{8.2}$$

is nondegenerate. Here (x, y) is a point of the total space of the tangent bundle $\pi : TM \to M$ of M. Denoting by $a^{ij}(x, y)$ the reciprocal of $a_{ij}(x, y)$ we obtain a known result:

Theorem 8.1. 1°. *The geodesics of the spaces L^n are given by the Euler-Lagrange equations:*

$$\frac{dx^i}{dt} = y^i,\ \frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = 0. \tag{8.3}$$

2°. *The functions*

$$N^i{}_j(x, y) = \frac{\partial G^i}{\partial y^j}, \qquad G^i = \frac{1}{4} a^{ij}\left(\frac{\partial^2 L}{\partial y^j \partial x^k} y^k - \frac{\partial L}{\partial x^j}\right) \tag{8.3)'$$

are the coefficients of a nonlinear connection N on TM, which is determined by the Lagrangian $L(x, y)$, only. $N(N^i{}_j(x, y))$ is the canonical nonlinear connection of the Lagrange space L^n. Let us consider the „moments", [2, 17], determined by the Lagrangian $L(x, y)$:

$$p_i = \frac{1}{2}\frac{\partial L}{\partial y^i}. \tag{8.4}$$

Then, p_i give us a global d-covector field on M. The map given by

$$\varphi : (x, y) \in \widetilde{TM} \to (x, p) \in \widetilde{T^*M},\ (\widetilde{TM} = TM/\{0\},\ \widetilde{T^*M} = T^*M/\{0\}), \tag{8.5}$$

where p_i are given by (8.4), has some important properties.

Proposition 8.2. *φ is a local diffeomorphism from $\widetilde{TM}$ to $\widetilde{T^*M}$, which preserves the fibers.*

Indeed, φ is of the class C^∞ and in every point $(x, y) \in \widetilde{TM}$ its Jacobian is the determinant of the matrix $\left\|\begin{matrix} \delta^i_j & 0 \\ \cdot & a_{ij} \end{matrix}\right\|$. Then $\pi(x, y) = \pi^*(x, p) = x$, $\forall (x, y) \in \widetilde{TM}$. q.e.d.

Therefore, for every fixed point $(x, y) \in \widetilde{TM}$, there exists an open set $D \subset \widetilde{TM}$, $(x, y) \in D$ such that $D_1 = \pi(D)$ is an open set in M and there exists an open set $D^* \subset \widetilde{T^*M}$, $D^* = \varphi(D)$ such that $\varphi_{|D}$ is invertible. We can write the inverse function φ^{-1} in the form:

$$x^i = x^i,\quad y^i = \Phi^i(x, p)\quad \forall (x, p) \in D^*. \tag{8.6}$$

We restrict our considerations to the sets D and D^* and denote by L^n_D the restriction of L^n to D and by H^{n*}_D the restriction of H^n to D^* if this exists. It is convenient to denote:

$$\mathcal{L} = \frac{1}{2} L,\qquad \mathcal{H} = \frac{1}{2} H. \tag{8.7}$$

Now we can formulate:

Theorem 8.2. *On the open set $D^* \subset \widetilde{T^*M}$, the Hamilton function*

$$\mathcal{H}(x, p) = -\mathcal{L}(x, y) + p_i y^i, \tag{8.7)'$$

where y^i is given by (8.6), *has a nondegenerate Hessian.*

Proof. From (8.6), (8.7) and (8.7)′, it results that, on D^*, we have

$$\det \|\dot\partial^i \dot\partial^j \mathcal{H}\| = (\det \|a_{ij}(x, \Phi(x, p))\|)^{-1}. \text{ q.e.d.}$$

By virtue of this theorem, $\mathcal{H}(x, p)$ determines on D^* a Hamilton space $\mathcal{H}^n_{D^*}$. The transformation (8.4), (8, 7)′ (which determines the local diffeomorphism φ) is called the Legendre transformation from L^n_D to H^{n*}_D. This is the reason for which H^{n*}_D is called the dual space of the Lagrange space L^n_D.

Proposition 8.3. *The Hamilton space H^{n*}_D, the dual space of the Lagrange space L^n_D, has the property:*

$$\partial \mathcal{H}/\partial x^i = -\partial \mathcal{L}/\partial x^i, \qquad \partial \mathcal{H}/\partial p_i = \Phi^i(x, p). \tag{8.8.}$$

Proof. The formulae (8.6), (8.7) and (8.7)′ lead us to

$$\frac{\partial \mathcal{H}}{\partial x^i} = -\frac{\partial \mathcal{L}}{\partial x^i} - \frac{\partial \mathcal{L}}{\partial y^k}\frac{\partial \Phi^k}{\partial x^i} + p_k \frac{\partial \Phi^k}{\partial x^i} = -\frac{\partial \mathcal{L}}{\partial x^i}$$

and

$$\frac{\partial \mathcal{H}}{\partial p_i} = -\frac{\partial \mathcal{L}}{\partial y^k}\dot\partial^i \Phi^k + y^i + p_k \dot\partial^i \Phi^k = y^i = \Phi^i(x, p). \text{ q.e.d.}$$

Conversely, let us consider a Hamilton space $H^n(M, H)$ having the fundamental tensor $g^{ij}(x, p)$ given by (8.1). Let y^i „the velocities" determined by the Hamilton function H:

$$y^i = \frac{1}{2}\frac{\partial H}{\partial p_i}. \tag{8.9}$$

Proposition 8.4. *The map*

$$\psi : (x, p) \in \widetilde{T^*M} \to (x, y) \in \widetilde{TM},\ (\widetilde{T^*M} = T^*M \setminus \{0\},\ \widetilde{TM} = TM \setminus \{0\}), \tag{8.10}$$

where y^i are given by (8.9), *is a local diffeomorphism which preserves the fibers.*

Proof. For every fixed point $(x, p) \in \widetilde{T^*M}$, there exists an open set $D^* \subset \widetilde{T^*M}$, $(x, p) \in D^*$, which is projected by π^* in an open set $D_1 = \pi^*(D^*) \subset M$ and there exists an open set $D \subset \widetilde{TM}$, $D = \psi(D^*)$ such that ψ is invertible on D^*. The inverse function $\psi^{-1} : D \to D^*$ can be written in the form:

$$x^i = x^i,\quad p_i = \Psi_i(x, y). \tag{8.11}$$

Theorem 8.3. *On the open set $D \subset \widetilde{TM}$, the Lagrange function*

$$\mathcal{L}(x, y) = -\mathcal{H}(x, p) + p_i y^i, \tag{8.7)'''$$

where p_i is given by (8.11), *has a nondegenerate Hessian.*

Of course, L^n_D is an Lagrange space, which is the dual of the Hamilton space H^{n*}_D.

Now, easily it follows:

Theorem 8.4. *The dual of the dual of L^n_D is L^n_D. The dual of the dual of H^{n*}_D is H^{n*}_D.*

Theorem 8.5. *The Legendre transformation carries the system of Euler-Lagrange equations* (8.3) $\left(\frac{1}{2}L=\mathcal{L}\right)$ *in the system of Hamilton equations*

$$dx^i/dt-\partial\mathcal{H}/\partial p_i=0,\quad dp_i/dt+\partial\mathcal{H}/\partial x^i=0. \tag{8.12}$$

Proof. By using (8.7)′ and the Proposition 8.3, from the Euler-Lagrange equations (8.3), we get the first equations (8.12). Then $p_i=\partial\mathcal{L}/\partial y^i$ shows that the equations (8.3) are carried in the second equations (8.12). q.e.d.

In the same way, we find:

Proposition 8.5. *The Legendre transformation carries the nonlinear connection $N(N^i{}_j(x,y))$ on $\widetilde{TM}_{|D}$ from* (8.3)′ *in the nonlinear connection $\overset{c}{N}(\overset{c}{N}_{ij}(x,p))$ on $\widetilde{T^*M}_{|D^*}$:*

$$\overset{c}{N}_{ki}(x,p)=-a_{ij}(x,p)N^j{}_k(x,y)+1/2\,\partial^2L(x,y)/\partial y^i\partial x^k,\ y^i=\Phi^i(x,p). \tag{8.13}$$

But $N(N^j_i(x,y))$ from (8.3)′ is the canonical nonlinear connection of the Lagrange space L^n, therefore the nonlinear connection $\overset{c}{N}(\overset{c}{N}_{ki}(x,p))$ from (8.13) will be taken as canonical for the Hamilton space H^n.

It follows:

Theorem 8.6. *There exists, locally, the nonlinear connection on T^*M, which depends on the fundamental function of the Hamilton space H^n, only.*

In the next section we shall prove the global existence of the canonical nonlinear connection on T^*M.

§ 9. Hamilton spaces. As we have remarked, the theory of the generalized Hamilton spaces can be applied to Hamilton spaces $H^n=(M,H)$. But some specifical properties will appear, especially, those related to the global existence of the canonical nonlinear connections.

Proposition 9.1. *The d-tensor field C^{ijk}* (see (7.5)) *of a Hamilton space H^n has the form:*

$$C^{ijk}=-\frac{1}{4}\dot\partial^i\dot\partial^j\dot\partial^k H. \tag{9.1}$$

Consequently, it is completely symmetric.

Theorem 9.1. *A generalized Hamilton space $M^n=(M,g^{ij}(x,p))$ is a Hamilton space $H^n=(M,H)$, if and only if the d-tensor field C^{ijk} from* (7.4), (7.5) *is completely symmetric.*

Proof. If $M^n=H^n$, the Proposition 9.1 shows that the property holds. Conversely, observing (7.4) and (7.5), we have

$$C^{ijk}=-\frac{1}{2}(\dot\partial^k g^{ji}+\dot\partial^j g^{ki}-\dot\partial^i g^{jk}).$$

As C^{ijk} is completely symmetric, it results that $\dot\partial^k g^{ij}$ is completely symmetric. Then the system of equations with partial derivations (8.1) has solutions with respect to the function $H(x,p)$. q.e.d.

In the following, we consider the Poisson bracket for the real functions on TM.

$$(f,\ g)=\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i},$$

where the summation convention holds.

A very important result is given by:

Theorem 9.2. *There exists a nonlinear connection $\overset{c}{N}$ on TM determined only by the fundamental function $H(x,p)$ of a Hamilton space $H^n=(M,H)$. The coefficients of $\overset{c}{N}$ are given by the formula:*

$$\overset{c}{N}_{ij}=-\frac{1}{2}g_{jh}\left\{\frac{1}{4}g_{ik}\dot\partial^k(H,\ \dot\partial^h H)+\dot\partial^h\partial_i H\right\} \tag{9.2}$$

Proof. Using Proposition 8.5 and Theorem 8.6, we see that the Legendre transformation carries the nonlinear connection (8.3)′ to the nonlinear connection (9.2). But it is necessary to prove the global character of this nonlinear connection $\overset{c}{N}$. To this aim it is sufficient to show that, up to a change of coordinates (1.1), $\overset{c}{N}_{ij}(x,p)$ has the transformation law (1.5). Taking into account that $g^{ij}(x,p)$ from (8.1) is the fundamental tensor of the Hamilton space H^n, using (1.2), after a tedious calculus, we obtain the transformation (1.5) for $\overset{c}{N}_{ij}(x,p)$. q.e.d.

Theorem 9.3. *The nonlinear connection $\overset{c}{N}$ given by* (9.2) *has the properties:*

1°. $\tau_{ij}=0$, 2°. $\underset{(i,j,k)}{\mathcal{S}} R_{ijk}=0.$

These properties result immediately from

Lemma 9.1. *The coefficients $\overset{c}{N}_{ij}$ of the nonlinear connection* (9.2) *can be written in the form:*

$$\overset{c}{N}_{ij}=-\frac{1}{4}\left(\frac{\partial g_{ij}}{\partial x^l}\frac{\partial H}{\partial p_l}+g_{jk}\frac{\partial^2 H}{\partial p_k\partial x^i}+g_{ik}\frac{\partial^2 H}{\partial p_k\partial x^j}+2\frac{\partial H}{\partial x^l}\frac{\partial^3 H}{\partial p_k\partial p_h\partial p_l}g_{ik}g_{jh}\right) \tag{9.3}$$

Applying now Theorems 2.3 and 9.3 we have:

Theorem 9.4. *The almost symplectic structure $\overset{c}{\theta}$, associated to the nonlinear connection $\overset{c}{N}$, is integrable.*

Of course, the Hamilton equations (8.12) have not an invariant form. However, we have:

Theorem 9.5. *The Hamilton equations (9.12) are equivalent to the system :*

$$dx^i/dt-\partial\mathcal{H}/\partial p_i=0, \qquad \overset{c}{\delta}p_i/dt+\overset{c}{\delta}\mathcal{H}/\delta x^i=0, \tag{9.4}$$

where $\overset{c}{\delta}p_i$ *and* $\overset{c}{\delta}_i$ *are given by* (1.4) and(1.4)′ *for* $N=\overset{c}{N}$.

Proof. Form $\overset{c}{N}_{ij}=\overset{c}{N}_{ji}$ and (8.12) we get $\overset{c}{\delta}p_i/dt+(\partial\mathcal{H}/\partial x^i+\overset{c}{N}_{ij}\,dx^j/dt)=0$. Or, by virtue of the first equations (8.12), we have the second equation (9.4).

Conversely, (9.4) implies (8.12). q.e.d.

Corollary 9.1. *If the Hamilton space H^n has the property $\overset{c}{\delta}H/\delta x^i=0$, then (9.4) give an 1-parameter geodesic family of support elements.*

Corollary 9.2. *The fundamental function H of the Hamilton space H^n is constant on the integral curves of the Hamilton equations (9.4).*

Indeed, if $X=dx^i/dt\;\overset{c}{\delta}_i+\delta p_i/dt\;\dot\partial^i$, then from (9.4) :

$$L_X\mathcal{H}=dx^i/dt\;\overset{c}{\delta}_i\mathcal{H}+\overset{c}{\delta}p_i/dt\;\dot\partial^i\mathcal{H}=\dot\partial^i\mathcal{H}\,.\,\overset{c}{\delta}_i\mathcal{H}-\overset{c}{\delta}_i\mathcal{H}\dot\partial^i\mathcal{H}=0.$$

Corollary 9.3. *If the canonical connection $C\Gamma(\overset{c}{N})$ is of the Cartan type, then the Hamilton equations (9.4) are equivalent to the equations:*

$$\frac{dx^i}{dt}-\frac{\partial\mathcal{H}}{\partial p_i}=0, \qquad \frac{\overset{c}{\nabla}p_i}{dt}+\frac{\delta\mathcal{H}}{\delta x^i}=0. \tag{9.4′}$$

Indeed, $p_{i|k}=0$, $p_i|^k=\delta_i^k$ and (3.4) implies $\overset{c}{\nabla}p_i/dt=\overset{c}{\delta}p_i/dt$.

Finally, we have :

Theorem 9.6. *The following d-connection $C\Gamma(\overset{c}{N})=(\overset{c}{H}{}^i_{jk},\ \overset{c}{C}{}_i{}^{jk})$, where $\overset{c}{N}$ is the canonical nonlinear connection (9.2) and where*

$$\begin{aligned}\overset{c}{H}{}^i_{jk}&=1/2(\overset{c}{\delta}_jg_{hk}+\overset{c}{\delta}_kg_{hj}-\overset{c}{\delta}_hg_{jk})g^{ih},\\ \overset{c}{C}{}_i{}^{jk}&=-1/2(\dot\partial^jg^{hk}+\dot\partial^kg^{hj}-\dot\partial^hg^{jk})g_{ih},\end{aligned} \tag{9.5}$$

for $\overset{c}{\delta}_j=\partial_j+\overset{c}{N}_{jk}\dot\partial^k$, has the properties

1°. *It depends on the fundamental function H of the Hamilton space H^n, only.*

2°. *It is a d-metrical connection ;*

3°. *Its torsions $\overset{c}{T}{}_{jk}{}^i$ and $\overset{c}{S}{}_i{}^{jk}$ vanish ;*

4°. $\overset{c}{P}{}_{jk}{}^i=\overset{c}{P}{}_{kj}{}^i$;

5°. $\overset{c}{S}{}^{ijkh}=\overset{c}{C}{}_r{}^{ji}\overset{c}{C}{}^{rih}-\overset{c}{C}{}_r{}^{jh}\overset{c}{C}{}^{rik}$;

6°. *It is the unique d-metrical connection $H\Gamma(\overset{c}{N})$ for which 3° holds.*

$C\Gamma(\overset{c}{N})$ is called the canonical connection of the Hamilton space H^n.

§ 10. Cartan spaces. The Cartan spaces can be looked as duals of the Finsler spaces. This remark allows us to determine a canonical nonlinear connection and a canonical connection on which are based their geometry. E. Cartan has defined the above mentioned spaces in [5], using the notion of areal metric and he has studied them from this point of view. His construction is very complicated because it is not based on the notion of nonlinear connection.

The dual point of view allows to get a lot of new properties of the Cartan spaces. This means that the Cartan spaces have the same importance as the Finsler spaces.

Definition 10.1. *A Hamilton space $H^n=(M,H)$ in which the fundamental function $H(x,p)$, $p\neq0$ is 2-homogeneous with respect to p_i is called a Cartan space. We denote it by $\mathcal{C}^n$.*

Proposition 10.1. *For a Cartan space $\mathcal{C}^n$ we have :*

1°. *The fundamental tensor field $g^{ij}(x,p)$ is 0-homogeneous with respect to p_i ;*

2° $\qquad 1/2\,\partial H/\partial p_i=g^{ij}(x,p)\,p_j$;

3° $\qquad C^{\circ jk}=0.$

Theorem 10.1. *A generalized Hamilton space M^n is a Cartan space $\mathcal{C}^n$ if and oly if $C^{ijk}\neq0$* is completely symmetric and $C^{\circ jk}=0$.

Proof. If $M^n=\mathcal{C}^n$ then C^{ijk} has the above mentioned properties. Conversely, C^{ijk} being completely symmetric, we have $\dot\partial^kg^{ij}=C^{ijk}+C^{jik}$. This means that $\dot\partial^kg^{ij}$ is completely symmetric. From $C^{\circ jk}=0$, we get $p_k\dot\partial^kg^{ij}=0$. Therefore $g^{ij}(x,p)$ is 0-homogeneous with respect to p_i. We consider the Hamilton function $H=g^{ij}(x,p)p_ip_j$. It results $g^{ij}(x,p)=\frac{1}{2}\dot\partial^i\dot\partial^jH$. q.e.d.

The Cartan space $\mathcal{C}^n$ being a particular Hamilton space, it has all properties exhibited in the previous paragraph. We prove that its canonical nonlinear connection $\overset{c}{N}$ from (9.2) has a very simple form. To this aim, we consider the Christoffel symbols :

$$\gamma^i_{jk}=\frac{1}{2}g^{ih}(\partial_kg_{jh}+\partial_jg_{kh}-\partial_hg_{jk}), \tag{10.1}$$

and denote

$$\gamma^0_{jk}=p_i\gamma^i_{jk},\quad \gamma^0_{j0}=\gamma^0_{jk}g^{kh}p_h. \tag{10.1′}$$

We have:

Theorem 10.2. *The canonical nonlinear connection $\overset{c}{N}$ of a Cartan space $\mathcal{C}^n$ has the coefficients :*

$$\overset{c}{N}_{ij}=\gamma^0_{ij}-\frac{1}{2}\gamma^0_{l0}\dot\partial^lg_{ij}. \tag{10.2}$$

Proof. Observing that in $\mathcal{C}^n$ we have

$$H=g^{ij}p_ip_j,\quad \frac{1}{2}\frac{\partial H}{\partial p_i}=g^{ij}p_j,\quad \frac{1}{2}\frac{\partial^2 H}{\partial p_i\partial x^k}=p_j\frac{\partial g^{ij}}{\partial x^k},\quad \frac{\partial H}{\partial x^k}=p_ip_j\frac{\partial g^{ij}}{\partial x^k}$$

and using (9.3) we get the equality (10.2). q.e.d.

The canonical connection $C\Gamma(\overset{c}{N})=(\overset{c}{H}{}^i_{jk}, \overset{c}{C}{}_i{}^{jk})$ of a Cartan space $\mathcal{C}^n$ is given by (9.5), where $\overset{c}{N}$ is the nonlinear connection (10.1).

Theorem 10.3. *The canonical connection $C\Gamma(\overset{c}{N})$ of a Cartan space $\mathcal{C}^n$ has the properties:*

1°. *It depends only on the fundamental function of the space $\mathcal{C}^n$;*

2°. *It is a d-metrical connection;*

3°. $\overset{c}{C}{}^{ijk}=-\frac{1}{4}\partial^i\partial^j\partial^k H,\quad \overset{c}{C}{}^{0jk}=0$;

4°. $\overset{c}{\tau}_{ij}=0,\ \underset{(i,\,j,\,k)}{\mathcal{S}}\ \overset{c}{R}_{ijk}=0$;

5°. $\overset{c}{\theta}=\overset{c}{\delta}p_i\wedge dx^i$ *is a symplectic structure on T^*M;*

6°. $\overset{c}{T}{}_{jk}{}^i=0,\quad \overset{c}{S}{}_i{}^{jk}=0,\quad \overset{c}{P}{}_{jk}{}^i=\overset{c}{P}{}_{kj}{}^i$;

7°. *It is a d-connection of the Cartan type: $p_{i|j}=0,\ p_i|^j=\delta^j_i$.*

8°. $H_{|k}=0,\quad H|^k=2\,g^{kj}p_j$;

9°. *Its deflexion tensor $\overset{\circ}{D}_{ij}$ vanishes;*

10°. *The Hamilton equation are $dx^i/dt-\partial\mathcal{H}/\partial p_i=0,\ \overset{c}{\delta}p_i/dt=0$. and it is equivalent to $dx^i/dt-\partial\mathcal{H}/\partial p_i=0,\ \overset{c}{\nabla}p_i/dt=0$;*

11°. *The integral curves of the Hamilton equations are h-paths.*

12°. $\overset{c}{R}_{ihl}+\overset{c}{R}_l{}^o{}_{ih}=0,\quad \overset{c}{P}_{ih}{}^l+\overset{c}{P}_i{}^o{}_h{}^l=0,\ \overset{c}{S}_i{}^{ohl}=0$;

13°. $\overset{c}{R}{}^{ij}{}_{hl}+\overset{c}{R}{}^{ji}{}_{hl}=0,\ \overset{c}{P}{}^{ij}{}_h{}^l+\overset{c}{P}{}^{ji}{}_h{}^l=0,\ \overset{c}{S}{}^{ijhl}+\overset{c}{S}{}^{jihl}=0$;

14°. $\overset{c}{S}{}^{ijhl}=\overset{c}{C}{}^{ril}\overset{c}{C}{}_r{}^{jh}-\overset{c}{C}{}^{rih}\overset{c}{C}{}_r{}^{jl}$.

The proofs of these statements are not difficult. Some of them are characteristic for the canonical connection. So we have:

Theorem 10.4. *In the Cartan spaces $\mathcal{C}^n$ there exists a unique d-connection $H\Gamma(N)$ which verifies the axioms:*

A_1. $g^{ij}{}_{|k}=0$; A_2. $g^{ij}|^k=0$; A^3. $T_{jk}{}^i=0$; A_4. $S_i{}^{jk}=0$; A_5. $D_{ij}=0$.

Proof. The canonical connection $C\Gamma(\overset{c}{N})$ — from the theorem 10.3 — has the properties A_1-A_5. It remains to prove its uniqueness. Assume, by reductio ad absurdum, that there exists a d-connection $H\Gamma(N)$, different from $C\Gamma(\overset{c}{N})$ which satisfies A_1-A_5. By the theorem 6.1 there exist the d-tensor fields X_{ij}, $\Omega^{is}_{rj}X^r_{sk}$, $\Omega^{is}_{rj}Y^{rk}_s$ not all equal to zero, such that

(a) $$N_{ij}=\overset{c}{N}_{ij}+X_{ij},\quad H^i_{jk}=\overset{c}{H}{}^i_{jk}+\overset{c}{C}{}_i{}^{ih}X_{kh}+\Omega^{is}_{rj}X^r_{sk},\quad C_i{}^{jk}=\overset{c}{C}{}_i{}^{jk}+\Omega^{is}_{ri}Y^{rk}_s.$$

$H\Gamma(N)$ verifies A_1, A_2. It satisfies A_4 iff

(b) $$\Omega^{is}_{rj}Y^{rk}_s=0.$$

Then A_3, A_5 give us

(c) $$X_{kj}=\frac{1}{2}(X^0_{jk}-g_{lj}X^l_{0k}).$$

(d) $$\overset{c}{C}{}_j{}^{ih}X_{kh}-\overset{c}{C}{}_k{}^{ih}X_{jh}=\Omega^{is}_{rk}X^r_{sj}-\Omega^{is}_{rj}X^r_{sk}.$$

Applying the Christoffel process in (d), we get

(d)′ $$g_{ir}X^r_{jk}-g_{jr}X^r_{ik}=2\,(\overset{c}{C}{}_{ik}{}^hX_{jh}-\overset{c}{C}{}_{jk}{}^hX_{ih}).$$

Contracting in (d)′ with $p^i=g^{ih}p_h$, we have

$$1/2\ \ (X^0_{jk}-g_{jr}X^r_{0k})=-\overset{c}{C}{}_{jk}{}^hX_{0h}.$$

This equality, via (c), implies

$$X_{kj}+\overset{c}{C}{}_{jk}{}^hX_{0h}=0.$$

A new contraction by p^j gives $X_{0j}=0$. From the previous equation, we obtain

(e) $$X_{kj}=0.$$

Finally, by Christoffel process, (d) leads us to

(f) $$\Omega^{is}_{rj}X^r_{sk}=0.$$

Summarizing, we see that (b), (e) and (f) are contradictory with our assumption. q.e.d.

§ 11. M^{*n}-spaces with regular metric. Our purpose is now to study some classes of generalized Hamilton spaces in which the N-canonical connection $C\Gamma(N)$ is determined only by the fundamental tensor $g^{ij}(x, p)$ of the space. Looking at the Theorem 7.2, we must prove the existence of some spaces M^{*n} in which the nonlinear connection are determined by $g^{ij}(x, p)$ only.

Proposition 11.1. *If $M^{*n}=(M, g^{ij}(x, p))$ is a generalized Hamilton space, then the function*

(11.1) $$\tilde{H}(x, p)=g^{ij}(x, p)p_ip_j,$$

is globally defined on T^*M.

Due to the tensorial character of g^{ij} and p_i, the property holds.

Proposition 11.2. *The Hessian of the function $\tilde{H}$*

(11.2) $$\tilde{g}^{ij}=\frac{1}{2}\dot{\partial}^i\dot{\partial}^j\tilde{H}$$

is a d-tensor field on M, which is of the type (2,0) and symmetric.

Now, we can give.:

Definition 11.1. *M^{*n} is called space with weakly regular metric if the d-tensor field $\tilde{g}^{ij}(x, p)$ is nondegenerate.*

If M^{*n} is with weakly regular metric we can associate to it the Hamilton space $\tilde{H}^n=(M, \tilde{H})$. The canonical nonlinear connection $\tilde{N}$ of the space $\tilde{H}^n$, given by the theorem 9.2. provides us with a nonlinear connection on T^*M which depends only on the fundamental tensor $g^{ij}(x, p)$ of M^{*n}.

Theorem 11.1. *In a generalized Hamilton space M^{*n} with weakly regular metric there exists a nonlinear connection $\tilde{N}$ on T^*M determined by the fundamental tensor $g^{ij}(x, p)$, only. The coefficients $\tilde{N}_{ij}$ of $\tilde{N}$ are given by*

$$\tilde{N}_{ij}=-\frac{1}{2}\tilde{g}_{jh}\left\{\frac{1}{4}\tilde{g}_{ik}\dot{\partial}^k(\tilde{H}, \dot{\partial}^h\tilde{H})+\dot{\partial}^h\partial_i\tilde{H}\right\}. \tag{11.3}$$

Theorem 11.2. *In a generalized Hamilton space M^{*n} with weakly regular metric the following d-connection $C\tilde{\Gamma}(\tilde{N})$*

$$\begin{aligned}\tilde{H}^i_{jk}&=\frac{1}{2}g^{ih}(\tilde{\delta}_j g_{hk}+\tilde{\delta}_k g_{hj}-\tilde{\delta}_h g_{jk}),\\ \tilde{C}_i{}^{jk}&=-\frac{1}{2}g_{ih}(\dot{\partial}^j g^{hk}+\dot{\partial}^k g^{hj}-\dot{\partial}^h g^{jk})\end{aligned} \tag{11.4}$$

has the properties:

1°. *It depends only on the fundamental tensor field $g^{ij}(x, p)$;*

2°. *It is a d-metrical connection with respect to the metric $g^{ij}(x, p)$;*

3°. *It is the unique d-metrical connection $H\Gamma(\tilde{N})$ whose torsions $T_{jk}{}^i=0$, $S_i{}^{jk}=0$.*

Consequently, $C\tilde{\Gamma}(\tilde{N})$ is the canonical connection of the generalized Hamilton space M^{*n}, with weakly regular metric.

Definition 11.2. *A generalized Hamilton space M^{*n} is said tobe with regular metric if it is with the weakly regular metric and*

$$\frac{1}{2}\dot{\partial}^i\tilde{H}=g^{ij}(x, p)\,p_j. \tag{11.5}$$

The Cartan spaces $\mathcal{C}^n$ are examples of the generalized Hamilton spaces with regular metric.

Theorem 11.3. *If the space M^{*n} is with regular metric, then the associated Hamilton function $\tilde{H}(x, p)$ is 2-homogeneous with respect to p_i.*

Proof. From (11.4) it result $p_i\dot{\partial}^i\tilde{H}=2\tilde{H}$. q.e.d.

Therefore the Hamilton space $\tilde{H}^n$ associate to M^{*n} is a Cartan space $\tilde{\mathcal{C}}^n$. We can consider the canonical nonlinear connection of the space $\tilde{\mathcal{C}}^n$. Let $\tilde{\gamma}^i_{jk}$ be the Christoffel symbols of the d-tensor field $\tilde{g}^{ij}$ from (11.2).

Theorem 11.4. *If the generalized Hamilton space M^{*n} is with regular metric, then*

$$\tilde{N}_{ij}=\tilde{\gamma}^0_{ij}-1/2\tilde{\gamma}^0_{i0}\dot{\partial}^l\tilde{g}_{ij} \tag{11.6}$$

are the coefficients of a nonlinear connection on T^*M determined only by the fundamental tensor $g^{ij}(x, p)$ of the space M^{*n}.

The d-canonical metrical connection of the space M^{*n} with regular metric is given by (11.4), where $\tilde{N}$ is given by (11.6).

Theorem 11.5. *The Hamilton equations of the space M^{*n} with weakly regular (or regular) metric are*

$$\frac{dx^i}{dt}-\frac{\partial\tilde{\mathcal{H}}}{\partial p_i}=0, \qquad \frac{\tilde{\delta}p_i}{dt}+\frac{\tilde{\delta}\tilde{\mathcal{H}}}{\delta x^i}=0,$$

where $\tilde{\mathcal{H}}=1/2\,\tilde{H}$ and $\tilde{\delta}p_i$, $\tilde{\delta}_i$ are expressed by means of $\tilde{N}$ from (11.6).

§ 12. Mink^{*n} spaces. Another important class of generalized Hamilton spaces is that of the local Minkowski type.

Definition 12.1. *A space $M^{*n}=(M, g^{ij}(x, p))$ will be called of local Minkowski type — and will be denoted by Mink^{*n} — if T^*M can be covered by a system of local charts in which $g^{ij}(x, p)$ does not depend on x^i.*

Clearely, we can speak about Mink^{*n} spaces with weakly regular metric or with the regular metric. Also we can meet the spaces H^n or $\mathcal{C}^n$ which are Mink^{*n} spaces.

A dual result of the one due to S. Watanabe, [11, 12], is given by:

Theorem 12.1. *If the N-canonical connection $C\Gamma(N)$ of a generalized Hamilton space M^{*n} has the property*

$$D_{ij}=0, \quad R_{ijk}=0, \quad R_j{}^i{}_{kh}=0, \quad P_{jk}{}^i=0, \quad C^{ijk}{}_h=0, \tag{12.1}$$

then it is a Mink^{*n} space.

Proof. From (3.7) we have

$$P_j{}^i{}_h{}^l=\partial^l H^i_{jk}. \tag{a}$$

Then $P_j{}^i{}_h{}^l-P_h{}^i{}_j{}^l=0$. Denoting $P^{jihl}=g^{jr}g^{hs}P_r{}^i{}_s{}^l$ and using the theorem 7.6 we obtain

$$P^{jihl}-P^{hijl}=0, \quad P^{jihl}+P^{ijhl}=0.$$

Applying the Christoffel process it results

$$P_j{}^i{}_h{}^l=0. \tag{b}$$

By means of (a) it follows

$$\dot{\partial}^l H^i_{jk}=0. \tag{b$'$}$$

Consequently, the coefficients H^i_{jk} from (7.4) does not depend on p_i. Using the hypothesis we get

$$R_j{}^i{}_{hl}=\underset{(h,\,l)}{\mathcal{A}}\{\partial_l H^i{}_{jh}+H^r_{jh}(x)H^i_{rl}(x)\}=0. \tag{c}$$

Identically as in the Riemannian case, at the point $x\in M$, we can determine a local chart (U, x^i) in which $H^i_{jk}(x)=0$. Then, in the chart $(\pi^{*-1}(U), (x^i, p_i))$ of T^*M, we have $D_{ij}=N_{ji}-p_r H^r_{ij}=N_{ji}=0$.

Consequently $\delta_j=\partial_j$ and $g^{ij}{}_{|k}=\partial g^{ij}/\partial x^k=0$. q.e.d.

Corollary 12.1. *If the N-canonical connection $C\Gamma(N)$ of the space M^{*n} is of the Cartan type and moreover it has the property:*

$$R_j{}^i{}_{kr}=0,\ P_{jk}{}^i=0,\ C_i{}^{jk}{}_{|h}=0,$$

then $M^{*n}=\mathrm{Mink}^{*n}$.

Indeed, the propositions 3.3 and 3.4 show that $D_{ij}=0$, $R_{ijk}=0$. It results that (12.1) are satisfied.

Theorem 12.2. *A generalized Hamilton space M^{*n} with weakly regular metric is a Mink^{*n} space with weakly regular metric if and only if its N-canonical connection $C\Gamma(N)$ has the property* (12.1).

Proof. If $C\Gamma(N)$ verifies (12.1), from Theorem 12.1, we have $M^{*n}=\mathrm{Mink}^{*n}$.

Conversely, let $M^{*n}=\mathrm{Mink}^{*n}$ be with the weakly regular metric. Assume that on the open set $\pi^{*-1}(U)$ we have $g^{ij}(x,p)=g^{ij}(p)$. The associated Hamilton function

$$\text{(a)}\qquad \tilde H=g^{ij}(p)p_ip_j$$

depends only on p_i. It follows, from (11.3), that

$$\text{(b)}\qquad \tilde N_{ij}=0.$$

Then $\tilde\delta_k=\tilde\partial_k$ and (11.4) gives us

$$\text{(c)}\qquad \tilde H^i_{jk}=0,\qquad C_i{}^{jk}{}_{|h}=0.$$

But (b) and (c) imply the conditions (12.1). q.e.d.

Theorem 12.3. *A Cartan space $\mathcal{C}^n$ is a Mink^{*n} space if and only if its canonical connection $C\Gamma(\overset{c}{N})$ has the properties:*

$$(12.2)\qquad \overset{c}{R}_i{}^j{}_{hk}=0,\qquad \overset{c}{C}_i{}^{jk}{}_{|h}=0.$$

Proof. If $\mathcal{C}^n=\mathrm{Mink}^{*n}$, then (12.1) holds because $\mathcal{C}^n$ is with regular metric and Theorem 12.2 can be applied.

Conversely, if $\mathcal{C}^n$ has the property (12.2), its canonical connection $C\Gamma(\overset{c}{N})$ being of the Cartan type, we have

$$\text{(a)}\qquad \overset{c}{D}_{ij}=0,\qquad \overset{c}{R}_{ijk}=0.$$

It remains to prove that $\overset{c}{P}_{ji}{}^k=0$.

From (3.7), we obtain:

$$\overset{c}{P}_j{}^i{}_h{}^l=\partial^l \overset{c}{H}{}^i_{jh}+\overset{c}{C}_j{}^{is}\overset{c}{P}_{sh}{}^l.$$

Interchanging j and h, we get

$$\overset{c}{P}_j{}^i{}_h{}^l-\overset{c}{P}_h{}^i{}_j{}^l=\overset{c}{C}_j{}^{is}\overset{c}{P}_{sh}{}^l-\overset{c}{C}_h{}^{is}\overset{c}{P}_{sj}{}^l$$

or

$$\overset{c}{P}{}^{jihl}-\overset{c}{P}{}^{hijl}=\overset{c}{C}{}^{jis}\overset{c}{P}_{sh}{}^l-\overset{c}{C}{}^{his}\overset{c}{P}_{sj}{}^l.$$

Applying the Christoffel process and observing that $\overset{c}{P}{}^{jihl}+\overset{c}{P}{}^{ijhl}=0$, we get

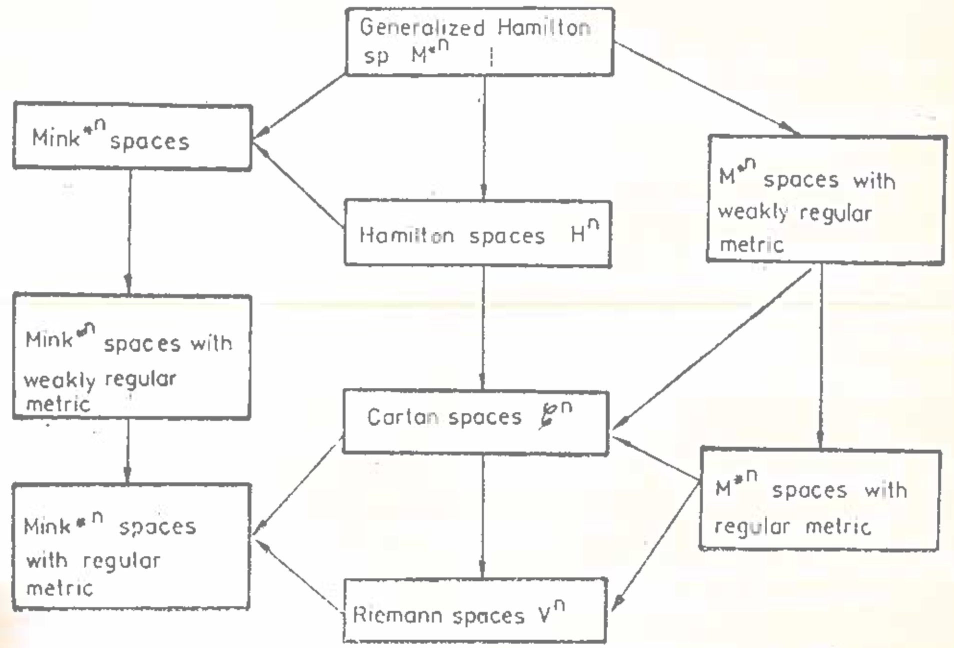
$$\text{(b)}\qquad \overset{c}{P}{}^{ihjl}=\overset{c}{C}{}^{jis}\overset{c}{P}_s{}^{hl}-\overset{c}{C}{}^{jhs}\overset{c}{P}_s{}^{il}$$

Now, using the Theorem 10.3, we have $\overset{c}{P}{}^{iojl}=-\overset{c}{P}{}^{ijl}$. Then (b) implies

$$\text{(c)}\qquad \overset{c}{P}_{ij}{}^l=g_{kj}\overset{c}{C}_i{}^{ks}\overset{c}{P}_s{}^{ol}$$

Contracting in (c) with $g^{jh}p_h$ weobtain $\overset{c}{P}_j{}^{ol}=0$. Finally (c) gives us $\overset{c}{P}_{jk}{}^i=0$. q.e.d.

Now we can make the scheme of the projections of the generalized Hamilton spaces M^{*n} on mentioned spaces



§ 13. The Hamilton spaces of the some physical fields. We consider some special classes of the Hamilton spaces, in which the fundamental function originates from the Hamiltonian theory of physical fields. All examples are take from the Pyragas-Alexandrov papers [15, 16].

A. Gravitational field. Let $H^n=(M,H)$ be the Hamilton spaces, in which $\mathcal{H}=\frac{1}{2}H$ is the Hamilton function

$$(13.1)\qquad \mathcal{H}=(2mc)^{-1}a^{ij}(x)p_ip_j,$$

where m and c have the known meanings, $a^{ij}(x)$ is a Riemann metric on M and p_i are the components of the cinetic momentum. H^n is a Cartan space $\mathcal{C}^n$. The velocity field is

$$\text{(a)}\qquad v^i=\partial^i\mathcal{H}=(mc)^{-1}a^{ij}(x)p_j.$$

and the fundamental tensor $g^{ij}(x,p)$ is:

$$\text{(b)}\qquad g^{ij}=(mc)^{-1}a^{ij}(x).$$

The Christoffel symbols of $g^{ij}(x)$ coincide to those of the tensor field $a^{ij}(x)$.

The canonical connection $C\Gamma(\overset{c}{N})$ is given by

(c) $$\overset{c}{N}_{ij}=\gamma^{0}_{ij},\quad \overset{c}{H}{}^{i}_{jk}=\gamma^{i}_{jk},\quad \overset{c}{C}{}_{i}^{jk}=0.$$

All properties of this $C\Gamma(\overset{c}{N})$ are mentioned in Theorem 10.3.

$\mathcal{C}^n$ is of the local Minkowski type if and only if the Riemann space $(M, a^{ij}(x))$ is flat.

The Hamilton equations of $\mathcal{C}^n$ are

$$\frac{dx^i}{dt}=\frac{1}{mc}a^{ij}(x)p_j,\qquad \frac{\overset{c}{\nabla} p_i}{dt}=0.$$

Their integral curves are h-paths of the space $\mathcal{C}^n$.

B. Electromagnetic field in the presence of a gravitational field. Let $H^n=(M, H)$ be the Hamilton space, in which $\mathcal{H}=\frac{1}{2}H$ is

(13.2) $$\mathcal{H}=(2mc)^{-1}a^{ij}(x)p_ip_j-e(mc^2)^{-1}p_iA^i(x)+e^2(2mc^3)^{-1}A^i(x)A_i(x),$$

where m, c, $a^{ij}(x)$ have the menings from A., e is the charge of a test-body, $A_i(x)$ is the vector-potential of an electromagnetic field.

The space H^n is not a Cartan space. The velocity field is given by

(a) $$v^i=\dot{\partial}^i\mathcal{H}=(mc)^{-1}a^{ij}(x)p_j-e(mc^2)^{-1}A^i(x).$$

The fundamental tensor of the space is

(b) $$g^{ij}=(mc)^{-1}a^{ij}(x).$$

It follows that $g_{ij}=mca_{ij}(x)$.

The canonical nonlinear connection (9.3) has the coefficients

(c) $$\overset{c}{N}_{ik}=\gamma^{0}_{ik}+(e/c)\,(A_{i,k}+A_{k,i}),$$

where $A_i=A^ja_{ji}$.

The d-tensor of the electromagnetic field is

$$F_{ij}=A_{i|j}-A_{j|i},$$

where h-covariant derivative is considered with respect to $C\Gamma(\overset{c}{N})=(\overset{c}{H}{}^{i}_{jk}, \overset{c}{C}{}_{i}^{jk})$

(d) $$\overset{c}{H}{}^{i}_{jk}=\gamma^{i}_{jk},\quad \overset{c}{C}{}_{i}^{jk}=0.$$

Therefore, $C\Gamma(\overset{c}{N})$ is a d-connection of the Rund type.

Observing that the covariant differential with respect to $C\Gamma(\overset{c}{N})$ is

$$\overset{c}{\nabla} p_i/dt=D_{i1}((mc)^{-1}a^{ik}p_k-e(mc^2)^{-1}A^i)+\overset{c}{\delta} p_i/dt$$

the Hamilton equations take a very simple form

$$dx^i/dt=(mc)^{-1}\ (p_k-(e/c)\,A_k)a^{ik}(x),$$

$$\overset{c}{\nabla} p_i/dt=e(mc^2)^{-1}\,(p_k-(e/c)\,A_k)\,A^k{}_{|i}.$$

C. Gravitational, potential and electromagnetic fields. In this case $\mathcal{H}=1/2H$ is given by

(13.3) $$\begin{aligned}\mathcal{H}(2c(m+u/c^2))^{-1}\,a^{ij}(x)p_ip_j-(c^2(m+u/c^2))^{-1}\,p_iA^i(x)+\\+e^2(2c^3(m+u/c^2))^{-1}\,A_i(x)A^i(x)-(2c)^{-1}\,U(x)\end{aligned}$$

where $U(x)$ is a force function.

The space $H^n=(M, H)$ is not a Cartan space. The velocity field is:

$$v^i=\dot{\partial}\mathcal{H}/\dot{\partial}p_i=(c(m+U/c^2))^{-1}\,(a^{ij}(x)p_j-(e/c)A^i(x))$$

The fundamental tensor field $g^{ij}(x, p)$ is

$$g^{ij}=(c(m+U/c^2))^{-1}\quad a^{ij}(x)$$

It follows that the Riemann spaces $(M, g^{ij}(x))$ and $(M, a^{ij}(x))$ are conformal.

The canonical nonlinear connection of the space H^n is

$$\overset{c}{N}_{ik}=\gamma^{0}_{ik}+e(2c^2(m+U/c^2))^{-1}\,\{g_{hi}A^h{}_{|k}+g_{hk}A^h{}_{|i}-(A_iU_{|k}+A_kU_{|i})\},$$

the h-covariant derivative being considered with respect to the canonical connection $C\Gamma(\overset{c}{N})$

$$\overset{c}{H}{}^{i}_{jk}=\gamma^{i}_{jk},\quad \overset{c}{C}{}_{i}^{jk}=0.$$

$C\Gamma(\overset{c}{N})$ is a d-connection of the Rund type.

The Hamilton equations are

$$dx^i/dt=(c(m+U/c^2))^{-1}\ (a^{ij}(x)p_j-(e/c)A^i(x)),$$

$$\begin{aligned}\overset{c}{\nabla} p_i/dt=e(c^2(m+U/c^2))^{-1}\,\{p_kA^k{}_{|i}-(e/c)\,a^{jk}A_{j|i}A_k-\\-e(c^2(m+U/c^2))^{-1}\ (p_lA^l-(2c)^{-1}eA_lA^l)U_{|i}\}+(2c)^{-1}U_{|i}.\end{aligned}$$

And $F_{ij}=A_{j|i}-A_{i|j}$ is the tensor of electromagnetic field.

These examples show us the necessity to apply the method of Jacobi [15, 16], for integration of Hamilton equations.

Part III. H^{2n} — MODEL OF A SPACE M^{*n}

The theory exposed in the previous two parts of this paper has an interesting geometrical interpretation on the total space of cotangent bundle (T^*M, π^*, M). Namely, the generalized Hamilton spaces M^{*n} are identified with the almost Hermitian spaces H^{2n}, of dimension $2n$ on T^*M.

The Hamilton spaces and the Cartan spaces have as models the Kählerian spaces K^{2n} on T^*M.

§ 14. N-connections of Hamilton type. As we have seen in the part one if we fix a nonlinear connection N on T^*M we get a local basis of the $\mathcal{F}(T^*M)$-module of the vector fields $\mathcal{X}(T^*M)$, given by $(\delta_i, \dot\partial^i)$ adapted to the horizontal distribution N and vertical distribution V. If $N_{ij}(x, p)$ are the coefficients of N we recall the notations

$$\delta_i = \partial_i + N_{ij}\dot\partial^j, \quad \delta p_i = dp_i - N_{ji}dx^j. \tag{14.1}$$

On the total space T^*M we have considered the almost symplectic structure θ associated to the nonlinear connection N:

$$\theta = \delta p_i \wedge dx^i. \tag{14.2}$$

The tensor field $\tilde\theta$ of θ in the adapted basis $(\delta_i, \dot\partial^i)$ has the components.

$$\tilde\theta(\delta_i, \delta_j) = 0, \ \tilde\theta(\delta_i, \dot\partial^j) = -\delta_i^j, \ \tilde\theta(\dot\partial^j, \delta_i) = \delta_i^j, \ 0 = \tilde\theta(\dot\partial^i, \dot\partial^j). \tag{14.2'}$$

Definition 14.1. *It is called N-connection of the Hamilton type on T^*M a linear connection ∇ on T^*M with the properties*

1°. ∇ preserves by parallelism each of the distributions N and V;

2°. $\tilde\theta$ is covariant constant with respect to ∇.

Theorem 14.1. *A linear connection ∇ on T^*M is an N-connection of the Hamilton type if and only if in the adapted basis $(\delta_i, \dot\partial^i)$ it is expressed as follows*

$$\begin{aligned} &\nabla_{\delta_k}\delta_j = H^i_{jk}\delta_i, \quad \nabla_{\delta_k}\dot\partial^j = -H^j_{ik}\dot\partial^i, \\ &\nabla_{\dot\partial^k}\delta_i = C_i{}^{jk}\delta_j, \quad \nabla_{\dot\partial^k}\dot\partial^j = -C_i{}^{jk}\dot\partial^i, \end{aligned} \tag{14.3}$$

where $(H^i_{jk}, C_i{}^{jk})$ are the coefficients of a d-connection $H\Gamma(N)$ on M.

Proof. If ∇ is an N-connection of the Hamilton type on T^*M, then, firstly, we have

$$\nabla_{\delta_k}\delta_j = H^i_{jk}\delta_i, \quad \nabla_{\delta_k}\dot\partial^j = \tilde H^j_{ik}\dot\partial^i, \quad \nabla_{\dot\partial^k}\delta_i = C_i{}^{jk}\delta_j, \quad \nabla_{\dot\partial^k}\dot\partial^j = \tilde C_i{}^{jk}\dot\partial^i. \tag{a}$$

Now writting $\nabla_{\delta_k}\tilde\theta = 0$, $\nabla_{\dot\partial^k}\tilde\theta = 0$, from (14.2)$'$, we obtain $\tilde H^j_{ik} = -H^j_{ik}$, $\tilde C_i^{jk} = -C_i{}^{jk}$. That means, by virtue of (a), that (14.3) holds.

Observing now that, with respect to (1.1), we have $\bar\delta_k = \partial x^i/\partial\bar x^k\delta_i$, $\bar{\dot\partial}^k = \partial\bar x^k/\partial x^i\dot\partial^i$, it results from (14.3) that $(H^i_{jk}, C_i{}^{jk})$ have the laws of transformations as the coefficients of a d-connection $H\Gamma(N)$ on M.

Conversely, if $(H^i_{jk}, C_i{}^{jk})$ are the coefficients of a d-connection $H\Gamma(N)$ on M, then (14.3) globally determines on T^*M a linear connection ∇ which has the properties 1° and 2° from the definition 14.1. q.e.d.

Corollary 14.1. *There exists a bijection between the set of d-connections $\{H\Gamma(N)\}$ on M and the set of N-connections of the Hamilton type $\{\nabla\}$ on T^*M.*

The N-connection of the Hamilton type ∇ which corresponds to the d-connection $H\Gamma(N)$ is called the lift to T^*M of the d-connection $H\Gamma(N)$.

Corollary 14.2. *The non linear connection N being fixed, a transformation of N-connections of the Hamilton type $\nabla \to \bar\nabla$ implies the transformations (6.4), with $A_{ij} = 0$, between their coefficients: $(H^i_{jk}, C_i{}^{jk}) \to (\bar H^i_{jk}, \bar C_i{}^{jk})$.*

Theorem 14.2. *If ∇ is the lift of d-connection $H\Gamma(N)$, then in the basis $(\delta_i, \dot\partial^i)$ the torsion T of ∇ is expressed by:*

$$T(\delta_k, \delta_j) = T_{jk}{}^i\delta_i + R_{jki}\dot\partial^i, \quad T(\dot\partial^k, \delta_j) = C_j{}^{ik}\delta_i + P_{ji}{}^k\dot\partial^i, \quad T(\dot\partial^k, \dot\partial^j) = S_i{}^{jk}\dot\partial^i, \tag{14.4}$$

where $T_{jk}{}^i$, $S_i{}^{jk}$, R_{jki}, $P_{jk}{}^i$ and $C_i{}^{jk}$ are the d-torsions of $H\Gamma(N)$.

Proof. By

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \ \forall X, Y \in \mathcal{X}(T^*M)$$

and (14.3), we get (14.4) where the coefficients are given by (3.6).

Theorem 14.3. *The lift ∇ of the d-connection $H\Gamma(N)$ is a symmetric connection if and only if all d-torsions of $H\Gamma(N)$ vanish.*

Theorem 14.4. *The curvature of the lift ∇ of the d-connection $H\Gamma(N)$ in the adapted basis $(\delta_i, \dot\partial^i)$ is expressed by*

$$\begin{aligned} &R(\delta_l, \delta_k)\delta_j = R_j{}^i{}_{kl}\delta_i, \quad R(\delta_l, \delta_k)\dot\partial^j = -R_i{}^j{}_{kl}\dot\partial^i, \quad R(\dot\partial^l, \delta_k)\delta_j = P_j{}^i{}_k{}^l\delta_i, \\ &R(\dot\partial^l, \delta_k)\dot\partial^i = -P_j{}^i{}_k{}^l\dot\partial^j, \quad R(\dot\partial^l, \partial^k)\delta_j = S_j{}^{ikl}\delta_i, \quad R(\dot\partial^l, \dot\partial^k)\dot\partial^i = -S_j{}^{ikl}\dot\partial^j, \end{aligned} \tag{14.5}$$

where $R_j{}^i{}_{kl}$, $P_j{}^i{}_k{}^l$ and $S_i{}^{jkl}$ are the curvature tensors of $H\Gamma(N)$.

Proof. By

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z, \quad \forall X, Y, Z \in \mathcal{X}(T^*M)$$

and (14.3), we get (14.5) where the coefficients are given by (3.7).

Theorem 14.5. *The lift ∇ of the d-connection $H\Gamma(N)$ is a flat linear connection if and only if all d-curvature tensors of $H\Gamma(N)$ vanish.*

Theorem 14.6. *The Ricci formulae satisfies by the lift ∇ of the d-connection $H\Gamma(N)$, in the basis $(\delta_i, \dot\partial^i)$ are given by the formulae (3.8).*

Proof. A vector field X can be written as $X = X^i(x, p)\delta_i + X_i(x, p)\dot\partial^i$, where $X^i(x, p)$, $(X_i(x, p))$ is a d-vector (d-covector) field on M. Then the Ricci formula for ∇

$$[\nabla_X, \nabla_Y]Z = R(X, Y)Z + \nabla_{[X, Y]}Z$$

in the basis $(\delta_i, \dot\partial^i)$ gives us just the formulae (3.8).

Theorem 14.7. *The Bianchi identities of the linear connection ∇ on T^*M, which is the lift of the d-connection $H\Gamma(N)$, in the adapted basis $(\delta_i, \dot\partial^i)$ are given by the formulae (4.4)–(4.8).*

Let $X = X^H = X^i\delta_i$ be a horizontal field of vectors. $X^i(x, p)$ is a d-vector field on M. We consider a curve S in T^*M

$$S: \begin{cases} x^i = x^i(t), \ \text{rank} \left\| \dfrac{dx^i}{dt} \right\| = 1, \ t \in (a, b) \\ p_i = p_i(t), \quad p(t) \neq 0 \end{cases} \tag{14.6}$$

and its tangent vector:

$$\dot S = dx^i/dt\ \delta_i + \delta p_i/dt\ \dot\partial^i. \tag{14.6'}$$

Then, the covariant derivative of $X^H = X$, with respect to the lift $\nabla_{\dot S}$
of the d-connection $H\Gamma(N)$, on S, is:

$$\nabla_{\dot S} X = (X^i{}_{|k} dx^k/dt + X^i|^k \delta p_k/dt)\delta_i. \tag{14.7}$$

But, the second member of (14.7) is just the covariant differential of the d-
vector field $X^i(x, p)$ along the 1-parameter family S, with respect to $H\Gamma(N)$.
So that the field $X = X^H$ is parallel on S with respect to ∇ if and only if $\nabla_{\dot S} X = 0$.
Thus we have obtained the geometrical interpretation of the parallelism of
d-vector field X^i on S with respect to $H\Gamma(N)$.

On this way we can give the geometrical interpretation for the notions
of h- and v-paths with respect to $H\Gamma(N)$.

§ 15. Almost Hermitian model of the space M^{*n}. Let N be a fixed nonli-
near connection on T^*M and $M^{*n} = (M, g^{ij})$ be a generalized Hamilton space.
The fundamental tensor $g^{ij}(x, p)$ determines, on total space T^*M, the tensor
field

$$J = g_{ij}\dot\partial^i \otimes dx^j - g^{ij}\delta_i \otimes \delta p_j \tag{15.1}$$

Theorem 15.1. 1°. *J is a tensor field of the type (1, 1) globally defined on T^*M.*

2°. *J is an almost complex structure on T^*M.*

Proof. 1°. From (15.1), we immediately see that J is a tensor field of
the type (1,1) on T^*M. Using (1.6), (1.6)' we have the global character of on
T^*M.

2°. By means of (15.1):

$$J(\delta_i) = g_{ij}\dot\partial^j, \quad J(\dot\partial^i) = -g^{ij}\delta_j, \tag{15.1$'$}$$

it follows

$$J \circ J = -I \quad \text{q.e.d.} \tag{15.1$''$}$$

The existence of d-tensor field $g^{ij}(x, p)$ of the generalized Hamilton
space M^{*n} allows us to consider the tensor field

$$G = g_{ij}(x, p)dx^i \otimes dx^j + g^{ij}(x, p)\delta p_i \otimes \delta p_j. \tag{15.2}$$

Theorem 15.2. 1°. *G is a tensor field of the type (0,2) globaly defined on
T^*M.*

2°. *G is a Riemann metric on T^*M.*

Proof. 1°. The equality (15.2) show us that G is a tensor field on T^*M
of the type (0,2). Again (1.6), (1.6)' and the law of transformation of $g^{ij}(x, p)$
prove the global character of G on T^*M.

2°. Since

$$G(\delta_i, \delta_j) = g_{ij}(x, p),\ G(\delta_i, \dot\partial^j) = 0,\ G(\dot\partial^i, \delta_j) = 0,\ G(\dot\partial^i, \dot\partial^j) = g^{ij}(x, p), \tag{15.2$'$}$$

the announced property holds.

Proposition 15.1. *The distributions N and V are orthogonal with respect
to the metric G.*

We say that G is the N-lift of the metrical structure $g^{ij}(x, p)$ of M^{*n}.

Theorem 15.3. 1°. *The pair (G, J) is an almost Hermitian structure on
T^*M.*

2°. *$\theta = \delta p_i \wedge dx^i$ is the associated almost symplectic structure to (G, J).*

Proof. 1°. The equation $G(JX, JY) = G(X, Y)$ is verified by means of
(15.1)' and (15.2)'.

2°. From (14.2)', (15.1)' and (15.2)' we easily see that $\tilde\theta(X, Y) = G(X,
JY)$, $\forall X, Y \in \mathcal{X}(T^*M)$ q.e.d.

Now we can state:

Theorem 15.4. *If G is a Riemann metric on T^*M, N is the orthogonal
supplementary distribution of V by G and (G, J) is the almost Hermitian structure
for which $\theta = \delta p_i \wedge dx^i$ is the associated almost symplectic structure, then there
exists a unique metrical structure $g^{ij}(x, p)$ on M such that G is its N-lift.*

Proof. Because N is the orthogonal supplementary distribution to the
vertical distribution V, in the adapted basis $(\delta_i, \dot\partial^i)$ of N and V, we have:

$$G = g_{ij}(x, p)dx^i \otimes dx^j + h^{ij}(x, p)\delta p_i \otimes \delta p_j.$$

It follows that $g_{ij}(x, p)$ and $h^{ij}(x, p)$ are two nondegenerate symmetric d-
tensor fields on M. From $G(JX, JY) = G(X, Y)$ and $\tilde\theta(X, Y) = G(X, JY)$ we
get without difficulties $h^{ij}(x, p) = g^{ij}(x, p)$, where $\|g^{ij}(x, p)\| = \|g_{ij}(x, p)\|^{-1}$.
Consequently, $g^{ij}(x, p)$ is uniquely determined, $M^{*n} = (M, g^{ij}(x, p))$ is a ge-
neralized Hamilton space and the lift of $g_{ij}(x, p)$ is just G. q.e.d.

Let $H^{2n} = (T^*M, (G, J))$ be the almost Hermitian space determined by
the structure (G, J).

Theorem 15.4 shows that, if the nonlinear connection N is fixed, there
exists a bijection between the set of the spaces M^{*n} and the set of the spaces
H^{2n}. Therefore H^{2n} is called the N-almost Hermitian model of the space M^{*n}.

Theorem 15.5. *The N-almost Hermitian model of the generalized Hamilton
space M^{*n} is an almost Kählerian space if and only if the nonlinear connection
N has the properties:*

$$\dot\partial^i \tau_{jk} = 0, \quad \mathop{\mathcal{S}}_{(i,j,k)} R_{ijk} = 0. \tag{15.3}$$

Indeed, the 2-form θ from (14.2) is a closed 2-form if and only if (15.3)
holds.

Theorem 15.6. *The generalized Hamilton spaces, with the weakly regular
metric, Hamilton spaces and Cartan spaces admit $\overset{c}{N}$-models which are the almost
Kählerian spaces.*

Now we examine the case when the N-almost Hermitian model H^{2n} is a
Hermitian space. Using the Nijenhuis tensor of the almost complex structure J

$$\mathcal{N}(X, Y) = [X, Y] + J[X, JY] + J[JX, Y] - [JX, JY],\ \forall X, Y \in \mathcal{X}(T^*M)$$

the formulae (2.4), (15.1)' and the notations $C_{ijk} = C_i{}^{rs}g_{rj}g_{sk}$ etc., we get:

Theorem 15.7. *Let $C\Gamma(N)$ be the N-canonical connection of the space M^{*n}.
The almost complex structure J is integrable if and only if*

$$R_{ijk} = C_{jki} - C_{ikj},\ P_{ikj} - P_{ijk} = 0. \tag{15.4}$$

Corollary 15.1. *If H^{2n} is a Hermitian space and N is an integrable distribution, then M^{*n} is a Hamilton space.*

Indeed, from $R_{ijk}=0$, $C^{ijk}\neq 0$ we have C^{ijk} completely symmetric.

Theorem 15.8. *The N-model H^{2n} of a Hamilton space H^n is a Kählerian space if and only if*

$$\overset{c}{R}_{ijk}=0,\quad \overset{c}{P}_{ijk} \text{ is completely symmetric.} \tag{15.5}$$

Proof. Theorem 15.6 shows that the model H^{2n} of the space H^n is an almost Kählerian space. If H^{2n} is a Kählerian space, Theorem 15.7 shows us that (15.4) hold. But C^{ijk} being totally symmetric it follows $R_{ijk}=0$ and P_{ijk} is completely symmetric.

Conversely, (15.5) implies (15.4), such that H^{2n} is a Kählerian space.

Theorem 15.9. *If M^{*n} a Mink^{*n} space with weakly regular metric, then its N-model H^{2n} is a Kähler space.*

Indeed, one applies the Theorems 12.2 and 15.7 and the result is immediate.

Corollary 15.2. *The Cartan space $\mathscr{C}^n$ of the local Minkowski type has as the $\overset{c}{N}$-model H^{2n} a Kähler space.*

§ 16. The almost Hermitian connections on T^*M. It is interesting to determine the linear connections on the total space of the cotangent bundle, T^*M, compatible with the almost Hermitian structure (G, J).

Definition 16.1. *A linear connection ∇ on T^*M is called N-almost Hermitian connection if:*

1°. *∇ is an N-connection of the Hamilton type.*

2°. $\nabla_X G=0$, $\forall X\in\mathscr{X}(T^*M)$.

Proposition 16.1. *If ∇ is an N-almost Hermitian connection on T^*M, then $\nabla_X J=0$, $\forall X\in\mathscr{X}(T^*M)$.*

Proof. $\nabla_X\tilde{\theta}=0$, $\nabla_X G=0$, and $\tilde{\theta}(X, Y)=G(X, JY)$, $G(JX, JY)=G(X, Y)$ imply $\nabla_X J=0$.

Theorem 16.1. 1°. *There exists a unique N-almost Hermitian connection on T^*M with the property $T_{jk}{}^{i}=0$, $S_i{}^{jk}=0$.*

2°. *N-almost Hermitian connection ∇ from 1° is the lift of the N-canonical connection $C\Gamma(N)$ of the space M^{*n}.*

Proof. 1°. By our hypothesis ∇ is the lift of a d-connection $H\Gamma(N)=(H^i_{jk}, C_i{}^{jk})$. The formulae (14.3) hold. But $\nabla_X G=0$ is verified if and only if $g^{ij}|_k=0$, $g^{ij}|^k=0$. Then, for $T_{jk}{}^{i}=0$, $S_i{}^{jk}=0$, it results $H\Gamma(\ )=C\Gamma(N)$.

2°. From 1°, it follows that ∇ is the lift of $C\Gamma(N)$ q.e.d.

Theorem 16.2. 1°. *If H^n is a Hamilton space and $\overset{c}{N}$ is its canonical nonlinear connection, then there exists a unique $\overset{c}{N}$-almost Kählerian connection $\overset{c}{\nabla}$ with the property $\overset{c}{T}_{jk}{}^{i}=0$, $\overset{c}{S}_i{}^{jk}=0$.*

2° *$\overset{c}{\nabla}$ is the lift of the canonical connection $C\Gamma(\overset{c}{N})$ from (9.5) of the space H^n.*

The properties are immediate consequences of the theorem 16.1.

Corollary 16.1. *In the case of the Cartan spaces, the properties expressed in the Theorem 16.2 are verified.*

Corollary 16.2. *There exists a unique N-almost Kählerian connection $\overset{c}{\nabla}$, which is the lift of the d-connection $H\Gamma(N)$, which verifies the axioms A_1-A_5 from the Theorem 10.4.*

This paper first appeared in „Preprint Series of University of Timişoara", Seminar of Mechanics, No. 3, 1987.

REFERENCES

1. Arnold V. I. — *Mathematical Methods of Classical Mechanics*, Springer-Verlag, 1978.
2. Asanov G. S. — *Finsler Geometry, Relativity and Gauge Theories*, Reidel Publ. Comp., 1985.
3. Brickell F. — *On areal spaces*, Tensor N. S. 13 (1963) 19—30.
4. Buchin Su. — *Extremal deviation in a geometry based on the notion of area*, Acta Math. 845 (1951), 99—116.
5. Cartan E. — *Les espaces métriques fondés sur la notion d'aire*, Actual. Sci. Industr. Nr. 72, Paris 1933, 47.
6. Cartan E. — *Les espaces de Finsler*. Actual. Sci. Industr. Nr. 79, Paris 1934, 49.
7. Fava F. — *Derivationi di algebre graduate e connessioni non lineari di ordine qualsiast*, Conf. del Sem. di Mat. delle Univ. di Bari, 1979, Nr. 170, 30.
8. Gheorghiev Gh., Oproiu V. — *Varietăţi diferenţiabile finit şi infinit dimensionale* vol. I, II. Ed. Academiei R.S.R., 1976.
9. Kawaguchi A. — *Space of n dimensions with a connection depending on m-dimensional plane elements*. Abh. Sem. Vektor und Tensor-analysis, 1941. Nr. 5, 290—300.
10. Matsumoto M. — *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaisheisa Press, Tokyo, 1986.
11. Miron R., Anastasiei M. — *Vector bundles. Lagrange Geometry. Applications in Relativity* (Roumanian). Ed. Academiei R.S.R., 1987.
12. Miron R. — *A Lagrangian theory of relativity*, Sem. de Geometrie şi Topologie, Univ. Timişoara, Nr. 84 1985, 53.
13. Miron R. — *Metrical Finsler structures and metrical Finsler connections*. Journal of Math. Kyoto Univ. 23 (1983), 219—221.
14. Miron R. — *Vector bundles Finsler geometry*. Proc. of the Nat. Seminar on Finsler spaces, Univ. Braşov (1982) 147—188.
15. Pyragas K. A. — *Covariant canonical equations of the test-particle motion in general relativity*. Tensor N.S. 23 (1972), 243—279.
16. Payragas K. A., Alexandrov A. N. — *Stable trajectories of dynamics and Schrödinger and Klein — Gordon equations*, Tensor N.S. 23 (1972), 337—343.
17. Rund H. — *The Hamilton-Jacobi theory in the calculus of variations*, London-New York, 1966.
18. Vrănceanu G., Popovici A. — *Fundamentele teoriei generale a relativităţii*, Studii şi Cerc. Mat. Bucureşti 15 (1964) 547—593.
19. Wagner V. — *Geometry of space with an areal metric and its applications to the calculus of variations*, Recueil Math. 19 (1946), 341—406.

Received 28.11.1987

University of Iaşi
Faculty of Mathematics
6600 Iaşi, Romania