$$R(X^H, Y^H)Z^H = DL^1(X^H, -\omega X^H)(Z^H, Y^H) - DL^1(Y^H, -\omega Y^H)(Z^H, X^H) +$$
$$+ L^1(L^1(Z^H, Y^H), X^H) - L^1(L^1(Z^H, X^H), Y^H) + C^1(Z^H, \omega'(X^H, -$$
$$-\omega Y^H)Y^H - \omega'(Y^H, -\omega Y^H)X^H).$$
$$R(X^V, Y^H)Z^H = DL^1(0, X^V)(Z^H, Y^H) - DC^1(Y^H, -\omega Y^H)(Z^H, X^V) -$$
$$-L^1(C^1(Z^H, X^V), Y^H) + C^1(L^1(Z^H, Y^H)X^V) + C^1(Z^H, \omega'(0, X^V)Y^H),$$
$$R(X^V, Y^V)Z^H = DC^1(0, X^V)(Z^H, Y^V) - DC^1(0, Y^V)(Z^H, X^V) +$$
$$+ C^1(C^1(Z^H, Y^V), X^V) - C^1(C^1(Z^H, X^V), Y^V)$$

for the horizontal component. The vertical component follows from the above three relations if we substitute in (*) Z^H with Z^V and L^1, C^1 with L^2, C^2.

The last two theorems generalize the results from [2], [3].

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FINSLER SPACES WITH RECTILINEAR EXTREMALS

BY

WEI XIANZHI

Introduction. Finsler spaces with rectilinear extremals have been considered by many authors and recently we have interesting results given by M. H a s h i g u c h i [1], M. M a t s u m o t o [2] and T. O k a d a [5]. As it is well-known, a Finsler space is with rectilinear extremals if and only if it is projectively flat [3], and it is characterized by vanishing of the so-called Weyl tensor and Douglas tensor in case of dimension more than two [2]. In terms of the Berwald connection are these tensors written by the h-curvature tensor H and hv-curvature tensor G respectively.

The purpose of the present paper is to study Finsler spaces P_n with rectilinear extremals in a rectilinear coordinate system [3]. After we are concerned with P_n of vanishing H and G, that is, locally Minkowski spaces [4, § 24], the subsequent two sections are devoted to considering P_n with $H=0$ and P_n with $G=0$ respectively.

The author wishes to express his sincere gratitude to Professor Dr. M. M a t s u m o t o for the invaluable suggestions and encouragement.

§ 1. Finsler spaces with rectilinear extremals. A Finsler space F_n is with rectilinear extremals if and only if F_n is covered by coordinate neighborhoods in every neighborhood of which there exists a function $p(x, y)$ (positively homogeneous of degree 1 in y^i) satisfying

$$G^i(x, y) = -p(x, y)y^i, \tag{1}$$

where $G^i(x, y) = \gamma^i_{jk} y^j y^k/2$ and γ^i_{jk} are Christoffel symbols. Following M a t s u m o t o [3] such a coordinate system will be called *rectilinear* and the function $p(x, y)$ is the *projective factor*. From [6] we have

Proposition 1. *A Finsler space F_n is with rectilinear extremals if and only if there exists a coordinate system in a neighborhood of every point such that one of the following conditions holds*

$$\partial_k l_j = \partial_j l_k ; \tag{2}$$

$$\partial_k l_j = -p_j l_k - p_k l_j - p l_{jk} - L p_{jk} ; \tag{3}$$

$$\partial_k L = \partial_0 l_k ; \tag{4}$$

$$\partial_k L = -\partial_k (Lp) ; \tag{5}$$

$$\partial_k g_{ij} = -2g_{ij} p_k - g_{jk} p_i - g_{ik} p_j - y_i p_{jk} - y_j p_{ik} - 4pC_{ijk} + 2C_{ijk} \text{[illegible]} ; \tag{6}$$

where $\partial_k=\partial/\partial x^k$, $\dot\partial_k=\partial/\partial y^k$, $p_k=\dot\partial_k p$, $p_{ik}=\dot\partial_i p_k$, $l_k=\dot\partial_k L$, $l_{ik}=\dot\partial_i l_k$, $C_{ijk;0}=C_{ijk;s}y^s$ *denotes covariant differentiation with respect to the Berwal connection* $B\Gamma$, *and* L *is the metric function of* F_n. *This coordinate system is rectilinear.*

Proposition 2 *A Finsler space* F_n *is with rectilinear extremals if and only if there exists a coordinate system in a neighborhood of every point such that one of the following conditions holds*

(a) $$\gamma_{ijk}=l_j\partial_k l_i+L\partial_k l_{ij}/2+(l_{ij}\partial_k L+l_{jk}\partial_i L-l_{ik}\partial_j L)/2;$$

(b) $\gamma_{ojk}=l_j\partial_k L+l_{jk}\partial_o L/2$; (c) $\gamma_{ijo}=l_j\partial_0 l_i+L\partial_o l_{ij}/2+l_{ij}\partial_o L/2$; (d) $\gamma_{ojo}=l_j\partial_o L$,

where

(7) $$\gamma_{ijk}=(\partial_i g_{jk}+\partial_k g_{ij}-\partial_j g_{ik})/2.$$

Proof. We consider the condition (a). By means of $g_{ij}=l_il_j+Ll_{ij}$ we get

(8) $$\partial_k g_{ij}=l_i\partial_k l_j+l_j\partial_k l_i+L\partial_k l_{ij}+l_{ij}\partial_k L.$$

(7) and (8) lead to

$$\gamma_{ijk}=[l_i(\partial_k l_j-\partial_j l_k)+l_k(\partial_i l_j-\partial_j l_i)+l_j(\partial_i l_k+\partial_k l_i)]/2+L(\partial_k l_{ij}+\partial_i l_{jk}-$$
$$-\partial_j l_{ik})/2+(l_{ij}\partial_k L+l_{jk}\partial_i L-l_{ik}\partial_j L)/2.$$

Using (2) we find the condition (a).

Conversely, contracting (a) by g^{is} we have

$$\gamma^s_{jk}=(\partial_k l_j)l^s+g^{si}(\partial_k l_{ij}+l_{ij}\partial_k L+l_{jk}\partial_i L-l_{ik}\partial_j L)/2.$$

Contraction by y^jy^k gives

$$2G^s(x,y)=y^s\partial_o L/L.$$

Putting $p=-\partial_0 L/2L$ we get (1), so that F_n is with rectilinear extremals.

By the same way discussed above we can prove the results (b)—(d).

By virtue of (6) and (7) we have

Proposition 3. F_n *is with rectilinear extremals if and only if there exists a coordinate system in a neighborhood of every point such that one of the following conditions holds,*

(a) $\gamma_{ijk}=-g_{ij}p_k-g_{kj}p_i-y_jp_{ik}-2C_{ijk}+C_{ijk;0}$;

(b) $\gamma_{ojk}=-y_jp_k-g_{kj}p$; (c) $\gamma_{ojo}=-2y_jp$.

Theorem 1. *A Finsler space* P_n *with rectilinear extremals is a locally Minkowski space if and only if in a rectilinear coordinate system we have*

(9) $$P_{i;k}=p_ip_k.$$

Proof. we have

$$p_{i;k}=\partial_kp_i-p_{ir}G^r_k-p_sG^s_{ik}.$$

We here consider a Finsler space with rectilinear extremals. Then it is easy to show

(10) $$G^r_k=-p\delta^r_k-p_ky^r.$$

and

$$G^s_{ik}=-p_i\delta^s_k-p_k\delta^s_i-p_{ik}y^s,$$

so that we have

(11) $$p_{i;k}=\partial_kp_i+2p_ip_k+2pp_{ik}.$$

On the other hand, (1.6) of [3] gives

(12) $$H^i_{jk}=H^i_{ojk}=[\partial_jp_ky^i-\delta^i_j(\partial_kp+pp_k)](-j/k),$$

where H^i_{ojk} is the contracted tensor of the h-curvature tensor H^i_{hjk} of $B\Gamma$ and $(-j/k)$ means the interchange of indices j, k and substration. From $y_iH^i_{jk}=0$, we see

$$\partial_jp_k-\partial_kp_j=l_j(\partial_kp+pp_k)/L(-j/k).$$

(12) reduces to

(13) $$H^i_{jk}=(\partial_jp+pp_j)h^i_k(-j/k).$$

Since a locally Minkowski space is a Finsler space of zero curvature, from (13) we get

$$(\partial_jp+pp_j)h^i_k(-j/k)=0.$$

Contraction with respect to i and k, gives

(14) $$(n-2)(\partial_jp+pp_j)+l_j(\partial_kp+pp_k)l^k=0.$$

Contracting (14) by l^j we have

$$(\partial_kp+pp_k)l^k=0.$$

(14) is written

(15) $$\partial_jp+pp_j=0.$$

Differentiation by y^k gives

(16) $$\partial_jp_k+p_jp_k+pp_{jk}=0.$$

(11) and (16) lead to

(17) $$p_{i;k}=p_ip_k+pp_{ik}.$$

As a locally Minkowski space is a Berwald space, it is easily seen that

(18) $$G^i_{hjk}=-p_{jh}\delta^i_k-p_{jk}\delta^i_h-p_{hk}\delta^i_j-p_{jhk}y^i=0,$$

where G^i_{hjk} is the hv-curvature tensor of $B\Gamma$. Contraction with respect to i and h gives

(19) $$p_{jk}=0.$$

Therefore (17) and (19) lead to (9). Conversely, (11) holds for P_n. From (9) and (11) we get

$$\partial_k p_i + p_i p_k + 2pp_{ik} = 0.$$

Contraction by y^i gives (15), which implies that P_n is a Finsler space of zero curvature. So that (17) holds, thus (9) leads to (19). Hence P_n is Berwald space. This completes the proof of Theorem 1.

§ 2. P_n with $H=0$.

Theorem 2. *P_n is of zero curvature if and only if in a rectilinear coordinate system we have, $\partial_j p = -pp_j$.*

Proof. If P_n is a Finsler spaces of zero curvature, we get (14) which implies (15).

Conversely, from (15) and (13) we get $H^i_{jk}=0$.

Theorem 3. *P_n is of zero curvature if and only if in a rectilinear coordinate system we have $H_0=0$*, where $H_0=H_s y^s$, $H_s=H^k_{sk}$.

Proof. Contraction with respect to i and k in (13) gives

$$H_j = n\partial_j p + (n-1)pp_j - \partial_0 p_j.$$

Contracting this by y^j we get

$$H_0=(n-1)(\partial_0 p + p^2). \tag{20}$$

On the other hand, contraction of (13) with y^j yields

$$H^i_k=(\partial_0 p + p^2)h^i_k. \tag{21}$$

where $H^i_k=H^i_{sk}y^s$.

(20), (21) and $H_0=0$ lead to $H^i_k=0$.

The converse is obvious.

Theorem 4. *P_n is of zero curvature if and only if in a rectilinear coordinate system we have*

$$3(\partial_0 L)^2 = 2L\partial_0\partial_0 L. \tag{22}$$

Proof. (22) is written

$$-2(-\partial_0 L/2L)\,[-L(-\partial_0 L/2L)+\partial_0 L]=\partial_0\partial_0 L,$$

which immediately implies

$$2Lp^2 - 2p\partial_0 L = \partial_0\partial_0 L. \tag{23}$$

For P_n we have

$$-2pL=\partial_0 L.$$

Differentiation by x^j gives

$$-(2L\partial_j p + 2p\partial_j L)=\partial_j\partial_0 L.$$

Contracting this by y^j we see

$$-2L\partial_0 p - 2p\partial_0 L = \partial_0\partial_0 L. \tag{24}$$

(23) and (24) lead to

$$\partial_0 p = -p^2. \tag{25}$$

By means of Theorem 3 and (20) it is easily seen that P_n is of zero curvature.

Conversely, from Theorem 3 we get (25). Then (25) and (24) lead to (23). So that (22) holds.

Theorem 5. *P_n is of zero curvature if and only if in a rectilinear coordinate system we have*

$$P_{|0} = p^2. \tag{26}$$

Proof. (26) is written

$$(\partial_k p - p_i G^i_k)y^k = p^2. \tag{27}$$

Using (10) we get (25). Hence P_n is of zero curvature.
Conversely, Theorem 3 leads to (25). So that (27) holds.

Theorem 6. *P_n is of zero curvature if and only if in a rectilinear coordinate system one of following conditions is satisfied:*

(a) $p_{k;0}=pp_k$, (b) $p_{i;k}=p_ip_k+pp_{ik}$, (c) $p_{;k}=pp_k$, (d) $\partial_i G^2=-2(n+1)\partial_i G$,

$$(e)\ \partial_0 G = -G^2/(n+1).$$

where $G=G^s_s$.

From [6] we have

Theorem 7. *P_n is of constant curvature if and only if in a rectilinear coordinate system one of following conditions is satisfied:*

(a) $\partial_k G_j = \partial_j G_k$, (b) $\partial_j G = y^k\partial_k G_j$, (c) $y^s\partial_s G_{ik}=0$, (d) $\partial_i G_{jk} = \partial G_{ik}$.

§ 3. P with $G=0$. (18) and (19) lead to

Theorem 8. *P_n is a Berwald space if and only if in a rectilinear coordinate system we have*

$$p = p_i(x)y^i \ (or\ \ p_{ik}=0,\ or\ G_{ik}=0).$$

Theorem 9. *P_n is a Berwald space if and only if in a rectilinear coordinate system we have*

$$\partial_k g_{ij} = -(2g_{ij}p_k + g_{ik}p_j + g_{jk}p_i). \tag{28}$$

Proof. By means of Theorem 30.7 of [4], it is easily seen that P_n is a locally Minkowski or Riemannian space. If it is a locally Minkowski space, (28) is obvious. If it is a Riemannian space, from $C_{ijk}=0$ and (6) we have

$$\partial_k g_{ij} = -2g_{ij}p_k - g_{jk}p_i - g_{ik}p_j - y_iP_{jk} - y_jp_{ik}.$$

Using Theorem 8 we get (28).
Conversely, (28) and (6) lead to

$$-y_ip_{jk} - y_jp_{ik} - 4pC_{ijk} + 2C_{ijk;0} = 0.$$

Contraction by y^j gives $p_{jk}=0$. Consequently the proof is complete by means of Theorem 8.

Theorem 10. *P_n is a Berwald space if and only if in a rectilinear coordinate system we have*

$$\gamma_{ijk} = -g_{ij}p_k - g_{kj}p_i. \tag{29}$$

Proof. From Proposition 3 we have

$$\gamma_{ijk}=-g_{ij}p_k-g_{kj}p_i-y_jp_{ik}-2C_{ijk}+C_{ijk\cdot 0}.$$

If P_n is a Berwald space, from Theorem 30.7 of [4] it is easily seen that P_n is a locally Minkowski or Riemannian space, so that we have

$$\gamma_{ijk}=-g_{ij}p_k-g_{kj}p_i-y_jp_{ik}.$$

Using Theorem 8 we get (29).

Conversely, (31) and Proposition 3 lead to

$$-y_jp_{ik}-2C_{ijk}+C_{ijk\cdot 0}=0.$$

Contraction by y^j gives $p_{ik}=0$. This completes the proof of Theorem.

Theorem 11. *P_n is a Berwald space if and only if in a rectilinear coordinate system we have*

$$\partial_i l_k=-p_il_k-p_kl_i-pl_{ik}. \tag{30}$$

Proof. By (3), (30) and Theorem 8 we can easily prove this Theorem.

From (11) we have

Theorem 12. *P_n is a Berwald space if and only if in a rectilinear coordinate system we have*

$$p_{i:k}=\partial_kp_i+2p_ip_k.$$

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RECENZII

I. J. SCHOENBERG, *Selected Papers*, Edited by CARL DE BOOR, Birkhäuser-Verlag, Boston, Basel, 1988, vol. I, XVIII+405 p., vol. II, XVI+441 p., $ 150.

Cele două volume de opere alese conţin 761 din cele peste 2000 pagini, de articole, publicate de autor pînă acum; acestea reflectă cu fidelitate vasta şi variata activitate a lui I. J. Schoenberg. Volumul I, pe lîngă o autobiografie care prezintă originea ideilor şi dezvoltarea lor, cuprinde următoarele teme: *Teoria numerelor* — conţinînd şi teza de doctorat referitoare la repartiţia assimptotică a numerelor reale (mod 1), *Inegalităţi lineare* (inclusiv metoda relaxării pentru inecuaţii liniare, în colaborare cu T. S. Motzkin), *Geometrie metrică* referitor la conceptul lui M. Fréchet (inclusiv un articol în colaborare cu J. von Neumann despre integrale Fourier şi geometrie metrică), *Analiză reală* (în particular problema lui Landau), Volumul al II-lea conţine *Spline cardinale* — care constituie una din operele fundamentale ale lui Schoenberg — inclusiv probleme de interpolare şi aproximare, *Regularizare*, *Pozitivitate totală* — o a doua preocupare fundamentală a lui Schoenberg, *Funcţii de frecvenţă Polya*, *Abordarea variaţională a funcţiilor spline*, *Interpolare Hermite-Birkhoff*. Fiecare capitol este urmat de un comentariu, făcut de o personalitate cunoscută în domeniul respectiv de activitate, comentariu care arată rolul pe care l-au avut lucrările lui Schoenberg şi dezvoltările ulterioare la care au dat loc. Printre autorii de comentarii — care reflectă preţuirea activităţii lui Schoenberg — notăm: P. Erdös, J. L. Goffin, R. Askey, S. Ruscheweyh, Carl de Boor, S. Karlin, G. G. Lorentz.

Adolf Haimovici

K. KURATOWSKI: *Selected papers*, PWN — Polish Scientific Publishers, Warszawa, 1988, LII+610 p.

K. Kuratowski este unul din creatorii topologiei moderne, influenţa sa în munca de cercetare şi educaţie depăşeşte hotarele Poloniei. Este unul din fondatorii periodicului polonez de matematică „Fundamenta Matematicae". Un deosebit de mare merit îi revine în restabilirea şi reorganizarea activităţii matematice în Polonia după cel de-al doilea război mondial. Activitatea ştiinţifică desfăşurată de K. Kuratowski între anii 1917—1979 (a murit la 18 iunie 1980 la Varşovia în vîrstă de 84 ani) s-a materializat în 172 lucrări ştiinţifice şi 2 monografii care tratează subiecte de topologie şi teoria mulţimilor. Între temele tratate în aceste articole amintim: teoria continuului, topologia planului, spaţii conexe, teoria dimensiunii, spaţii de funcţii, hiperspaţii, teoria descriptivă a mulţimilor, multifuncţii şi selectori. Cele două monografii ale sale se referă la aceleaşi subiecte: Topologie (în două volume) şi Teoria mulţimilor (în colaborare cu A. Mostowski). Menţionăm că o formă prescurtată a cursului de topologie „Introducere în teoria mulţimilor şi în topologie" a apărut în limba română la Ed. Tehnică în 1969.

În prezenta carte sînt selectate un număr de 49 de lucrări ale lui K. Kuratowski, unele scrise în colaborare cu B. Knaster, W. Sierpiński, S. Banach, S. Mazurkiewicz, A. Tarski, S. Ulam, J. v. Neumann, H. Steinhaus, C. Ryll-Nardzewski, ş.a.

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