Using the above local coordinate expression of K we get by covariant differentiation:

$$\nabla_l P^{ki}_{hj} = \bar{\nabla}_l P^{ki}_{hj} \tag{47}$$

$$\nabla_l R^k_{hij} = \bar{\nabla}_l R^k_{hij} - R_{lim} P^{mk}_{hj} + R_{ljm} P^{mk}_{hi} \tag{48}$$

$$\nabla_l \dot{R}_{hkij} = \bar{\nabla}_l \dot{R}_{hkij} - R_{lkm} R^m_{hij} - R_{lim} R^m_{jkh} + R_{ljm} R^m_{ikh} + R_{lhm} R^m_{kij}. \tag{49}$$

Using these formulas we may get the essential coordinate expressions of the second Bianchi identity:

$$\begin{aligned}
&\text{(i)}\ \partial P^{ih}_{kl}/\partial p_l = \partial P^{lh}_{kl}/\partial p_j, \quad \text{(ii)}\ \partial R^j_{hkl}/\partial p_i = \partial R^i_{hkl}/\partial p_j,\\
&\text{(iii)}\ \partial R^k_{hij}/\partial p_l = \nabla_i P^{lk}_{hj} - \nabla_j P^{lk}_{hi},\\
&\text{(iv)}\ \partial \dot{R}_{hkij}/\partial p_l = \nabla_i R^l_{jkh} - \nabla_j R^l_{ikh} - R_{ikm} P^{ml}_{jh} + R_{ihm} P^{ml}_{jk} + R_{jkm} P^{ml}_{ih} - R_{jhm} P^{ml}_{ik},\\
&\text{(v)}\ \sum_{(i,j,k)} \{\nabla_k R^l_{hij} + R_{mij} P^{ml}_{hk}\} = 0,\\
&\text{(vi)}\ \sum_{(i,j,k)} \{\nabla_k \dot{R}_{hlij} - R_{mij} R^m_{khl} + R_{lhm} R^m_{kij} - R_{klm} R^m_{hij}\} = 0.
\end{aligned} \tag{50}$$

Remark that the identities (50) (i), (ii) are quite obvious if we take into account the expressions (34), (35). The identities (50) (iii), (iv) give the partial derivatives of the tensor fields R^k_{hij}, $\dot{R}_{hkij}$ with respect to the (co-)tangential coordinates. The last identities (50) (v), (vi) are quite similar to the classical local coordinate expressions of the Bianchi identities.

REFERENCES

1. Kern J. — *Lagrange geometry*, Archiv. Math., 25 (1974), 438—443.
2. Lanczos C. — *The Variational Principles of Mechanics*, Univ. of Toronto Press, 1964.
3. Miron R. — *Hamilton geometry*, An. şt. Univ. Iaşi, XXXV, s. I a, f. 1 (1989), 33—67.
4. Oproiu V. — *A Riemannian structure in Lagrange geometry*, Rend. Sem. Fac. Sci. Univ. Cagliari, 55(2), 1985, 1—20.
5. Oproiu V. — *A pseudo-Riemannian structure in Lagrange geometry*, An. şt. Univ. Iaşi, XXXIII, s. I a, f. 3 (1987) 239—251.

Received 18.III.1987

Faculty of Mathematics
University of Iaşi
Iaşi, Romania
Department of Mathematics
Polytechnic Institute of
Iaşi, Romania

An. şt. Univ. „Al. I. Cuza" Iaşi
Matematica, 35 (1989)

ON THE GEOMETRY OF THE TOTAL SPACE OF AN INFINITE DIMENSIONAL VECTOR BUNDLE

BY

DRAGOŞ HRIMIUC

The study of the differential geometry of vector bundles by classical methods is not convenient because some important geometrical objects defined on the total space have complicated local forms. Using the ideas from the Finsler geometry, Professor R. Miron has considerably simplified the theory [2], [3]. The object of this paper is to extend some of his results to vector and manifolds modelled on Banach spaces.

1. Connections. Let M be a smooth Banach manifold, modelled on Banach space $\mathbf{M}$. A chart for M is given by a diffeomorphism $\varphi: U \to \varphi(U)$ where U is open in M and $\varphi(U)$ is open in $\mathbf{M}$. Given another chart for M we have the transition map

$$t: \varphi(U \cap V) \to \psi(U \cap V), \quad t(x) = \psi\varphi^{-1}(x).$$

Let $\pi: E \to M$ be a smooth vector bundle and (Φ, φ, U) a local trivialization for π:

$$\Phi: \pi^{-1}(U) \to \varphi(U) \times \mathbf{E}, \quad \Phi(w) = (\varphi(p), \Phi_p(w)), \quad w \in \pi^{-1}(p) = E_p$$

where (φ, U) is a chart for M and $\Phi_p: E_p \to \mathbf{E}$ is a toplinear isomorphism of Banach spaces.

Given another trivialization (Ψ, ψ, V) we get the transition map

$$U: (\varphi U \cap V) \times \mathbf{E} \to \psi(U \cap V) \times \mathbf{E}, \quad U(x, a) = (t(x), T(x)a)$$

where $T: \varphi(U \cap V) \to L(\mathbf{E}, \mathbf{E})$, $T(x) = \psi_{\varphi^{-1}(x)} \Phi^{-1}_{\varphi^{-1}(x)}$ is smooth.

For the tangent bundle $\tau: TM \to M$ we have the trivializations $T\varphi: TU \to \varphi(U) \times \mathbf{M}$, where $T\varphi$ is the tangent map. In this case we have

$U(x, \lambda) = (t(x), t'(x)\lambda)$ (t' denote the Fréchet differential of t).

For the tangent bundle $\sigma: TE \to E$ the trivializations are $(T\Phi, \Phi, \pi^{-1}(U))$

$$T\Phi: T(\pi^{-1}(U)) \to \varphi(U) \times \mathbf{E} \times \mathbf{M} \times \mathbf{E}$$

and the change of charts

$$U_*(x, a, \lambda, b) = (t(x), T(x)a, t'(x)\lambda, T'(x)(\lambda)a + T(x)b).$$

Definition 1.1. *A connection map for the bundle $\pi: E\to M$ (shortly a connection) is a map $N: TE\to E$ such that for any local trivialization (Φ, φ, U) of π there is a smooth map $\omega_\varphi: \varphi(\omega)\times \mathbf{E}\to L(\mathbf{M}, \mathbf{E})$ which gives the local reprezentative of N by the formula*

$$(N_\varphi(x, a, \lambda, b)=(x, b+\omega_\varphi(x, a)\lambda) \tag{1.1}$$

Proposition 1.1. *If ω_φ, ω_ψ are the local components of N in the trivializations (Φ, φ, U) respectively (Ψ, ψ, V), $U\cap V\neq 0$ then*

$$T(x)\omega_\varphi(x, a)\lambda=T'(x)(\lambda)a+\omega_\psi(l(x), \quad T(x)a)l'(x)\lambda. \tag{1.2}$$

Proof: We have : $UN_\varphi=N_\psi U_*$

$$\Rightarrow(l(x), T(x)b+T(x)\omega_\varphi(x, a)\lambda=(l(x), T'(x)(\lambda)a+T(x)b+\omega_\psi(l(x), T(x)a)l'(x)\lambda).$$

Remark 1.1. In definition 1.1 it is sufficient to demand the local experession (1.1) for a trivializing covering of M because if exist a local representation with respect to (Φ, φ, U) then according to (1.2) there is a local representation with respect to (Ψ, ψ, V).

Remark 1.2. If for a trivializing covering of M there is a smooth map $\omega_\varphi: \varphi(U)\times\mathbf{E}\to L(\mathbf{M}, \mathbf{E})$ which verifies (1.2) then there is a connection map $N: TE\to E$ which has the local representation (1.1).

Remark 1.3. The morphism $N: TE\to E$ is continuous linear on the σ fibres but not in general on $\pi_*: TE\to TM$ fibres. If N is linear on the π_* fibres, then N is called a linear connection map (shortly linear connection). We denote it with K in this case.

In the trivializations generated by (U, φ) we get

$$K_p(x, a, \lambda, b)=(x, b+\Gamma(x)(a, \lambda))$$

where $\Gamma_\varphi: \varphi(U)\to L(\mathbf{M}, \mathbf{E}; \mathbf{E})$ is smooth. The local connector Γ_φ corresponds to the classical Christoffel symbols.

Theorem 1.1. *Let $\pi: E\to M$ be a smooth vector bundle. Then the following properties are equivalent*

(i) *There is a connection map $N: TE\to E$,*

(ii) *VE splits TE ($VE=\mathrm{Ker}\,\pi^*$, the vertical subbundle),*

(iii) *$(\exists)g: TE\to VE$ a smooth morphism such that $g\circ i=1_{VE}$($i: VE\to TE$ is the canonical inclusion).*

Proof:

(i)$\Rightarrow$(ii). Let $F: TE\to\pi^*(TM)\oplus\pi^*(E)$ be the map given locally by

$$F_\varphi(x, a, \lambda, b)=(x, a, \lambda, b+\omega_\varphi(x, a)\lambda).$$

Then F is vector bundle isomorphism. ($\pi^*(E)$, $\pi^*(TM)$ denotes the vector bundles induced by π).

We have immediately that F_φ is is a toplinear isomorphism on the TE and $\pi^*(TM)\oplus\pi^*(E)$ fibres.

The changes of charts on $\pi^*(TM)\oplus\pi^*(E)$ are

$$\tilde{U}(x, a, \lambda, b)=(l(x), T(x)a, l'(x)\lambda, T(x)b).$$

It only remains to show that : $\tilde{U}F_\varphi=F_\psi U_*$. We have

$$\tilde{U}F_\varphi(x, a, \lambda, b)=\tilde{U}(x, a, \lambda, b+\omega_\varphi(x, a)\lambda)=(l(x), T(x)a, l'(x)\lambda, T(x)b+$$
$$+T(x)\omega_\varphi(x, a)\lambda)$$

and $F_\psi U_*(x, a, \lambda, b)=(l(x), T(x)a, l'(x)\lambda, T'(x)(\lambda)a+T(x)b$

$$+\omega_\psi(l(x), T(x)a)l'(x)\lambda).$$

Therefore, according to (1.2), $\tilde{U}F_\varphi=F_\psi U_*$ which is what we had to show. From this follows that $F^{-1}(\pi^*(TM))$ and $F^{-1}(\pi^*(E))$ are subbundles of TE. But $F^{-1}(\pi^*(E))=VE$. This completes the proof of the first assertion.

(ii)$\Rightarrow$(iii). We can take g just $p: TE\to VE$.

(iii)$\Rightarrow$(i). Locally in the trivialization generated by (U, φ) g has the following form

$$g_\varphi(x, a, \lambda, b)=(x, a, 0, g^{\#}_\varphi(x, a)(\lambda, b))=(x, a, 0, g^1_\varphi(x, a)+g^2_\varphi(x, a)b),$$

but

$$g_\varphi(x, a, 0, b)=(x, a, 0, b)\Rightarrow g^2_\varphi(x, a)b=b,$$
$$g^{\#}_\varphi: \varphi(U)\times\mathbf{E}\to L(\mathbf{M}, \mathbf{E}; \mathbf{E}).$$

Hence $g_\varphi(x, a, \lambda, b)=(x, a, 0, b+\omega_\varphi(x, a)\lambda)$

$$\omega_\varphi=g^1_\varphi: \varphi(U)\times\mathbf{E}\to L(\mathbf{M}, \mathbf{E}).$$

Since g is a vector bundle morphism, in the trivializations generated by (U, φ) and (V, ψ), ω_φ and ω_ψ are related by (1.2). Therefore, we have found a connection $N: TE\to E$.

Theorem 1.2. *Let $\pi: E\to M$ a smooth vector bundle and M a C^∞-manifold admitting smooth partitions of unity. Then there exists $N: TE\to E$ a connection map.*

The proof follows from the preceding theorem and Proposition 8, p. 59, [1].

Remark. These results are known in the finite dimensional case and in some different forms, in the infinite dimensional case [5], [6].

Proposition 1.2. *Let $\pi: E\to M$ a smooth vector bundle and $\mathcal{A}=\{(\Phi, \varphi, U), (\varphi, U)$ a chart for $M\}$ a vector bundle atlas of trivialization charts. Suppose that for every $(\Phi, \varphi, U)\in\mathcal{A}$, $(\exists)F_\varphi: \varphi(U)\times\mathbf{E}\to\varphi(U)\times\mathbf{E}$ such that $F_\varphi(x, a)=$ $=(x, F^{\#}_\varphi(x, a))$ with $F^{\#}_\varphi: \varphi(U)\to(\mathbf{E}, \mathbf{E})$ smooth and $F^{\#}_\varphi(x): \mathbf{E}\to\mathbf{E}$ a toplinear isomorphism. Then $\mathcal{A}^\circ=\{(F_\varphi\Phi, \varphi, U), (\Phi, \varphi, U)\in\mathcal{A}\}$ is a smooth vector bundle atlas for π.* The proof is immediate.

Let $\pi: E\to M$ be a smooth vector bundle, $N: TE\to E$ a connection and $F_\varphi: \varphi(U)\times\mathbf{E}\times\mathbf{M}\times\mathbf{E}\to\varphi(U)\times\mathbf{E}\times\mathbf{M}\times\mathbf{E}$

$$F_\varphi(x, a, \lambda, b)=(x, a, \lambda, b+\omega_\varphi(x, a)\lambda).$$

Proposition 1.3. *Let $\mathcal{A}^0_*=\{(F_\varphi T\Phi, \Phi, \pi^{-1}(U)), \quad (T\Phi, \Phi, \pi^{-1}(U))\in\mathcal{A}_*\}$ be a smooth vector bundle atlas of trivialization charts on TE. ($\mathcal{A}_*$ is the usual vector bundle atlas on TE). Then the changes of charts in A^0_* are given by*

$$U^0_*(x, a, \lambda, b)=(l(x), T(x)a, l'(x)\lambda, T(x)b).$$

Proof.

$$F_\psi T_\psi(F_\varphi T\Phi)^{-1}(x, a, \lambda, b) = F_\psi U_* F_\varphi^{-1}(x, a, \lambda, b) = F_\psi U_*(x, a, \lambda, b - \omega_\varphi(x, a)\lambda) =$$
$$= F_\psi(l(x),\ T(x)a,\ l'(x)\lambda,\ T'(x)(\lambda)a + T(x)b - T(x)\omega_\varphi(x, a)\lambda) =$$
$$= (l(x),\ T(x)a,\ l'(x)\lambda,\ T'(x)(\lambda)\, a + T(x)b - T(x)\omega_\varphi(x, a)\lambda +$$
$$+ \omega_\psi(l(x),\ T(x)a)l'(x)\lambda) = (l(x),\ T(x)a,\ l'(x)\lambda,\ T(x)b).$$

Remark. In the new atlas the changes of charts are simpler. We obtain thus the same results as in [2, 3], in a general setting.

2. Linear connections on the total space E. Next we suppose that there exists $N : TE \to E$ a connection map.

We now want to construct a smooth atlas of trivialization charts on T^2E induced by N.

We have the maps

$$\tilde{F}_\varphi : \varphi(U) \times \mathbf{E} \times \mathbf{M} \times \mathbf{E} \times \mathbf{M} \times \mathbf{E} \times \mathbf{M} \times \mathbf{E} \to \varphi(U) \times \mathbf{E} \times \mathbf{M} \times \mathbf{E} \times \mathbf{M} \times \mathbf{E} \times \mathbf{M} \times \mathbf{E}$$

$$\tilde{F}_\varphi(x, a, \lambda, b\,;\ \mu, c, \nu, d) = (x, a, \lambda, b\,;\ \mu, c + \omega_\varphi(x, a)\mu, \nu, d)$$

$\tilde{F}_\varphi$ satisfies the assertions of Prop. 1.2.

Proposition 2.1. *Let* $\mathcal{A}^0_{**} = \{(\tilde{F}_\varphi T(F_\varphi T\Phi),\ F_\varphi T\Phi,\ TU)\,;\ (T\Phi, \Phi, \pi^{-1}(U)) \in \mathcal{A}_*\}$ *be a smooth atlas on* T^2E. *Then the changes of charts are given by*

$$U^0_{**}(x, a, \lambda, b, \mu, c, \nu, d) = (l(x),\ T(x)a,\ \ l'(x)\lambda,\ T(x)b\,;\ l'(x)\mu,\ T(x)c,\ l'(x)(\mu)\lambda +$$
$$+ l'(x)\nu,\ \ T'(x)(\mu)b + T(x)d).$$

Proof.

We have:

$$U^0_{**} = \tilde{F}_\psi T(F_\psi T_\psi)(\tilde{F}_\varphi T(F_\varphi T\Phi))^{-1} = \tilde{F}_\psi T U^0_* F_\varphi^{-1}$$

and

$$TU^0_*(x, a, \lambda, b\,;\ \ \mu, c, \nu, d) = ((l(x),\ \ T(x)am\, l'(x)\lambda,\ T(x)b\,;\ \ l'(x)\mu,\ T'(x)(\mu)a +$$
$$+ T(x)c,\ l''(x)(\mu)\lambda + l'(x)\nu,\ \ T(x)(\mu)b + T(x)d).$$

Thus

$$U^0_{**}(x, a, \lambda, b\,;\ \ \mu, c, \nu, d) = \tilde{F}_\psi T U^0_*(x, a, \lambda, b\,;\ \ \mu, c - \omega_\varphi(x, a)\mu, \nu, d) =$$
$$= \tilde{F}_\psi(l(x),\ \ T(x)a,\ l'(x)\lambda,\ T(x)b\,;\ l'(x)\mu,\ \ T'(x)(\mu)a + T(x)c -$$
$$- T(x)\omega_\varphi(x, a)\mu,\ l''(x)(\mu)\lambda + l'(x)\nu,\ T'(x)(\mu)b + T(x)d =$$
$$= (l(x),\ T(x)a,\ l'(x)\lambda,\ T(x)b,\ l'(x)\mu,\ \ T'(x)(\mu)a + T(x)c -$$
$$- T(x)\omega_\varphi(x, a)\mu + \omega_\psi(l(x),\ T(x)a)l'(x)\mu,\ l''(x)(\mu)\lambda +$$
$$+ l'(x)\upsilon,\ T'(x)(\mu)b + T(x)d).$$

According to (1.2) the assertion follows.

Next we study linear connections in the vector bundle. $\sigma : TE \to E$

A connection in this bundle, $K : T^2E \to TE$, is locally given by

$$K_\varphi(x, a, \lambda, b\,;\ \mu, c, \nu, d) = (x, a, \nu + \Omega^1_\varphi(x, a)(\lambda, b)(\mu, c), d + \tag{2.1}$$
$$+ \Omega^2_\varphi(x, a)(\lambda, b)(\mu, c)),$$

where the local components

$$\Omega^1_\varphi : \varphi(U) \times \mathbf{E} \to L(\mathbf{M} \times \mathbf{E},\ \mathbf{M} \times \mathbf{E}\,;\ \ \mathbf{M}),\ \ \Omega^2_\varphi : \varphi(U) \times \mathbf{E} \to L(\mathbf{M} \times \mathbf{E},\ \mathbf{M} \times$$
$$\times \mathbf{E}\,;\ \ \mathbf{E})$$

are smooth maps. The changes of charts on T^2E are given by

$$U_{**}(x, a, \lambda, b\,;\ \mu, c, \nu, d) = (l(x),\ T(x)a,\ l'(x)\lambda,\ T'(x)(\lambda)a +$$
$$+ T(x)b,\ l'(x)\mu,\ \ T'(x)(\mu)a + T(x)c,\ l''(x)(\mu) + l'(x)\nu,$$
$$T''(x)(\lambda, \mu)a + T'(x)(\nu)a + T'(x)(\lambda)c + T'(x)(\mu)b + T(x)d).$$

The local components K_φ and K_ψ are related by

$$K_\psi U_{**} = U_* K_\varphi,$$

and thus we obtain

$$l'(x)\Omega^1_\varphi(x, a)(\lambda, b)(\mu, c) = l''(x)(\mu)\lambda + \Omega^1_\psi(l(x),\ T(x)a)(l'(x)\lambda, \tag{2.2}$$
$$T'(x)(\lambda)a + T(x)b)(l'(x)\mu,\ \ T'(x)(\mu)a + (T(x)c),$$

$$T'(x)(\Omega^1_\varphi(x, a)(\lambda, b)(\mu, c))a + T(x)\Omega^2_\varphi(x, a)(\lambda, b)(\mu, c) =$$
$$T''(x)(\mu, \lambda)a + T'(x)(\lambda)c + T'(x)(\mu)b + \Omega^2_\psi(l(x),\ T(x)a)(l'(x)\lambda, \tag{2.3}$$
$$T'(x)(\lambda)a + T(x)b)(l'(x)\mu,\ \ T'(x)(\mu)a + T(x)c).$$

But these relations are not simple.

Let us consider on T^2E and TE the $\mathcal{A}^0_{**}$ and $\mathcal{A}^0_*$ atlas, respectively. Then we have

Proposition 2.2. *Let* $K : T^2E \to TE$ *be a linear connection in the smooth vector bundle* $\sigma : TE \to E$. *Then in the local trivializations of* $\mathcal{A}^0_{**}$, $\mathcal{A}^0_*$, *K has the following local representation*

$$\bar{K}^2_\varphi(x, a, \lambda, b\,;\ \ \mu, c, \upsilon, d) = (x, a, \nu + \bar{\Omega}^1_\varphi(x, (x, a)(\lambda, b)(\mu, c), \tag{2.1$'$}$$
$$d + \bar{\Omega}^2_\varphi(x, a)(\lambda, b)(\mu, c)),$$

where

$$\bar{\Omega}^1_\varphi(x, a)(\lambda, b)(\mu, c) = \Omega^1_\varphi(x, a)(\lambda, b - \omega\,(x, a)\lambda)(\mu, c - \omega_\varphi(x, a)\mu)$$

and

$$\bar{\Omega}^2_\varphi(x, a)(\lambda, b)(\mu, c) = \Omega^2_\varphi(x, a)(\lambda, b - \omega_\varphi(x, a)\lambda)(\mu, c - \omega_\varphi(x, a)\mu) +$$
$$\omega_\varphi(x, a)\Omega^1_\varphi(x, a)(\lambda, b - \omega_\varphi(x, a)\lambda)(\mu, c - \omega_\varphi(x, a)\mu) - \omega'_\varphi(x, a)(\mu, c)\lambda,$$

$$\bar{\Omega}^1_\varphi : \varphi(U) \times \mathbf{E} \to L^2(\mathbf{E} \times \mathbf{M},\ \ \mathbf{E} \times \mathbf{M}\,;\ \ \mathbf{M}),\ \ \bar{\Omega}^2_\varphi : \varphi(U) \times \mathbf{E} \to L^2(\mathbf{E} \times \mathbf{M},\ \ \mathbf{E} \times \mathbf{M}\,;\ \ \mathbf{E}),$$

(Ω^1_φ, Ω^2_φ are the local connectors in the natural trivializations).

Proof. Using the formula $\bar{K}_\varphi = F_\varphi T\Phi K(\tilde{F}_\varphi T(F_\varphi \Phi))^{-1}$ we get easily the results after some calculations.

Theorem 2.1. *Let $K: T^2E \to TE$ be a connection map in the smooth vector bundle $TE \xrightarrow{\sigma} E$ locally given by (2.1)′. Then in the trivialization charts $(\tilde{F}_\varphi T(F_\varphi T\Phi), F_\varphi T\Phi, TU)$ and $(\tilde{F}_\psi T(F_\psi T\psi), F_\psi T\psi, TU)$ we get the following transformation formulae for the local connectors*

$$(2.2') \quad \begin{aligned} t'(x)\bar{\Omega}^1_\varphi(x, a)(\lambda, b)(\mu, c) = t''(x)(\mu)\lambda + \bar{\Omega}^1_\psi(t(x), \ T(x)a)(t'(x)\lambda, \\ T(x)b)(t'(x)\mu, \ T(x)c), \end{aligned}$$

$$(2.3') \quad \begin{aligned} T(x)\bar{\Omega}^2_\varphi(x, a)(\lambda, b), (\mu, c) = T'(x)(\mu)b + \bar{\Omega}^2_\psi(t(x), \ T(x)a)(t'(x)\lambda, \\ T(x)b)(t'(x)\mu, T(x)c). \end{aligned}$$

Proof. We get easily from the equality: $U^0_* \bar{K}_\varphi = \bar{K}_\psi U^0_{**}$.

Remark 2.1. We have

$$\bar{\Omega}^1_\varphi(x, a)(\lambda, b)(\mu, c) = {}^{11}\bar{\Omega}^1_\varphi(x, a)(\lambda, \mu) + {}^{12}\bar{\Omega}^1_\varphi(x, a)(\lambda, c) +$$

$${}^{21}\bar{\Omega}^1_\varphi(x, a)(b, \mu) + {}^{22}\bar{\Omega}^1_\varphi(x, a)(b, c) \text{ where } {}^{11}\bar{\Omega}^1_\varphi, {}^{12}\bar{\Omega}^1_\varphi,$$

${}^{21}\bar{\Omega}^1_\varphi$, ${}^{22}\bar{\Omega}^1_\varphi$ are smooth maps from $\varphi(U) \times \mathbf{E}$ onto $L^2(\mathbf{M}, \mathbf{M}; \mathbf{M})$, $L^2(\mathbf{M}, \mathbf{E}; \mathbf{M})$, $L^2(\mathbf{E}, \mathbf{M}; \mathbf{M})$, $L^2(\mathbf{E}, \mathbf{E}; \mathbf{M})$, respectively.

For $\bar{\Omega}^2_\varphi$ we have an analogous decomposition.

Let $u \in TE$. Then $u = u^h + u^v$, $u^h \in HE$, $u^v \in VE$. In the local trivialization $(F_\varphi T\Phi, \Phi, \pi^{-1}(U))$, $u \in T\pi^{-1}(U)$. We have

$$F_\varphi T\Phi(u^h) = (x, a, \lambda, 0) \in \varphi(U) \times \mathbf{E} \times \mathbf{M} \times 0,$$

$$F_\varphi T\Phi(u^v) = (x, a, 0, b) \in \varphi(U) \times \mathbf{E} \times 0 \times \mathbf{E}.$$

Let $w \in T^2E$. We have $w = w^h + w^v$, $w^h \in T(HE)$, $w^v \in T(VE)$ and in a local trivialization $(\tilde{F}_\varphi T(F_\varphi T\Phi), F_\varphi T\Phi, TU)$, $w \in T^2U$

$$\tilde{F}_\varphi T(F_\varphi T\Phi)(w^h) = (x, a, \lambda, 0; \mu, c, \nu, 0),$$

$$\tilde{F}_\varphi T(F_\varphi T\Phi)(w^v) = (x, a, 0, b; \mu, c, 0, d).$$

Definition 2.1. *A connection $K: T^2E \to TE$ in the smooth vector bundle $TE \xrightarrow{\sigma} E$ is a d-connection if $K^H(w^v) = K^V(w^h) = 0$ (K^H and K^V are projections on HE and VE).*

Proposition 2.3. *A connection $K: T^2E \to TE$ is a d-connection if and only if ${}^{21}\bar{\Omega}^1_\varphi = {}^{22}\bar{\Omega}^1_\varphi = {}^{11}\bar{\Omega}^2_\varphi = {}^{12}\bar{\Omega}^2_\varphi = 0$.*

Proof.

$$K_\varphi(x, a, \lambda, b; \mu, c, \nu, d) = (x, a, \nu + {}^{11}\bar{\Omega}^1_\varphi(\lambda, \mu) + {}^{12}\bar{\Omega}^1_\varphi(\lambda, c) + {}^{21}\bar{\Omega}^1(b, \mu) +$$

$$+ {}^{22}\bar{\Omega}^2_\varphi(b, c), \ d + {}^{11}\bar{\Omega}^2_\varphi(\lambda, \mu) + {}^{12}\bar{\Omega}^2_\varphi(\lambda, c) + {}^{21}\bar{\Omega}^2_\varphi(b, c) + {}^{22}\bar{\Omega}^2_\varphi(b, c)),$$

$$K^v(w^h) = 0, \ {}^{11}\bar{\Omega}^2_\varphi = {}^{12}\bar{\Omega}^2_\varphi = 0, \quad K^h(w^v) = 0, \ {}^{21}\bar{\Omega}^1_\varphi = {}^{22}\bar{\Omega}^1_\varphi = 0.$$

Thus a d-connection has the following local form

$$K_\varphi(x, a, \lambda, b: \mu, c, \nu, d) = (x, a, \nu + L^1_\varphi(x, a)(\lambda, \mu) + C^1_\varphi(x, a)(\lambda, c),$$

$$d + L^2(x, a)(b, \mu) + C^2(x, a)(b, c)),$$

where we denote $L^1 = {}^{11}\bar{\Omega}^1$, $L^2 = {}^{12}\bar{\Omega}^1$, $C^1 = {}^{12}\bar{\Omega}^1$, $C^2 = {}^{22}\bar{\Omega}^2$.

Remark 2.2. We get the following transformation formulae for the local connectors

$$(2.4) \quad \begin{aligned} t'(x)L^1_\varphi(x, a)(\lambda, \mu) &= t''(x)(\mu)\lambda + L^1_\psi(t(x), \ T(x)a)(t'(x)\lambda, t'(x)\mu), \\ t'(x)C^1_\varphi(x, a)(\lambda, c) &= C^1_\psi(t(x), \ T(x)a)(t'(x), \ T(x)c), \\ T(x)L^2_\varphi(x, a)(b, \mu) &= T'(x)(\mu)b + L^2_\psi(t(x), \ T(x)a)(T(x)b, t'(x)\mu), \\ T(x)C^2_\varphi(x, a)(b, c) &= C^2_\psi(t(x), \ T(x)a)(T(x)b, \ T(x)c) \end{aligned}$$

These relations generalize results from [2] to the infinite dimensional manifolds.

Theorem 2.2. *Let M be a smooth Banach manifold admitting smooth partitions of unity. Then the maps*

$$K_\varphi(x, a, \lambda, b; \mu, c, \nu, d) = (x, a, \nu + \Gamma_\varphi(x)(\lambda, \mu), \ d + D_2\omega_\varphi(x, a)(b)\mu)$$

(where $\Gamma_\varphi : \varphi(U) \to L^2(\mathbf{M}, \mathbf{M}; \mathbf{M})$ is a local connector for a linear connection on M) are local components for a d-connection $K: T^2E \to TE$.

Proof. We have the following transformation formula for the local connector

$$t'(x)\Gamma_\varphi(x)(\lambda, \mu) = t''(x)\mu\lambda + \Gamma_\psi(t(x))(t'(x)\lambda, t'(x)\mu),$$

Differentiating partially with respect to a in direction b we get

$$T(x)D_2\omega_\varphi(x, a)(b)\mu = T'(x)(\mu)b + D_2\omega_\psi(t(x), \ T(x)a)(T(x)b)t'(x)\mu.$$

Remark 2.3. The same technique may be used for the study of linear connections in the vertical vector bundle $VE \to E$ (Finsler connection).

A linear connection $K: T(VE) \to VE$ is given by the local formula

$$K_\varphi(x, a, 0, b; \mu, c, 0, d) = (x, a, 0, d + \Omega_\varphi(x, a)(b, (\mu, c)) =$$

$$(x, a, 0, d + \Gamma_\varphi(x, a)(b, \mu) + C_\varphi(x, a)(b, c)) \ (\lambda = \nu = 0 \text{ in } (2.1)),$$

$$\Gamma_\varphi : \varphi(U) \times \mathbf{E} \to L(\mathbf{E}, \mathbf{M}; \mathbf{E}), \ C_\varphi : \varphi(U) \times \mathbf{E} \to L(\mathbf{E}, \mathbf{E}; \mathbf{E}).$$

The transition maps are

$$U_*(x, a, 0, b) = (t(x), \ T(x)a, 0, \ T(x)b),$$

$$U_{**}(x, a, 0, b; \mu, c, 0, d) = (t(x), \ T(x)a, 0, \ T(x)b; \ t'(x)\mu, \ T'(x)(\mu)a +$$

$$+ T(x)c, 0, \ T'(x)(\mu)b + T(x)d).$$

The local components Γ_φ and Γ_ψ are related by the equations

$$T(x)\Gamma_\varphi(x, a) = T'(x)(\mu)b + \Gamma_\psi(t(x), T(x)a)(T(x)b, t'(x)\mu) \tag{2.5}$$
$$+ C_\psi(t(x), T(x)a)(T(x)b, T'(x)(\mu)a),$$

$$T(x)C_\varphi(x, a)(b, c) = C_\psi(t(x), T(x)a)(T(x)b, T(x)c). \tag{2.6}$$

In the new trivializations K had the local form

$$\bar{K}_\varphi = T\Phi K(\tilde{F}_\varphi T^2\Phi)^{-1} = K_\varphi \tilde{F}_\varphi^{-1}.$$

$$\bar{K}_\varphi(x, a, 0, b) = (x, a, 0, d + \Gamma_\varphi(x, a)(b, \mu) + C_\varphi(x, a)(b, c - \omega_\varphi(x, a)\mu) =$$
$$= (x, a, 0, d + \bar{\Gamma}_\varphi(x, a)(b, \mu) + \bar{C}_\varphi(x, a)(b, c)),$$

where

$$\bar{\Gamma}_\varphi(x, a)(b, \mu) = \Gamma_\varphi(b, \mu) - C_\varphi(\ , a)(b, \omega_\varphi(x, a)\mu),$$

and

$$\bar{C}_\varphi(x, a)(b, c) = C_\varphi(x, a)(b, c).$$

If we change the trivializations, $U_*\bar{K}_\varphi = \bar{K}_\psi U^0_{**}$(2.5) reduces to

$$T(x)\bar{\Gamma}_\varphi(x, a)(b, \mu) = T'(x)(\mu)b + \bar{\Gamma}_\psi(t(x), T(x)a)(T(x)b, t'(x)\mu), \tag{2.5$'$}$$

which is more simple. One obtains in a general setting the same results as in [4].

3. The torsion and the curvature of a d-connection. Let $X \in \mathcal{X}(L)$ be a smooth vector field. In natural charts X has the local expression.

$$X(x, a) = (x, a, X^1(x, a), X^2(x, a)).$$

We write shortly

$$X = (X^1, X^2) \text{ (principal parts).}$$

In the new trivializations charts

$$\bar{X}(x, a) = (x, a, X^H(x, a), X^V(x, a)$$

Shortly

$$\bar{X} = (X^H, X^V).$$

We have

$$X^H = X^1, \quad X^V = X^2 + \omega X^1.$$

Proposition 3.1. *Let $X, Y, \in \mathcal{X}(E)$. In the new trivialization charts we have the following local expressions*

$$\bar{X}_*(x, a, \lambda, b) = (x, a, X^H, X^V;\ \lambda,\ b, DX^H(\lambda, b - \omega\lambda), DX^V(\lambda, b - \omega\lambda))$$
$$\overline{[X, Y]}(\ \ ) = (x, a, DY^H(X^H, X^V - \omega X^H) - DX^H(Y^H, Y^V - \omega Y^H),$$
$$DY^V(X^H, X^V - \omega X^H) - DX^H(Y^H, Y^V - \omega Y^H) + \omega'(Y^H, Y^V - \omega Y^H)X^H -$$
$$- \omega'(X^H, X^V - \omega X^H)Y^V).$$

The proof is immediate

Let $X, Y \in \mathcal{X}(E)$ and let $K : T^2E \to TE$ be a connection map.

We define the covariant derivative by

$$\nabla : \mathcal{X}(E) \times \mathcal{X}(E) \to \mathcal{X}(E),\ (X, Y) \to \nabla_X Y := \nabla(X, Y),$$

where

$$\nabla(X, Y) = KY_*X.$$

Proposition 3.1. *In the new trivialization charts $\nabla_X Y$ has the following local form*

$$\bar{\nabla}_X Y = (DY^H(X^H, X^V - \omega X^H) + \bar{\Omega}_1(Y^H, Y^V)(X^H, X^V),\ \ DY^V(X^H, X^V - \omega X^H) +$$
$$+ \bar{\Omega}_2(Y^H, Y^V)(X^H, X^V)) \textit{ (principal part of } \nabla_X Y).$$

Proof. We use the relation $\bar{\nabla}_X Y = K\bar{Y}_*\bar{X}$.

Next we study the torsion and the curvature of a d-connection. If $K : T^2E \to TE$ is a d-connection we have the following local form for the covariant derivative.

$$\nabla_X Y = (DY^H(X^H, X^V - \omega X^H) + L^1(Y^H, X^H) + C^1(Y^H, X^V), DY^V(X^H, X^V - \omega X^H) +$$
$$+ L^2(Y^V, X^H) + C^2(Y^V, X^V)).$$

Clearly $(\nabla_X Y^H)^V = (\nabla_X Y^V)^H = 0$. Thus the horizontal and vertical bundles are parallel with respect to ∇.

The torsion for a d-connection is defined by the usual formula

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Since $X = X^H + X^V$, $Y = Y^H + Y^V$ then

$$T(X, Y) = T(X^H, Y^H) + T(X^H, Y^V) + T(X^V, Y^H) + T(X^V, Y^V)$$

Theorem 3.1. *In the new trivialization charts we have the following local forms*

$$\bar{T}(X^H, Y^H) = (L^1(Y^H, X^H) - L^1(X^H, Y^H), \omega^1(Y^H, \omega Y^H)X^H - \omega'(X^H, \omega X^H)Y^H),$$

$$\bar{T}(X^H, Y^H) = (C^1(X^V, Y^H), \omega^1(0, X^V)X^H - L^2(X^V, Y^H)),$$

$$\bar{T}(X^V, Y^V) = (0, C^2(Y^V, X^V) - C^2(X^V, Y^V)).$$

Remark 3.1. The d-connection is without torsion if and only if L^1, C^2 are symmetric bilinear maps, $C^1 = 0$,

$$\omega'(\lambda, \omega\lambda)\mu = \omega'(\mu, \omega\mu)\lambda,\ \omega'(0, b)\mu = L^2(b, \mu).$$

For the curvature we have the formula

$$R(X, Y, Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X, Y]}Z.$$

Theorem 3.2. *In the new trivialization charts we have the following local forms*

$$R(X^H, Y^H)Z^H = DL^1(X^H, -\omega X^H)(Z^H, Y^H) - DL^1(Y^H, -\omega Y^H)(Z^H, X^H) + $$
$$+ L^1(L^1(Z^H, Y^H), X^H) - L^1(L^1(Z^H, X^H), Y^H) + C^1(Z^H, \omega'(X^H, -$$
$$-\omega Y^H)Y^H - \omega'(Y^H, -\omega Y^H)X^H).$$

$$R(X^V, Y^H)Z^H = DL^1(0, X^V)(Z^H, Y^H) - DC^1(Y^H, -\omega Y^H)(Z^H, X^V) -$$
$$-L^1(C^1(Z^H, X^V), Y^H) + C^1(L^1(Z^H, Y^H)X^V) + C^1(Z^H, \omega'(0, X^V)Y^H),$$
$$R(X^V, Y^V)Z^H = DC^1(0, X^V)(Z^H, Y^V) - DC^1(0, Y^V)(Z^H, X^V) +$$
$$+ C^1(C^1(Z^H, Y^V), X^V) - C^1(C^1(Z^H, X^V), Y^V)$$

for the horizontal component. The vertical component follows from the above three relations if we substitute in (*) Z^H with Z^V and L^1, C^1 with L^2, C^2.

The last two theorems generalize the results from [2], [3].

REFERENCES

1. L a n g S. — *Differential manifolds*, Springer-Verlag, New York, 1985.
2. M i r o n R. — *Vector bundles Finsler geometry*, in : Proceedings of the Second National Seminar on Finsler spaces Braşov, 1982, 147—188.
3. M i r o n R. — *Techniques of Finsler geometry in the theory of vector bundles*, Acta Sci. Math., 49 (1985), 119—129.
4. O p r o i u V. — *Some properties of the tangent bundle related to the Finsler geometry*, Proceed. Nat. Sem. Finsler Spaces, Braşov, 1980, 195—207.
5. V i l m s J. — *Connections on tangent bundles*, J. Differential Geometry 1 (1967), 235—243.
6. V i l m s J. — *Curvature of nonlinear connections*, Proc. Amer. Math. Soc., 19 (1968), 1125—1129

Received 19.III.1987

Seminarul Matematic „Al. Myller"
Universitatea din Iaşi
6600 Iaşi, România

An. şt. Univ. „Al. I. Cuza" Iaşi
Matematica, 35 (1989)

FINSLER SPACES WITH RECTILINEAR EXTREMALS

BY

WEI XIANZHI

Introduction. Finsler spaces with rectilinear extremals have been considered by many authors and recently we have interesting results given by M. H a s h i g u c h i [1], M. M a t s u m o t o [2] and T. O k a d a [5]. As it is well-known, a Finsler space is with rectilinear extremals if and only if it is projectively flat [3], and it is characterized by vanishing of the so-called Weyl tensor and Douglas tensor in case of dimension more than two [2]. In terms of the Berwald connection are these tensors written by the h-curvature tensor H and hv-curvature tensor G respectively.

The purpose of the present paper is to study Finsler spaces P_n with rectilinear extremals in a rectilinear coordinate system [3]. After we are concerned with P_n of vanishing H and G, that is, locally Minkowski spaces [4, § 24], the subsequent two sections are devoted to considering P_n with $H=0$ and P_n with $G=0$ respectively.

The author wishes to express his sincere gratitude to Professor Dr. M. M a t s u m o t o for the invaluable suggestions and encouragement.

§ 1. Finsler spaces with rectilinear extremals. A Finsler space F_n is with rectilinear extremals if and only if F_n is covered by coordinate neighborhoods in every neighborhood of which there exists a function $p(x, y)$ (positively homogeneous of degree 1 in y^i) satisfying

$$G^i(x, y) = -p(x, y)y^i, \tag{1}$$

where $G^i(x, y) = \gamma^i_{jk} y^j y^k/2$ and γ^i_{jk} are Christoffel symbols. Following M a t s u m o t o [3] such a coordinate system will be called *rectilinear* and the function $p(x, y)$ is the *projective factor*. From [6] we have

Proposition 1. *A Finsler space F_n is with rectilinear extremals if and only if there exists a coordinate system in a neighborhood of every point such that one of the following conditions holds*

$$\partial_k l_j = \partial_j l_k ; \tag{2}$$

$$\partial_k l_j = -p_j l_k - p_k l_j - p l_{jk} - L p_{jk} ; \tag{3}$$

$$\partial_k L = \partial_0 l_k ; \tag{4}$$

$$\partial_k L = -\partial_k (Lp) ; \tag{5}$$

$$\partial_k g_{ij} = -2g_{ij} p_k - g_{ik} p_j - g_{jk} p_i - y_i p_{jk} - y_j p_{ik} - 4pC_{ijk} + 2C_{ijk} y^j ; \tag{6}$$