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THE LEGENDRE TRANSFORMATION AND A PSEUDO-RIEMANNIAN METRIC

BY

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A Lagrange space is a smooth manifold carrying a regular Lagrangian, i.e. a smooth function defined on its tangent bundle such that the Hessian with respect the tangential coordinates is nondegenerate. The Legendre transformation is a (local) diffeomorphism between the tangent and cotangent bundles of a Lagrange space. In [5] the first author studied the properties of the Levi Civita connection of a certain pseudo-Riemannian metric on the tangent bundle of a Lagrange space.

In the present paper we study the properties of the Levi Civita connection of the image by the Legendre transformation of this pseudo-Riemannian metric. The obtained formulas are remarkably simple. The vertical distribution is preserved by the parallel transport defined by this connection. The term giving the deviation from the parallel transport for the horizontal distribution is strongly related to the nonholonomy of this distribution. The covariant differentiation with respect to the (co-) tangential coordinates does coincide with the usual partial differentiation. We obtain the local coordinate expression of the curvature tensor field and the local coordinate expressions of the Bianchi identities. To get our formulas we have used a certain nonlinear connection on the cotangent bundle of a Lagrange space. This is the image by the Legendre transformation of the nonlinear connection on the tangent bundle defined by the Euler-Lagrange equations associated with the given regular Lagrangian. Remark that the nonlinear connection on the tangent bundle was obtained in [1] while its image by the Legendre transformation was obtained in [3].

Functions, vector fields, tensor fields and geometric objects we consider are assumed to be smooth (i.e. of class C^∞). The vector fields on T^*M are denoted by $X, Y, Z, \ldots$, and the set of these vector fields is denoted by $\mathcal{X}(T^*M)$. The well known summation convention is used and the range for the indices $h, i, j, k, l, m, \ldots$ is $\{1, \ldots, n\}$.

1. The nonlinear connection on T*M. Let M be an n-dimensional Lagrange space and denote by $L: TM \to \mathbf{R}$ the corresponding regular Lagrangian. If $(\tau^{-1}(U), x^1, \ldots, x^n, y^1, \ldots, y^n)$ is a local chart on TM induced from the local chart $(U, x^1, \ldots, x^n)$ on M by the natural projection $\tau: TM \to M$, then the matrix with the entries

$$g_{ik} = \partial^2 L / \partial y^i \partial y^k, \tag{1}$$

is nondegenerate. Denote by g^{ik} the entries of the inverse of the matrix given by (1). Let VTM be the vertical distribution on TM defined as the kernel of the tangent mapping $D\tau : TTM \to TM$ of the natural projection τ. The Euler-Lagrange equations of the Lagrangian L define a complementary distribution HTM to VTM in TTM. Namely, if

(2) $$N_k^i = \frac{1}{2}\frac{\partial G^i}{\partial y^k}, \quad G^i = g^{ih}\left(\frac{\partial^2 L}{\partial x^l \partial y^h} y^l - \frac{\partial L}{\partial x^h}\right),$$

then the local vector fields on TM

(3) $$\delta/\delta x^k = \partial/\partial x^k - N_k^i \partial/\partial y^i, \quad k = 1, \ldots, n,$$

give a local frame of HTM. Then

(4) $$TTM = VTM \oplus HTM,$$

and $(\partial/\partial y^1, \ldots, \partial/\partial y^n, \delta/\delta x^1, \ldots, \delta/\delta x^n)$ is a local frame of TTM adapted to this decomposition, i.e. $\partial/\partial y^i$, $i = 1, \ldots, n$, define a local frame of VTM and $\delta/\delta x^i$, $i = 1, \ldots, n$, define a local frame of HTM. The corresponding dual local frame of T^*TM is $(\delta y^i, dx^i)$, $i = 1, \ldots, n$, where

(5) $$\delta y^i = dy^i + N_k^i dx^k.$$

The complete integrability of the partial differential system on TM defined by $HTM \subset TTM$ is given by the vanishing of the tensor field R_{ik}^h where

(6) $$R_{ik}^h = \delta N_k^h / \delta x^i - \delta N_i^h / \delta x^k.$$

This is the torsion tensor field of the nonlinear connection considered above. Let $\tau^* : T^*M \to M$ be the cotangent bundle of the Lagrange space M. If $(U, x^1, \ldots, x^n)$ is a local chart on M it induces a local chart $(\tau^{*-1}(U), q^1, \ldots, q^n, p_1, \ldots, p_n)$ on T^*M, where $q^i = x^i \circ \tau$, $i = 1, \ldots, n$ and p_i, $i = 1, \ldots, n$, are the coefficients of the elements in T^*M when they are expressed with the help of $dx^1, \ldots dx^n$. The Legendre transformation defined by L is a smooth mapping $\Lambda : TM \to T^*M$ given in local coordinates induced on TM, T^*M by

(7) $$q^i = x^i, \quad p_i = \partial L / \partial y^i, \quad i = 1, \ldots, n,$$

(see e.g. [2]). Since L is regular Λ is a diffeomorphism onto its image. In fact, by the inverse function theorem, Λ is a local diffeomorphism. Then consider the Hamiltonian

(8) $$H = p_i y^i - L,$$

thought of as a function on T^*M. Then Λ^{-1} is given by

(9) $$x^i = q^i, \quad y^i = \partial H / \partial p_i; \quad i = 1, \ldots, n,$$

whenever (q^i, p_i) is in the image of Λ. If M is thought of as a submanifold of TM, given by the null section, then Λ is a diffeomorphism of a tubular neighborhood of $M \subset TM$ onto a tubular neighborhood of $\Lambda(M) \subset T^*M$. From (7), (9) we get easily

(10) $$g^{ik} = \partial^2 H / \partial p_i \partial p_k$$

whenever $(q^i, p_i) = \Lambda(x^i, y^i)$. For the tangent mappings $D\Lambda$, $D\Lambda^{-1}$ we have the local coordinate expressions

(11) $$D\Lambda\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial q^i} + \frac{\partial^2 L}{\partial x^i \partial y^k}\frac{\partial}{\partial p_k}, \quad D\Lambda\left(\frac{\partial}{\partial y^i}\right) = g_{ik}\frac{\partial}{\partial p_k}.$$

(11′) $$D\Lambda^{-1}\left(\frac{\partial}{\partial q^i}\right) = \frac{\partial}{\partial x^i} + \frac{\partial^2 H}{\partial q^i \partial p_k}\frac{\partial}{\partial y^k}, \quad D\Lambda^{-1}\left(\frac{\partial}{\partial p_i}\right) = g^{ki}\frac{\partial}{\partial y^k}.$$

We have also, by using (8):

(12) $$-\partial H / \partial q^i = \partial L / \partial x^i$$

whenever $(q^i, p_i) = \Lambda(x^k, y^k)$. We shall no longer write $D\Lambda$, $D\Lambda^{-1}$ in the formulas of the type (11), (11′). Then

(13) $$\frac{\partial}{\partial x^i} = \frac{\partial}{\partial q^i} - \frac{\partial^2 H}{\partial q^i \partial p_k} g_{kh}\frac{\partial}{\partial p_h}, \quad \frac{\partial}{\partial y^i} = g_{ik}\frac{\partial}{\partial p_k},$$

(13′) $$\frac{\partial}{\partial q^i} = \frac{\partial}{\partial x^i} - \frac{\partial^2 L}{\partial x^i \partial y^k} g^{kh}\frac{\partial}{\partial y^h}, \quad \frac{\partial}{\partial p_i} = g^{ih}\frac{\partial}{\partial y^h}.$$

The images of the local vector fields (3) by $D\Lambda$ are

(14) $$D\Lambda\left(\frac{\delta}{\delta x^i}\right) = \frac{\partial}{\partial q^i} - \left(N_i^h + \frac{\partial^2 H}{\partial q^i \partial p_h}\right) g_{kh}\frac{\partial}{\partial p_k}.$$

Hence the image of the nonlinear connection defined by (2) on TM gives us a nonlinear connection on $\Lambda(TM) \subset T^*M$

(15) $$N_{ik} = g_{kh}(N_i^h + \partial^2 H / \partial q^i \partial p_h).$$

From now on we shall suppose that Λ is *bijective* and use the notation

(16) $$\delta / \delta q^i = D\Lambda(\delta / \delta x^i) = \partial / \partial q^i - N_{ik} \partial / \partial p_k.$$

Thus we get the following direct sum decomposition

(17) $$TT^*M = VT^*M \oplus HT^*M,$$

and $(\partial/\partial p_1, \ldots, \partial/\partial p_n, \delta/\delta q^1, \ldots, \delta/\delta q^n)$ is a local frame in TT^*M adapted to this decomposition. The corresponding dual local frame in T^*T^*M is given by $(\delta p_1, \ldots, \delta p_n, dq^1, \ldots, dq^n)$ where $\delta p_i = dp_i + N_{ki} dq^k$.

A straightforward computation by using (15), (8), (13) gives us:

(18) $$N_{ij} = \frac{1}{2}\left(g_{ih}\frac{\partial^2 H}{\partial p_h \partial q^j} + g_{jh}\frac{\partial^2 H}{\partial p_h \partial q^i}\right) + \frac{1}{2}\left(\frac{\partial H}{\partial p_h}\frac{\partial g_{ij}}{\partial q^h} - \frac{\partial H}{\partial q^h}\frac{\partial g_{ij}}{\partial p_h}\right).$$

It follows

(19) $$N_{ij} = N_{ji}.$$

The complete integrability of the partial differential system defined by $HT^*M \subset TT^*M$ is given by the vanishing of the corresponding torsion tensor field

(20) $$R_{kij} = \delta N_{kj}/\delta q^i - \delta N_{ki}/\delta q^j.$$

In fact we have $[\delta/\delta q^i, \delta/\delta q^j] = -R_{kij}\partial/\partial p_k$.

Proposition 1. *The torsion tensor fields R^h_{ij}, R_{kij} of the nonlinear connections (2), (15) on TM, T^*M respectively are related by*

(21) $$R_{kij} = g_{kh}R^h_{ij}.$$

Proof. Using (14), (15) we get

$$R_{kij} = \frac{\delta}{\delta x^i}\left(g_{kh}N^h_j - \frac{\partial^2 L}{\partial x^j \partial y^k}\right) - \frac{\delta}{\delta x^j}\left(g_{kh}N^h_i - \frac{\partial^2 L}{\partial x^i \partial y^k}\right).$$

Next (21) follows by a straightforward computation. Thus the torsion tensor field of the nonlinear connection (15) is obtained from the torsion tensor field of the nonlinear connection (2) by lowering an index.

Proposition 2. *The nonlinear connection (15) satisfies the following identity*

(22) $$\delta g_{jk}/\delta q^i - \delta g_{ik}/\delta q^j = g_{ih}\partial N_{jk}/\partial p_h - g_{jh}\partial N_{ik}/\partial p_h.$$

Proof. From (15) we have

$$\frac{\partial N^h_i}{\partial y^j} + \frac{\partial^3 H}{\partial q^i \partial p_h \partial p_k} g_{kj} = g_{jk}\frac{\partial}{\partial p_k}(N_{il}g^{lh}).$$

Next, using the formula (10) and the obvious property $\partial N^h_i/\partial y^j - \partial N^h_j/\partial y^i = 0$ of the nonlinear connection (2) we get

$$g_{jk}\frac{\partial g^{hk}}{\partial q^i} - g_{ik}\frac{\partial g^{hk}}{\partial q^j} = g_{jk}\frac{\partial}{\partial p_k}(N_{il}g^{lh}) - g_{ik}\frac{\partial}{\partial p_k}(N_{jl}g^{lh}).$$

Then the formula (22) follows by a straightforward computation.

Remark. The identity (22) is the counterpart of a similar identity satisfied by the nonlinear connection (2) in the Lagrange geometry (see [4]). Remark also the following quite trivial identity

(23) $$\sum_{(i,j,k)} R_{ijk} = 0,$$

where $\sum_{(i,j,k)}$ denotes the sum by the cyclic permutations of (i, j, k). The corresponding identity in the Lagrange geometry is not so obvious.

2. The pseudo-Riemannian metric in T^*M. In [5] the first author studied the properties of a certain pseudo-Riemannian metric defined on the tangent bundle of a Lagrange space. Remark that this pseudo-Riemannian metric was thought of as a kind of a „complete lift" of (1). We may consider the image by Λ of this pseudo-Riemannian metric. Using adapted local frames we get the following local coordinate expression

(24) $$G = 2\delta p_i dq^i$$

i.e.

(25) $$G(\partial/\partial p_i, \partial/\partial p_k) = 0, \quad G(\delta/\delta q^i, \delta/\delta q^k) = 0, \quad G(\delta/\delta q^i, \partial/\partial p_k) = G(\partial/\partial p_k, \delta/\delta q^i) = \delta^k_i.$$

It follows that the distributions VT^*M, HT^*M are both isotropic with respect to G. Moreover the local vector fields $\partial/\partial p_i$, $i = 1, \ldots, n$, from VT^*M are the images by G of the local vector fields $\delta/\delta q^i$, $i = 1, \ldots, n$, from HT^*M and vice versa. The signature of the pseudo-Riemannian metric (24) is (n, n).

We are interested in finding the Levi Civita connection of the pseudo-Riemannian metric G. We shall see that the distribution VT^*M is preserved by this connection and the local coordinate formulas are remarkably simple.

Let ∇ be the Levi Civita connection of the pseudo-Riemannian metric (24) and denote:

$$\nabla^i = \nabla_{\partial/\partial p_i}, \quad \nabla_i = \nabla_{\delta/\delta q^i}.$$

Let the local coordinate expression of ∇ be given by

(26) $$\nabla^i \frac{\partial}{\partial p_j} = -D^{ij}_k \frac{\partial}{\partial p_k} + b^{ijk}\frac{\delta}{\delta q^k}, \quad \nabla_i \frac{\partial}{\partial p_j} = -\Gamma^j_{ik}\frac{\partial}{\partial p_k} + E^{jk}_i \frac{\delta}{\delta q^k},$$
$$\nabla^i \frac{\delta}{\delta q^j} = E^{ik}_j \frac{\delta}{\delta q^k} + P^i_{jk}\frac{\partial}{\partial p_k}, \quad \nabla_i \frac{\delta}{\delta q^j} = \Phi^k_{ij}\frac{\delta}{\delta q^k} + a_{ijk}\frac{\partial}{\partial p_k}.$$

We shall obtain the unknown coefficients in (26) from the usual properties of the Levi Civita connection. First, from the condition for ∇ to be torsion free we get

(27) $$\text{(i) } D^{ij}_k = D^{ji}_k, \quad \text{(ii) } b^{ijk} = b^{jik}, \quad \text{(iii) } \bar{E}^{ik}_j = E^{ik}_j,$$
$$\text{(iv) } P^i_{jk} + \Gamma^i_{jk} + \frac{\partial N_{jk}}{\partial p_i} = 0, \quad \text{(v) } \Phi^k_{ij} = \Phi^k_{ji}, \quad \text{(vi) } a_{ijk} - a_{jik} = -R_{kij}.$$

Next, from the condition $\nabla^i G = 0$ we have

(28) $$\text{(i) } b^{ijk} + b^{ikj} = 0, \quad \text{(ii) } E^{ij}_k = D^{ij}_k, \quad \text{(iii) } P^i_{jk} + P^i_{kj} = 0.$$

Finally, from the condition $\nabla_i G = 0$ we have

(29) $$\text{(i) } E^{jk}_i + E^{kj}_i = 0, \quad \text{(ii) } \Phi^k_{ij} = \Phi^k_{ij}, \quad \text{(iii) } a_{ijk} + a_{ikj} = 0.$$

Theorem 3. *The local coordinate expression of the Levi Civita connection ∇ is*

(30) $$\nabla^i \frac{\partial}{\partial p_j} = 0, \quad \nabla_i \frac{\partial}{\partial p_j} = \frac{\partial N_{ik}}{\partial p_j}\frac{\partial}{\partial p_k}$$

(31) $$\nabla^i \frac{\delta}{\delta q^j} = 0, \quad \nabla_i \frac{\delta}{\delta q^j} = -\frac{\partial N_{ij}}{\partial p_k}\frac{\delta}{\delta q^k} + R_{ijk}\frac{\partial}{\partial p_k}$$

i.e.

(32) $$\Gamma^k_{ij} = \Phi^k_{ij} = -\partial N_{ij}/\partial p_k, \quad a_{ijk} = R_{ijk},$$

the other coefficients in (26) being trivial.

Proof. From the symmetry property (27) (ii) and the skew symmetry property (28) (i) we get by a classical argument $b^{ijk}=0$. Next, from (27) (i), (28) (ii) and (29) (i) we have $E^{ij}_k=D^{ij}_k=0$. Then from (27) (vi), (29) (iii) we obtain by using (23) $a_{ijk}=R_{ijk}$. Finally, from (27) (iv), (27) (v), (28) (iii) and (29) (ii) we get $P^k_{ij}=0$, $\Gamma^k_{ij}=\Phi^k_{ij}=-\partial N_{ij}/\partial p_k$.

Remark. The vertical distribution VT^*M is preserved by the parallel transport defined by the Levi Civita connection ∇ on T^*M. However the horizontal distribution HT^*M is not preserved by this connection and the term $R_{ijk}\partial/\partial p_k$ in (31) gives the deviation from parallelism for HT^*M.

Remark. The covariant differentiation with respect to the (co-) tangential coordinates p_i on T^*M does coincide with the usual partial differentiation.

3. The curvature tensor field. The curvature tensor field K of the connection ∇ is obtained from its well known invariant formula $K(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z$; $X,Y,Z\in\mathcal{X}(T^*M)$. Taking for X,Y,Z the vector fields of the local frame $(\partial/\partial p_i, \delta/\delta q^i)$, $i=1,\dots,n$, of TT^*M, adapted to the direct sum decomposition (17) we get the local coordinate expression of K. First, from (30) we get

$$S^{kij}_h\partial/\partial p_h+S^{hkij}\,\delta/\delta q^h=K(\partial/\partial p_i,\ \partial/\partial p_j)\partial/\partial p_k=0,$$

thus

(33) $$S^{kij}_h=0,\qquad S^{hkij}=0.$$

Then,

$$P^{ki}_{hj}\partial/\partial p_h+P^{hki}_j\delta/\delta q^h=K(\partial/\partial p_i,\ \delta/\delta q^j)\partial/\partial p_k=\nabla^i\nabla_j\partial/\partial p_k,$$

from which we have

(34) $$P^{ki}_{hj}=\partial^2N_{jh}/\partial p_i\partial p_k,\qquad P^{hki}_j=0.$$

Next

$$R^k_{hij}\partial/\partial p_h+R^{hk}_{ij}\delta/\delta q^h=K(\delta/\delta q^i,\ \delta/\delta q^j)\partial/\partial p_k=\nabla_i\nabla_j\partial/\partial p_k-\nabla_j\nabla_i\partial/\partial p_k,$$

thus

(35) $$R^k_{hij}=\partial R_{hij}/\partial p_k,\qquad R^{hk}_{ij}=0.$$

From the formulas (31) we get $\dot{S}^{hij}_k\delta/\delta q^h+\dot{S}^{ij}_{hk}\partial/\partial p_h=K(\partial/\partial p_i,\partial/\partial p_j)\,\delta/\delta q^k=0$ from which

(36) $$\dot{S}^{hij}_k=0,\qquad \dot{S}^{ij}_{hk}=0.$$

Then

$$\dot{P}^{hi}_{kj}\frac{\delta}{\delta q^h}+\dot{P}^i_{hkj}\frac{\partial}{\partial p_h}=K\left(\frac{\partial}{\partial p_i},\ \frac{\delta}{\delta q^j}\right)\frac{\delta}{\delta q^k}=\nabla^i\left(-\frac{\partial N_{jk}}{\partial p_h}\frac{\delta}{\delta q^h}+R_{jkh}\frac{\partial}{\partial p_h}\right)$$

so that

(37) $$\dot{P}^{hi}_{kj}=-\partial^2N_{jk}/\partial p_i\partial p_h=-P^{hi}_{kj},\qquad \dot{P}^i_{hkj}=\partial R_{jkh}/\partial p_i=R^i_{jkh}.$$

$$\text{Finally}\ \dot{R}^h_{kij}\frac{\delta}{\delta q^h}+\dot{R}_{hkij}\frac{\partial}{\partial p_h}=K\left(\frac{\delta}{\delta q^i},\ \frac{\delta}{\partial q^j}\right)\frac{\delta}{\delta q^k}=\nabla_i\nabla_j\frac{\delta}{\delta q^k}-\nabla_j\nabla_i\frac{\delta}{\delta q^k}$$

thus

(38) $$\dot{R}^h_{kij}=-R^h_{kij},\qquad \dot{R}_{hkij}=-\bar{\nabla}_iR_{jhk}+\bar{\nabla}_jR_{ihk}$$

where

(39) $$\bar{\nabla}_iR_{jhk}=\frac{\delta R_{jhk}}{\delta q^i}+\frac{\partial N_{ij}}{\partial p_l}R_{lhk}+\frac{\partial N_{ih}}{\partial p_l}R_{jlk}+\frac{\partial N_{ik}}{\partial p_l}R_{jhl}.$$

The nontrivial local coordinate expressions obtained from the first Bianchi identity $K(X,Y)Z+K(Y,Z)X+K(Z,X)Y=0$, $X,Y,Z\in\mathcal{X}(T^*M)$ are

(40)
$$\text{(i)}\ P^{kh}_{ij}=P^{hk}_{ij},\quad \text{(ii)}\ P^{kh}_{ij}=P^{kh}_{ji},\quad \text{(iii)}\ \sum_{(i,j,k)}R^h_{kij}=0,$$
$$\text{(iv)}\ \sum_{(i,j,k)}\dot{R}_{hkij}=-\sum_{(i,j,k)}\bar{\nabla}_kR_{hij}=0.$$

Remark that the relations (40) (i), (40) (ii) are quite obvious from the expression (34) of P^{ki}_{hj}. If we use (35) we see that the identity (40) (iii) is a corollary of (23).

To get the local coordinate expressions of the second Bianchi identity

$$(\nabla_XK)(Y,Z)+(\nabla_YK)(Z,X)+(\nabla_ZK)(X,Y)=0,\ X,Y,Z\in\mathcal{X}(T^*M),$$

satisfied by K we need the local coordinate expression on the covariant derivatives of K with respect to ∇. Let

$$K=(P^{ki}_{hj}\delta p_i\otimes dq^j\otimes\delta p_k-P^{ki}_{hj}dq^j\otimes\delta p_i\otimes\delta p_k+R^k_{hij}dq^i\otimes dq^j\otimes\delta p_k+$$
$$+\dot{P}^i_{hkj}\delta p_i\otimes dq^j\otimes dq^k-\dot{P}^i_{hkj}dq^j\otimes\delta p_i\otimes dq^k+\dot{R}_{hkij}dq^i\otimes dq^j\otimes dq^k)\otimes\frac{\partial}{\partial p_h}+$$
$$+(\dot{P}^{hi}_{kj}\delta p_i\otimes dq^j\otimes dq^k-\dot{P}^{hi}_{kj}dq^j\otimes\delta p_i\otimes dq^k+\dot{R}^h_{kij}dq^i\otimes dq^j\otimes dq^k)\otimes\frac{\delta}{\delta q^h}$$

be the local coordinate expression of K. Then we compute the covariant derivatives of the dual local frame $(\delta p_i, dq^i)$, $i=1,\dots,n$.

(41) $$\nabla^i\delta p_k=0,\qquad \nabla_i\delta p_k=-(\partial N_{ik}/\partial p_j)\delta p_j+R_{ikj}dq^j$$

(42) $$\nabla^idq^k=0,\qquad \nabla_idq^k=(\partial N_{ij}/\partial p_k)dq^j.$$

From the Levi Civita connection ∇ we get the Schouten connection $\bar{\nabla}$ defined by the direct sum decomposition (17). Its local coordinate expression is obtained from (30), (31) by projections on VT^*M, HT^*M respectively

(43) $$\bar{\nabla}^i\partial/\partial p_j=0,\qquad \bar{\nabla}_i\partial/\partial p_j=(\partial N_{ik}/\partial p_j)\partial/\partial p_k,$$

(44) $$\bar{\nabla}^i\delta/\delta q_j=0,\qquad \bar{\nabla}_i\delta/\delta q^j=-(\partial N_{ij}/\partial p_k)\,\delta/\delta q^k.$$

From (41), (42) we get

(45) $$\bar{\nabla}^i\delta p_k=0,\qquad \bar{\nabla}_i\delta p_k=-(\partial N_{ik}/\partial p_j)\,\delta p_j$$

(46) $$\bar{\nabla}^idq^k=0,\qquad \bar{\nabla}_idq^k=(\partial N_{ij}/\partial p_k)dq^j.$$

Using the above local coordinate expression of K we get by covariant differentiation:

$$\nabla_l P^{ki}_{hj}=\bar{\nabla}_l P^{ki}_{hj} \tag{47}$$

$$\nabla_l R^k_{hij}=\bar{\nabla}_l R^k_{hij}-R_{lim}P^{mk}_{hj}+R_{ljm}P^{mk}_{hi} \tag{48}$$

$$\nabla_l \dot{R}_{hkij}=\bar{\nabla}_l \dot{R}_{hkij}-R_{lkm}R^m_{hij}-R_{lim}R^m_{jkh}+R_{ljm}R^m_{ikh}+R_{lhm}R^m_{kij}. \tag{49}$$

Using these formulas we may get the essential coordinate expressions of the second Bianchi identity:

$$\begin{aligned}
&\text{(i)}\ \partial P^{ih}_{kl}/\partial p_j=\partial P^{jh}_{kl}/\partial p_i, \quad \text{(ii)}\ \partial R^j_{hkl}/\partial p_i=\partial R^i_{hkl}/\partial p_j,\\
&\text{(iii)}\ \partial R^k_{hij}/\partial p_l=\bar{\nabla}_i P^{lk}_{hj}-\bar{\nabla}_j P^{lk}_{hi},\\
&\text{(iv)}\ \partial \dot{R}_{hkij}/\partial p_l=\bar{\nabla}_i R^l_{jkh}-\bar{\nabla}_j R^l_{ikh}-R_{ikm}P^{ml}_{jh}+R_{ihm}P^{ml}_{jk}+R_{jkm}P^{ml}_{ih}-R_{jhm}P^{ml}_{ik},\\
&\text{(v)}\ \sum_{(i,j,k)}\{\bar{\nabla}_k R^l_{hij}+R_{mij}P^{ml}_{hk}\}=0,\\
&\text{(vi)}\ \sum_{(i,j,k)}\{\bar{\nabla}_k \dot{R}_{hlij}-R_{mij}R^m_{khl}+R_{ikm}R^m_{lij}-R_{klm}R^m_{hij}\}=0.
\end{aligned} \tag{50}$$

Remark that the identities (50) (i), (ii) are quite obvious if we take into account the expressions (34), (35). The identities (50) (iii), (iv) give the partial derivatives of the tensor fields R^k_{hij}, $\dot{R}_{hkij}$ with respect to the (co-) tangential coordinates. The last identities (50) (v), (vi) are quite similar to the classical local coordinate expressions of the Bianchi identities.

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ON THE GEOMETRY OF THE TOTAL SPACE OF AN INFINITE DIMENSIONAL VECTOR BUNDLE

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The study of the differential geometry of vector bundles by classical methods is not convenient because some important geometrical objects defined on the total space have complicated local forms. Using the ideas from the Finsler geometry, Professor R. M i r o n has considerably simplified the theory [2], [3]. The object of this paper is to extend some of his results to vector and manifolds modelled on Banach spaces.

1. Connections. Let M be a smooth Banach manifold, modelled on Banach space $\mathbf{M}$. A chart for M is given by a diffeomorphism $\varphi: U\to\varphi(U)$ where U is open in M and $\varphi(U)$ is open in $\mathbf{M}$. Given another chart for M we have the transition map

$$t:\varphi(U\cap V)\to\psi(U\cap V),\quad t(x)=\psi\varphi^{-1}(x).$$

Let $\pi: E\to M$ be a smooth vector bundle and (Φ, φ, U) a local trivialization for π:

$$\Phi:\pi^{-1}(U)\to\varphi(U)\times\mathbf{E},\ \Phi(w)=(\varphi(p),\ \Phi_p(w)),\ w\in\pi^{-1}(p)=E_p$$

where (φ, U) is a chart for M and $\Phi_p: E_p\to\mathbf{E}$ is a toplinear isomorphism of Banach spaces.

Given another trivialization (Ψ, ψ, V) we get the transition map

$$U:(\varphi U\cap V)\times\mathbf{E}\to\psi(U\cap V)\times\mathbf{E},\ U(x, a)=(t(x),\ T(x)a)$$

where $T:\varphi(U\cap V)\to L(\mathbf{E}, \mathbf{E})$, $T(x)=\Psi_{\varphi^{-1}(x)}\Phi^{-1}_{\varphi^{-1}(x)}$ is smooth.

For the tangent bundle $\tau: TM\to M$ we have the trivializations $T\varphi: TU\to\to\varphi(U)\times\mathbf{M}$, where $T\varphi$ is the tangent map. In this case we have

$U(x, \lambda)=(t(x), t'(x)\lambda)$ (t' denote the Fréchet differential of t).

For the tangent bundle $\sigma: TE\to E$ the trivializations are $(T\Phi,\ \Phi,\ \pi^{-1}(U))$

$$T\Phi: T(\pi^{-1}(U))\to\varphi(U)\times\mathbf{E}\times\mathbf{M}\times\mathbf{E}$$

and the change of charts

$$U_*(x,\ a,\ \lambda,\ b)=(t(x),\ T(x)a,\ t'(x)\lambda,\ T'(x)(\lambda)a+T(x)b).$$