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# QUASI-CONTINUITY IN H-CONES

BY

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**Introduction.** The property of quasi-continuity for superharmonic functions was discovered by H. C a r t a n. This notion was extended to abstract harmonic spaces by C o n s t a n t i n e s c u-C o r n e a [7] and further to $H$-cones in [8]. A different definition was proposed by B o b o c-B u c u r [2], [3]. In [4] the relation between quasi-continuity and nearly continuity was established.

The purpose of the present paper is to study the relation between the two definitions of quasi-continuity, as well as to develop further properties of quasi-continuous elements, especially those related to the change of base space. The main result is Theorem 12, which asserts the equivalence of the facts that any element in a $H$-cone, satisfying some additional assumptions, satisfies one or other definition of quasi-continuity. The last paragraph includes an application of the previous results to $H$-maps.

The notions, notations and results about $H$-cones from [6] are used without special mention. All the topological notions, not otherwise qualified, reffer to the natural topology. fcl $A$, fint $A$ represent the closure, resp. the interior of the set $A$, in the fine topology.

**§1. Definition and first results.** Let $S$ be a standard $H$-cone of functions on the nearly saturated space $X$. We say that $s \in S$ satisfies the condition $(K)$ if $\forall p \in S_0 : \bigwedge \{B^{X \setminus K} p \mid K \text{ compact}, \ s|_K \text{ continuous}\} = 0$.

We say that $s \in S$ satisfies the condition $(QC)$ if $\exists (G_n)_{n \in N}$ a decreasing sequence of open sets, such that $s|_{X \setminus G_n}$ is continuous, $\forall n \in \mathbf{N}$ and $\bigwedge_{n \in N} B^{G_n} 1 = 0$.

$\mathcal{K}$, resp. $Q$ will denote the set of elements of $S$, which satisfy the condition $(K)$, resp. $(QC)$.

As technical conditions, we consider also the following ones $(K_1)$ $\bigwedge \{B^{X \setminus K} 1 \mid K \text{ compact}, \ s|_K \text{ continuous}\} = 0$ $(QC_0)$ $\exists (G_n)_{n \in N}$ a decreasing sequence of open sets, such that : $s|_{X \setminus G_n}$ is continuous, $\forall n \in N$ and $\forall p \in S_0$; $\bigwedge_{n \in N} B^{G_n} p = 0$.

*Comments.* 1) Conditions $(K)$ appears in [8], as a direct adaptation of the definition from harmonic spaces. It proved useful in studying the fine openess of a $H$-map. Condition $(QC)$ appears in [2], [3], [4].

2) It is clear that $(K_1) \Rightarrow (K)$ and $(QC) \Rightarrow (QC_0)$. Conversely, if 1 is nearly bounded (see [4]), then $(K) \Rightarrow (K_1)$ and $(QC_0) \Rightarrow (QC)$. We remark that in any standard $H$-cone, there exists weak units, which are nearly continuous, hence nearly bounded.

3) The following implications also hold: $(K)\Rightarrow(QC_0)$ and $(K_1)\Rightarrow(QC)$.

Indeed, since there exists a countable, increasingly dense part of $S_0$, we can choose an increasing sequence $(K_n)_{n\in\mathbf{N}}$ of compact sets, such that $\forall p\in S_0: \bigwedge_{n\in\mathbf{N}} B^{X\setminus K_n}p=0$ (resp. $\bigwedge_{n\in\mathbf{N}} B^{X\setminus K_n}1=0$) and $s|_{K_n}$ is continuous, $\forall n\in\mathbf{N}$. It suffices to take $G_n=X\setminus K_n$.

We recall the next three propositions from [8].

**Proposition 1.** *$\mathcal{K}$ is a convex sub-cone, min-stable and specifically complete. If $F\subseteq S$ is an increasing part, such that*

$$\sup\{t(x)\mid t\in F\}=+\infty, \quad \forall x\in X$$

*then: $s\wedge t\in\mathcal{K}$, $\forall t\in F\Rightarrow s\in\mathcal{K}$.*

**Proposition 2.** *For each $s\in\mathcal{K}$ there exists a finely closed, polar set $A\subseteq X$ such that each $x\in X\setminus A$ has a fine neighbourhood, $K$ which is also naturally compact, such that $s|_K$ is continuous.*

**Proposition 3.** *If $\mathcal{K}\neq\emptyset$, then any nearly continuous element of $S$ belongs to $\mathcal{K}$. Particularly, if $S$ satisfies the axiom of nearly continuity, then either $\mathcal{K}=\emptyset$ or $\mathcal{K}=S$.*

*Remark.* The condition $\mathcal{K}\neq\emptyset$ is fulfilled, for exemple, if there exists "*Evans function*" in the following sense $\forall p\in S_0$ $\exists s\in S$ such that $\forall\alpha>0$: $K_\alpha=\mathrm{cl}[s<\alpha p]$ is a compact set.

Indeed, we have $B^{X\setminus K_\alpha}p\leq\alpha^{-1}\cdot s$, hence $\bigwedge\{B^{X\setminus K}p\mid K \text{ compact}\}\leq$ $\leq\bigwedge_{\alpha>0}B^{X\setminus K_\alpha}p\leq\bigwedge_{\alpha>0}\alpha^{-1}s=0$.

The next two results are proved in [4].

**Theorem 4.** *Let $p\in Q$ be a bounded potential. Then $p$ is nearly continuous.*

**Corollary 5.** *If there exists a strictly positive potential in $S$, then $S$ satisfy the axiom of nearly continuity iff $S=Q$.*

Next, we improve slightly Proposition 2.

**Lemma 6.** *For each $s\in\mathcal{K}$, there exists a finely closed, polar set $A\subseteq X$ and a sequence $(K_n)_{n\in\mathbf{N}}$ of compact sets, such that: $s|_{K_n}$ is continuous, $\forall n\in\mathbf{N}$ and: $X=A\cup\bigcup_{n\in\mathbf{N}}\mathrm{fint}\,K_n$.*

*Proof.* Let $\{p_n\}_{n\in\mathbf{N}}$ be an increasingly dense, countable family from $S_0$. Using [6, cor. 4.3.10.], there exists a family $\{K_{nm}\}$ of compact sets, such that $s|_{K_{nm}}$ is continuous, $\forall n,m\in\mathbf{N}$ and $\bigwedge_{m\in\mathbf{N}}B^{X\setminus K_{nm}}p_n=0$, $\forall n\in\mathbf{N}$.

Let us denote $A=X\setminus\bigcup_{n,m\in\mathbf{N}}\mathrm{fint}\,K_{nm}$. Then, for each $n\in\mathbf{N}$ $B^A p_n=$ $=B^{X\setminus\bigcup\mathrm{fint}\,K_{im}}p_n=B^{\bigcap\mathrm{fcl}(X\setminus K_{im})}p_n\leq\bigwedge_{m\in\mathbf{N}}B^{X\setminus K_{nm}}p_n=0$

**Proposition 7.** *If $\mathcal{K}\neq\emptyset$, then the following hold a) There exists a strictly positive potential in $S$. b) There exists a finely closed polar set $A\subseteq X$ such that each point from $X/A$ possesses naturally compact, fine neighbourhood.*

*Proof.* a) We show that each $p\in S_0$ is a potential. Let $(G_n)_{n\in\mathbf{N}}$ be a sequence of open sets, such that $\mathrm{cl}\,G_n\subseteq G_{n+1}$ and $X=\bigcup_{n\in\mathbf{N}}G_n$. Hence for each compact $K\subseteq X$ there exists $n\in\mathbf{N}$ such that $K\subseteq G_n$. It results: $\bigwedge_{n\in\mathbf{N}}B^{X\setminus G_n}p\leq$ $\leq\bigwedge_k B^{X\setminus K}p=0$.



b) Let us denote: $A=X\setminus\bigcup\{\mathrm{fint}\,K\mid K \text{compact}\}$. $\forall p\in S_0$ we have

$$B^A p=B^{\mathrm{fcl}\bigcap_k(X\setminus K)}p=B^{\bigcap_k(X\setminus K)}p\leq\bigwedge_k B^{X\setminus K}p=0.$$

**§2. The main theorem.** Let $S$ be a standard $H$-cone of functions on the nearly saturated space $X$.

**Proposition 8.** *Let us suppose that*

*(*) $\forall G\subseteq X$ finely open, $Q$ satisfies $(K)$ in $\mathcal{S}'(G)$. Then $Q\subseteq\mathcal{K}$.*

*Proof.* Let $s\in Q$, hence there exists a sequence $(G_n)_{n\in\mathbf{N}}$ of open sets, such that $s|_{X\setminus G_n}$ is continuous, $\forall n\in\mathbf{N}$ and: $\bigwedge_n B^{G_n}1=0$. Let $n\in\mathbf{N}$ be fixed and denote $G=X\setminus\mathrm{fcl}\,G_n$.

**Lemma 9.** *If $x\in\mathrm{fcl}\,G_n$ and $\mathrm{fcl}\,G_n$ is thin at $x$, then $x\in\mathrm{fcl}(G\setminus K)$, where $K\subseteq G$ is a compact set.*

*Proof.* Let, by contradiction, $V$ be a fine neighbourhood for $x$, such that $V\subseteq\mathrm{fcl}\,G_n\cup K$. Since $x\in\mathrm{fcl}\,G_n$ it follows $x\in K$, hence we can suppose that $V\cap K=\emptyset$. Then $V\subseteq\mathrm{fcl}\,G_n$ contradicts the fact that $\mathrm{fcl}\,G_n$ is thin at $x$.

The lemma above enables us to apply [1, prop. 2.3] in order to obtain, for any $p\in S_0$ and any compact $K\subseteq G$

$$B^{X\setminus K}p=B^{X\setminus G}p+\hat{B}^{G\setminus K}(p-B^{X\setminus G}p) \text{ on } G$$

(where $\hat{B}$ stands for the balayage in $\mathcal{S}'(G)$).

**Lemma 10.** *If $p\in S_0$, then $p-B^{X\setminus G}p$ is nearly continuous in $\mathcal{S}'(G)$.*

*Proof.* By [6] it suffices to show that $p-B^{X\setminus G}p$ is 1-continuous in $\mathcal{S}'(G)$. Let $(s_i)_{i\in I}$ be an increasing family from $\mathcal{S}'(G)$ such that $p-B^{X\setminus G}p=\bigvee_{i\in I}s_i$. By [1, th. 2.1] there exists a family $(v_i)_{i\in I}$ from $S$, such that $s_i=v_i-B^{X\setminus G}p$ on $G$, and $v_i\leqslant p$, $\forall i\in I$. Moreover, if $s_i\leqslant s_j$ we get $v_i\leqslant v$ on $G$, hence $B^G v_i\leqslant B^G v_j$ in $S$. But $v_i=B^G v_i$ on $G$, hence we can suppose that $(v_i)_{i\in I}$ is increasing to $p$ in $S$. For $\varepsilon>0$, there exists $i_\varepsilon$ such that $v_i\leqslant p+\varepsilon$, hence $s_i\leqslant p-B^{X\setminus G}p+\varepsilon$.

Using the above results and the hypothesis (*) we obtain for any $p\in S_0$

$$\bigwedge_{\substack{K \text{ compact}\\ K\subseteq G}}B^{X\setminus K}p=B^{G_n}p \text{ on } G.$$

But $B^{G_n}p=p$ on $b(G_n)=\mathrm{fcl}\,G_n=X\setminus G$, hence:

$$\bigwedge_k B^{X\setminus K}p\leqslant\bigwedge_n B^{G_n}p\leqslant\bigwedge_n B^{G_n}1=0.$$

*Remark.* There are situations when (*) does not holds, even for $G=X$. Indeed, let $X$ be a $B$-harmonic space, with countable base, finely connected [7]. Let $\emptyset\neq U\subseteq X$ be an open set, such that $U=\mathrm{int}\,\mathrm{cl}\,U$, but such that there exists irregular points in $\partial U$ for $U$ (for example the Lebesgue spine). Let $A=b(X/\mathrm{cl}\,U)$. $A$ is a basic set, hence finely closed. Hence $A$ is not finely open. On the other hand, $A$ is not closed: if $x\in\partial U$ is irregular, then $x\in$ $\in b(X/U)$, hence $x\notin A\subset b(X/U)$. But $X/U\subset\mathrm{cl}\,A$ shows that $x\in\mathrm{cl}\,A$.

If we consider the $H$-cone of the restrictions to $A$ of the positive, superharmonic functions on $X$, then we have a standard $H$-cone of functions on the semi-saturated space $A$: the natural and fine topologies on $A$, with respect to this cone are simply the traces on $A$ of the natural, resp. fine topologies on $X$.

We show that there exists points in $A$, which has no naturally compact, fine neighbourhood. Indeed, let $x \in \mathrm{fcl}\ (\mathrm{cl}\ A \setminus A) \setminus (\mathrm{cl}\ A \setminus A)$. Suppose that $K \subset A$ is a compact set and a fine neighbourhood for $x$. There exists a finely open set $V \subset X$ such that $x \in V$ and $V \cap A = \mathrm{fint}\ K$. Then

$$V \cap \mathrm{cl}\ A \subset \mathrm{cl}(V \cap A) \subset \mathrm{cl\ fint}\ K \subset K \subset A$$

while $V \cap (\mathrm{cl}\ A \setminus A) \neq \emptyset$.

A stronger result is available, choosing a basic set $A$, such that $A \subset$ $\subset \mathrm{fcl}(\mathrm{cl}\ A / A)$. In this case no point of $A$ has a naturally compact, fine neighbourhood.

The carrier theory used in the sequel refers to cl $X$, with the natural topology. The notations : $\mathcal{D}(X)$, $\mathcal{D}_d(X)$, $p_A$ are used accordingly (cf. [5], [6]).

**Key Lemma 11.** *Let us suppose that $X$ is semi-saturated and that there exists a strictly positive potential in $S$. If $s \in \mathcal{K}$ is nearly bounded, then $s \in \mathcal{D}_d$ ; hence $s$ is nearly continuous and $s \in Q$ .*

*Proof.* We use the method from [4, th. 2.1]. We can suppose that $s$ is universally bounded since $\mathcal{K}$ is specifically complete. Hence there exists $p \in S_0$ such that $s \leq p$. Let $(K_n)_{n \in \mathbf{N}}$ be an increasing sequence of compact sets, such that : $s|_{K_n}$ is continuous, $\forall n \in \mathbf{N}$ and : $\bigwedge_{n \in \mathbf{N}} B^{X \setminus K_n} p = 0$.

Since $s$ is bounded, it follows that $s$ is superharmonic. Being also a natural potential, we get $s \in \mathcal{D}(X)$. We prove next that : $s_{\bigcap_{n \in \mathbf{N}} (X \setminus K_n)} = 0$. Indeed, for any closed $F \subseteq \bigcap_{n \in \mathbf{N}} (X \setminus K_n)$ we have : carr $s_F \subseteq F \subseteq X \setminus K_n$, $\forall n \in \mathbf{N}$. Since $s \in \mathcal{D}$ [6, prop. 3.4.3] implies that $s_F = B^{X \setminus K_n} s_F$, $\forall n \in \mathbf{N}$. We get : $s_F \leq \bigwedge_{n \in \mathbf{N}} B^{X \setminus K_n} p = 0$, hence the conclusion.

It follows that $s = \bigvee_{n \in \mathbf{N}} s_{K_n}$ and it remains to prove that each $s_{K_n} (n \in \mathbf{N})$ satisfies the domination principle. $n \in \mathbf{N}$ being fixed, let $q \leq s_{K_n}$ and $u \in S$, $u \geq q$ on carr $q$. Since $s|_{K_m}$ is continuous, $\forall m \in \mathbf{N}$, it follows that $q|_{K_m}$ is also continuous, $\forall m \in \mathbf{N}$. For each $m \in \mathbf{N}$ there exists an open set $G_m \subset X$ such that $[u > q] \cap K_m = G_m \cap K_m$. Now : $G_m \cup (X \setminus K_m)$ is an open set which contains carr $q$. Let $t \in S$ be such that $t > p$ on $X \setminus K_m$. We have $u + t > q$ on $G_m \cup (X / K_m)$ and : $u + B^{X \setminus K_m} p \geqslant q$ on $X$. It results : $q \leq u + \bigwedge_{m \in \mathbf{N}} B^{X \setminus K_m} p = u$.

**The Main Theorem 12.** *Let $S$ be a standard $H$-cone on the set $X$ such that : (i) there exists a strictly positive potential on $X$ : (ii) $X$ is suslinean, semi-saturated and there exists an increasing sequence $(K_n)_{n \in \mathbf{N}}$ of compact sets, such that $X = \bigcup_{n \in \mathbf{N}} \mathrm{fint}\ K_n$. Then, the following properties are equivalent :*

1) $S = \mathcal{K}$ ; 2) $S = Q$ ; 3) *$S$ satisfies the axiom of nearly continuity.*

*Proof.* $2 \Rightarrow 3$ are proved in [3] or [4], and $1 \Rightarrow 3$ is the key lemma 11. $3 \Rightarrow 1$. Using (ii) and prop. 3, it suffices to show that $0 \in \mathcal{K}$. Let us denote $G_n = d(X \setminus K_n)$. $\{G_n\}_{n \in \mathbf{N}}$ is an increasing sequence of Borel, finely open sets. Moreover : fint $K_n \subset G_n \subset K_n$. Since each $p \in S_0$ is also a fine potential [3, th. 2.1] it follows that : $\bigwedge_{n \in \mathbf{N}} B^{X \setminus G_n} p = 0$.



*Remarks.* The other implications can be established directly ; $1 \Rightarrow 2$ holds if 1 is nearly bounded ; and $2 \Rightarrow 1$ holds if : $\forall G \subset X$ finely open, 0 satisfies $(K)$ in $\mathcal{S}'(G)$.

The condition (ii) from the theorem may be replaced by (ii′) $S$ satisfies the axiom of polarity, $X$ is suslinean, semisaturated and there exists a polar set $A \subset X$, such that each point of $X \setminus A$ has a naturally compact, fine neighbourhood.

**§ 3. The localisation and the changing of the representation space.**

**Proposition 13.** *If $s \in Q$ and $U \subset X$ is open, then $s|_U$ satisfies $(QC)$ in $\mathcal{S}'(U)$.*

*Proof.* Let $(G_n)_{n \in \mathbf{N}}$ be the sequence corresponding to $s$. Then $(G_n \cap U)_{n \in \mathbf{N}}$ is the sequence for $s|_U$ since : $\hat{B}^{G_n \cap U} \leq B^{G_n}\ 1$ on $U$.

*Comments.* The implication : $s$ satisfies $(K) \Rightarrow s|_U$ satisfies $(K)$ in $\mathcal{S}'(U)$ holds if one supposes that : There exists a strictly positive potential in $S$ ; 0 satisfies $(K)$ in $\mathcal{S}'(G)$, $\forall G \leq X$ finely open ; and $s$ is nearly bounded.

**Proposition 14.** *Let us suppose that there exists a strictly positive potential in $S$. Let $u \in S$ be such that there exists an open covering $(U_i)_{i \in I}$ for $X$, such that $u|U_i$ satisfies $(K)$ in $\mathcal{S}'(U_i)$, $\forall i \in I$. Then $u$ satisfies $(K)$ in $S$.*

*Proof.* Using lemma 10 we get

$$\bigwedge_{\substack{K \subseteq U_i \\ K \text{compact}}} \hat{B}^{U_i \setminus K}(p - B^{X \setminus U_i} p) = 0$$

for any $p \in S_0$. Hence

$$\bigwedge_{\substack{K \subseteq U_i \text{ compact} \\ u|_K \text{coutinuous}}} B^{X \setminus K} p = B^{X \setminus U_i} p \text{ on } U_i$$

It follows that

$$\bigwedge_{\substack{K \text{ compact} \\ u|_K \text{continuous}}} B^{X \setminus K} p \leq B^{X \setminus U_i} p \text{ on } U_i.$$

Since the last inequality clearly holds on $b(X \setminus U_i)$, it follows that it holds everywhere. Since $p$ is a potential, we get the conclusion.

**Corollary 15.** *Let us suppose that there exists a strictly positive potential in $S$. Let $(U_i)_{i \in I}$ be an open covering for $X$ such that any element of $\mathcal{S}'(U_i)$ satisfies $(K)$, $\forall i \in I$. Then $S = \mathcal{K}$.*

*Remark.* The implication : each element of $\mathcal{S}'(U_i)$ satisfies $(QC)$, $\forall_i \in I \Rightarrow$ $\Rightarrow S = Q$ holds if one supposes moreover that 0 satisfies $(K)$ in $\mathcal{S}'(G)$, $\forall G \subset X$ finely open.

**Theorem 16.** *Suppose that there exists a strictly positive potential in $S$ and that 0 satisfies $(K)$ in $\mathcal{S}'(G)$, $\forall G \subset X$ fienly open. Then the following are equivalent*

1) *$S = \mathcal{K}$, resp. $S = Q$.*

2) *$\forall U \subset X$ open, any element of $\mathcal{S}'(U)$ satisfies $(K)$, resp. $(QC)$.*

3) *There exists an open covering $(U_i)_{i \in I}$ of $X$ such that any element of $\mathcal{S}'(U_i)$ satisfies $(K)$, resp. $(QC)$, $\forall i \in I$.*

**Corollary 17.** *Suppose that there exists a strictly positive potential in $S$ and that 0 satisfies $(K)$ in $\mathcal{S}'(G)$, $\forall G \subset X$ finely open. Then the following are equivalent :*

1) *$S$ satisfies the axiom of nearly continuity.*

2) $\forall U \subset X$ open, $\mathcal{S}'(U)$ *satisfies the axiom of nearly continuity.*

3) *There exists an open covering* $(U_i)_{i\in I}$ *for* $X$, *such that* $\mathcal{S}'(U_i)$ *satisfies the axiom of nearly continuity,* $\forall i \in I$.

**Lemma 18.** *If* $A \subset X$ *is finely closed, then*

$$ {}^1B^{X_1\setminus A}\bar{s}|_X = B^{X\setminus A}s, \qquad \forall s\in S.$$

*Proof.* The relation follows from the inclusion: $X_1\setminus A \subseteq \mathrm{fcl}(X\setminus A)$.

**Proposition 19.** *If* $s\in\mathcal{K}$, *then* $\bar{s}$ *satisfies* $(K)$ *on* $X_1$.

*Proof.* Let $p\in S_c$ and denote $u=\bigwedge_{\substack{K\subset X_1 \text{ compact}\\ \bar{s}|_K \text{continuous}}} {}^1B^{X_1\setminus K}p$. From the preceding lemma we obtain:

$$u|_X=\bigwedge_{\substack{K\subset X\\ \bar{s}|_K \text{continuous}}} B^{X\setminus K}p \leqslant \bigwedge_{\substack{K\subset X\\ s|_K \text{continuous}}} B^{X\setminus K}p=0.$$

**Proposition 20.** $s$ *satisfies* $(QC)$ *in* $X$ *iff it satisfies* $(QC)$ *in* $X_1$.

*Proof.* Only one implication needs proof. Suppose that $s$ satisfies $(QC)$ in $X$ and let $(G_n)_{n\in\mathbf{N}}$ be the corresponding sequence. For each $n\in\mathbf{N}$, let $(t_k)_{k\in\mathbf{N}}$ be such that

$$\bigwedge_{k\in\mathbf{N}} t_k={}^1B^{G_n\cup(X_1\setminus X)}1$$

Denoting $D_m=[t_n+m^{-1}>1]$ the sets $G_n\cup D_m$ are open, $\bar{s}|X_1\setminus(G_n\cup D_m)$ is continuous and

$$\bigwedge_{n,m} {}^1B^{G_n\cup D_m}1=\bigwedge_n {}^1B^{G_n\cup(X_1\setminus X)}1=\bigwedge_n B^{G_n}1 \text{ on } X.$$

*Remarks.* The implication: $\bar{s}$ satisfies $(K)$ on $X_1 \Rightarrow s$ satisfies $(K)$ on $X$ holds for $s$ nearly bounded, if there exists a strictly positive potential in $S$ and 0 satisfies $(K)$ in $\mathcal{S}'(G)$, $\forall G\subset X$ finely open.

Next we study the modification of quasi-continuity at the change of the unit. Let $S$ be a standard $H$-cone and $u, v\in S$ be weak units. We denote: $X_{uv}=\{x\in X_u \mid x(v)<\infty\}$. $X_u\setminus X_{uv}$ is a polar set; $X_{uv}$ and $X_{vu}$ are finely homeomorphic through the map $x\to\dfrac{1}{x(v)}\cdot x$. If $u\leq\alpha v$, then $X_{vu}\equiv X_v$. Particularly, $X_{uv}$ is finely homeomorphic to $X_{u+v}$.

**Proposition 21.** *If* $v$ *satisfies* $(K)$ *on* $X_{uv}$, *then any element which satisfies* $(K)$ *on* $X_{uv}$ *satisfies also* $(K)$ *on* $X_{vu}$.

*Proof.* Let $K\leq X_{uv}$ be a compact set, such that $v|_K$ is continuous. Then, there exists a compact set $K'\subset X_{vu}$, such that $K$ and $K'$ are naturally homeomorphic through the canonical map. Let $s$ satisfies $(K)$ on $X_{uv}$ and $L\subset X_{uv}$ be a compact set such that $s|_L$ is continuous. $L\cap K$ is naturally homeomorphic to a compact $L'\subset X_{vu}$, and $s|_{L'}$ is continuous. Since

$$B^{X_{vu}\setminus L'}p\leq B^{X_{vu}\setminus K}p+B^{X_{uv}\setminus L}p$$

the conclusion follows.

**Corollary 22.** *If any element satisfies* $(K)$ *on* $X_{uv}$, *then any element satisfies* $(K)$ *on* $X_{vu}$.



*Particularly, if any element satisfies* $(K)$ *on any semi-saturated space of* $X_1$, *then any element satisfies* $(K)$, *on any semi-saturated space, with respect to any other weak unit.*

**Proposition 23.** *If* $u\leq\alpha v$ *and* $v$ *satisfies* $(QC)$ *on* $X_u$, *then:* $s$ *satisfies* $(QC)$ *on* $X_v \Rightarrow s$ *satisfies* $(QC)$ *on* $X_u$.

*Proof.* Let $D_m\subset X_{uv}$, $(m\in\mathbf{N})$ be open sets, such that $v|_{X_{uv}\setminus D_m}$ is continuous and $\bigwedge_{m\in\mathbf{N}} B^{D_m}1=0$. Let $G_n\leq X_v (n\in\mathbf{N})$ be open sets such that $s|_{X_v\setminus G_n}$ is continuous and $\bigwedge_{n\in\mathbf{N}} B^{G_n}1=0$. Let $\varphi: X_{uv}\to X_v=X_{vu}$ be the canonical mapping. Since $\varphi|_{X_{uv}\setminus D_m}$ is naturally continuous, it follows that $\varphi^{-1}(X_v\setminus G_n)\cap(X_{uv}\setminus D_m)$ is closed in $X_{uv}\setminus D_m$, hence in $X_{uv}$ $\forall m, n\in\mathbf{N}$. Hence there exists open sets $H_{mn}\subseteq X_{uv}$ such that $X_{uv}\setminus H_{mn}=\varphi^{-1}(X_v\setminus G_n)\cap(X_{uv}\setminus D_m)$. Also $s|_{X_{uv}\setminus H_{mn}}$ is continuous. By hypothesis: $B^{H_{mn}}1\leq\alpha B^{G_n}1+B^{D_m}1$, which proves the assertion.

**Proposition 24.** *If there exists a strictly positive potential in* $S$ *and any element satisfies* $(QC)$ *on a representation space, then any element satisfies* $(QC)$ *on any other representation space (with respect to any other weak unit).*

*Proof.* One uses the fact that the axiom of nearly continuity is fulfilled.

**§ 4. H-maps and quasi-continuity.** Let $S$ and $T$ be $H$-cones of functions on the sets $X$, $Y$. Let $\varphi: Y\to X$ be a $H$-map (i.e. $\forall s\in S$ bounded, $s\circ\varphi\in T$). We say that $\varphi$ satisfies $(QC)$ if: there exists an increasing sequence of open sets $(G_m)_{m\in\mathbf{N}}$ of $Y$ such that $\varphi|_{Y\setminus G_n}$ is continuous and $\bigwedge_{n\in\mathbf{N}} B^{G_n}1=0$.

**Theorem 25.** *The following properties are equivalent*

1) $\varphi$ *satisfies* $(QC)$;

2) $\forall s\in S_0$, $s\circ\varphi$ *satisfies* $(QC)$ *in* $T$.

3) $\forall s\in S$ *bounded and satisfying* $(QC)$ *in* $S$, $s\circ\varphi$ *satisfies* $(QC)$ *in* $T$.

*Proof.* $1\Rightarrow2$ and $3\Rightarrow2$ are clear.

$2\Rightarrow1$ Let $\{s_n \mid n\in\mathbf{N}\}$ be a countable, increasingly dense part of $S_0$. Let $\{y_m \mid m\in\mathbf{N}\}$ be a countable, dense part of $Y$, such that each $s_n\circ\varphi$ is quasi-continuous with respect to the weight $\psi y_j$ (cf. [2]). One can choose, for each $m, n\in\mathbf{N}$ an open set $G_m^n$ such that $s_n\circ\varphi|_{Y\setminus G_m^n}$ is continuous and

$$B^{G_m^n}1(y_j)<m^{-1}\cdot2^{-(n+1)}, \quad \forall j=\overline{1,m}.$$

For each fixed $n\in\mathbf{N}$, the sequence $(G_m^n)_{m\in\mathbf{N}}$ is decreasing. Let us denote $D_m=\bigcup_{n\in\mathbf{N}} G_m^n$. $(D_m)_{m\in\mathbf{N}}$ is a decreasing sequence of open sets and, for each $n\in\mathbf{N}$, $s_n\circ\varphi|_{Y\setminus D_m}$ is continuous. Finally, since: $B^{D_m}1(y_j)<m^{-1}$, $\forall j=\overline{1,m}$ we get $\bigwedge_{m\in\mathbf{N}} B^{D_m}1(y_j)=0$, $\forall j\in\mathbf{N}$, hence the conclusion.

$1\Rightarrow3$ Let $(G_n)_{n\in\mathbf{N}}$ be a decreasing set of open sets of $Y$, such that $\varphi|_{Y\setminus G_n}$ is continuous, $\forall n\in\mathbf{N}$. Let $(D_m)_{m\in\mathbf{N}}$ be the sequence associated with $s$. Then $A_{nm}=(Y\setminus G_n)\cap\varphi^{-1}(X\setminus D_m)$ is closed and $s\circ\varphi|_{A_{nm}}$ is continuous, $\forall n, m\in\mathbf{N}$. Finally $Y\setminus A_{nm}=G_n\cup\varphi^{-1}(D_m)$ and: $B^{Y\setminus A_{nm}}1\leq B^{G_n}+B^{\varphi^{-1}(D_m)}1\leq B^{G_n}1+B^{D_m}1\varphi$, hence the conclusion.

*Remark.* If any element of $T$ satisfies $(QC)$, then any $H$-map from $S$ to $T$ satisfies $(QC)$. Using the results above, we establish finally the connection with the definition used in [8].

**Theorem 26.** *If there exists a strictly positive potential in $S$, then the following are equivalent*

1) $\forall p \in T_\theta : \bigwedge \{B^{Y \setminus K} p \mid K \subset Y$ compact, $\varphi|_K$ continuous$\} = 0$
2) $\varphi$ *satisfies* $(QC)$.

## BIBLIOGRAPHY


1. Boboc N., Bucur Gh. — *Natural Sheaf Property in Standard H-cones of Functions*, Preprint Series in Math. 32/1982 INCREST
2. Boboc N., Bucur Gh. — *Potentials and Supermedian Functions on Fine Open Sets in Standard H-cones*, Rev. Roum. Math. Pures Appl. 31 (1986), 745—774
3. Boboc N., Bucur Gh. — *Fine Potentials and Supermean Functions on Standard H-cones*, Rev. Roum. Math. Pures Appl. 32 (1987) 881—890
4. Boboc N., Bucur Gh. — *Potentials in Standard H-cones of Functions*, Preprint INCREST, 6/1988
5. Boboc N., Bucur Gh., Cornea A. — *Carrier Theory and Negligible Sets in a Standard H-cone of functions*, Rev. Roum. Math. Pures Appl. 25 (1980), 163—197
6. Boboc N., Bucur Gh., Cornea A. — *Order and Convexity in Potential Theory: H-cones*, Lect. Notes in Math. Springer Verlag, Berlin-Heidelberg-New York, vol. 853, 1981
7. Constantinescu C., Cornea A. — *Potential Theory on Harmonic Spaces*, Springer Verlag, Berlin-Heidelberg-New York, 1972
8. Popa E. — *Natural Continuity of a H-map*, An. şt. Univ. Iaşi, t. XXXIII (1987), 189—192


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Matematică, 36 (1990)

# EXTENSIONS OF SOME RESULTS OF M. PAL

BY

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**Introduction.** In [3] S. Kurepa proved the following theorem.

**Theorem K.** *If $A \subset \mathbf{R}^n$ ($n$-dimensional Euclidean space) is Lebesgue measurable and has positive measure and $b_1, b_2, \ldots, b_p$ ($b_k \neq 0$) are $p$ real numbers then there exists a ball $K_r$ of radius $r$ with center at the origin such that for each $x \in K_r$, there are vectors $a_0(x), a_1(x), \ldots, a_p(x)$ in $A$ such that*

$$x = (a_1(x) - a_0(x))b_1^{-1} = (a_2(x) - a_0(x))b_2^{-1} = \ldots = (a_p(x) - a_0(x))b_p^{-1}.$$

M. Pal [11] extended Kurepa's result to the following.

**Theorem P.** *Let $A$ be a set of positive Lebesgue measure in $\mathbf{R}^n$. Also let $b_1, b_2, \ldots, b_p (b_k \neq 0)$ be a system of $p$ real numbers and let $(c_n)$ be a sequence of non-zero real numbers such that $\lim_{n\to\infty} c_n = 1$. Then there exists a positive integer $N_0$ and a ball $K$ with center at the origin such that for every vector $x$ in $K$ and system of $p$ real numbers $c_{n_1}, c_{n_2}, \ldots, c_{n_p}$, of the sequence $(c_n)$ with $n_i > N_0$ theere exist vectors $a_0(x; c_{n_1}, c_{n_2}, \ldots, c_{n_p})$ and $a_k(x; c_{n_1}, c_{n_2}, \ldots, c_{n_p})$ ($k = 1, 2, \ldots, p$) in $A$ such that $x = [c_{n_k}^{-1} a_k(x; c_{n_1}, c_{n_2}, \ldots, c_{n_p}) - a_0(x; c_{n_1}, c_{n_2}, \ldots, c_{n_p})]b_k^{-1}$, $k = 1, 2, \ldots, p$.*

The mapping $T_c$, where $c$ is a real number, $T_c : \mathbf{R}^n \to \mathbf{R}^n$, defined by $T_c(x) = cx$, where $cx = (cx_1, cx_2, \ldots, cx_n)$ for each $x = (x_1, x_2, \ldots, x_n)$, is an extremely simple mapping. It is natural to ask if Theorem P can be extended to a theorem about more general mappings than those of type $T_c$. In addition one might wonder about extending Theorem P in such a way that each $x \in K$ is expressible in infinitely many ways.

Borwein and Ditor [2] have proved the following theorem, answering a question of P. Erdös.

**Theorem B and D.** (i) *If $A$ is a measurable set of real numbers having positive measure and $(d_n)$ is a sequence converging to zero, then, for almost all $x \in A$, $x + d_n \in A$ for infinitely many $n$;* (ii) *There is a measurable set $A$ of positive measure and a monotonic sequence $(d_n)$ of positive numbers converging to zero such that, for all $x$, $x + d_n \notin A$ for infinitely many $n$.*

Recently the current author has generalized the above result of Borwein and Ditor using general 2-place functions in place of $+$ (see [9]).

In [8] the present author observed that the Baire set analogue of (ii) is not true. In that paper the author also considers a certain property $(C)$

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