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CONDITIONS FOR QUATERNIONIC SECTIONAL CURVATURE TO BE CONSTANT

BY

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1. Introduction. Let M^{4m} be a quaternion Kaehlerian manifold with metric g. Then there exists a 3-dimensional vector bundle V of type (1, 1) tensors with local basis of almost Hermitian structures I, J, K such that (1) $IJ=-JI=K$, and (2) for any local cross-section ψ of V and any vector X tangent to M, $\nabla_X\psi$ is also a local cross-section of V, where ∇ denotes the covariant differentiation on M, [1].

Let X be a unit vector of M. Then X, IX, JX and KX form an orthonormal frame in M. Let us denote by $Q(X)$ the 4-plane spanned by them. A 2-dimensional section $\Pi(X, Y)$ of T_pM spanned by X and Y is called half quaternionic if $Q(X)=Q(Y)$. The sectional curvature for a half-quaternionic section is called quaternionic. A quaternion Kaehlerian manifold of constant quaternionic sectional curvature is called a quaternion-space-form. For examples of quaternion-space-forms see [4] and [6].

Let M be a quaternion Kaehlerian manifold and $\Pi(X, Y)$ a 2-dimensional section of T_pM spanned by orthonormal vectors X and Y. We shall say that $\Pi(X, Y)$ is 0-quaternionic respect to the local basis $\{I, J, K\}$ of V at p if $|g(\Phi X, Y)|=\cos\theta=\alpha$ for any $\Phi=I, J, K$. It is clear that if $4m\geq 12$ such sections do exist and are not half-quaternionic for $0\leq\alpha^2<1/3$.

The purpose of the present paper is to prove the following theorems:

Theorem 1. *Let M be a quaternion Kaehlerian manifold of real dimension $4m\geq 12$. Then M is a quaternion-space-form if and only if for any $p\in M$ and some local basis $\{I, J, K\}$ of V at p, the sectional curvature for 0-quaternionic sections respect to $\{I, J, K\}$ at p such that $0\leq\alpha^2<1/3$ are constant.*

Theorem 2. *Let M be a quaternion Kaehlerian manifold of real dimension $4m\geq 12$. Then M is a quaternion-space-form if and only if for any $p\in M$ and some local basis $\{I, J, K\}$ of V at p the curvatures $R(X, \Phi X, \Phi Y, Y)$ for any $\Phi=I, J, K$ and any 0-quaternionic section $\Pi(X, Y)$ respect to $\{I, J, K\}$ at p such that $0\leq\alpha^2<1/3$ are constant.*

Note that if $\alpha=0$, $Q(X)\perp Q(Y)$, that is, $\Pi(X, Y)$ is a totally real section. Thus Theorems 1 and 2 generalize results obtained in [3].

In the proof of such theorems we shall use the following lemmas

Lemma A. [1]. *Let M be a quaternion Kaehlerian manifold. If the real dimension of M is $4m\geq 8$ then*

$$[R(X, Y), I]=-\frac{\rho}{m+2}\{g(KX, Y)J-g(JX, Y)K\}$$

$$[R(X,Y),J] = -\frac{\rho}{m+2}\{-g(KX,Y)I+g(IX,Y)K\}$$

$$[R(X,Y),K] = -\frac{\rho}{m+2}\{g(JX,Y)I-g(IX,Y)J\}$$

for any unit vector fields X, Y tangent to M, where R denotes the curvature tensor of M and $4m\rho$ is the constant scalar curvature of M.

Lemma B. [3]. *Let M be a quaternion Kaehlerian manifold of real dimension $4m \geq 8$ and X a unit vector in T_pM. If there exists a local basis $\{I,J,K\}$ of V such that*

$$\sigma(X,IX)=\sigma(X,JX)=\sigma(X,KX)$$

$$R(X,IX,X,JX)=R(X,IX,X,KX)=R(X,JX,X,KX)$$

then the quaternionic sectional curvature in $Q(X)$ is constant, where σ denotes the sectional curvature.

2. *Proof of Theorem* 1. Let us suppose that for any 0-quaternionic section $\Pi(X,Y)$ respect to $\{I,J,K\}$ at p $\sigma(X,Y)=\lambda$. If we suppose that $g(IX,Y)=\alpha$ then $\Pi(X,-Y+2\alpha IX)$ is also a 0-quaternionic section. This gives, by applying Lemma *A*,

$$(2.1)\qquad R(X,Y,IX,X)=\alpha\sigma(X,IX).$$

From the section $\Pi(Y,X+2\alpha IY)$ we obtain

$$(2.2)\qquad R(X,Y,IY,Y)=\alpha\sigma(Y,IY).$$

But the section $\Pi\left(\frac{\alpha}{\sqrt{1-\alpha^2}}X+\frac{1}{\sqrt{1-\alpha^2}}IY,\ \frac{1}{\sqrt{1-\alpha^2}}IX-\frac{\alpha}{\sqrt{1-\alpha^2}}Y\right)$ is also 0-quaternionic, thus $\lambda=\sigma\left(\frac{\alpha}{\sqrt{1-\alpha^2}}X+\frac{1}{\sqrt{1-\alpha^2}}IY,\ \frac{1}{\sqrt{1-\alpha^2}}IX-\frac{\alpha}{\sqrt{1-\alpha^2}}Y\right)$.

Applying Lemma *A* we get

$$\frac{1}{(1-\alpha^2)^2}\Big\{\alpha^2\sigma(X,IX)+\alpha^2\sigma(Y,IY)+(\alpha^4+2\alpha^2+1)\lambda+(2\alpha^3+$$

$$(2.3)\quad +2\alpha)R(X,Y,Y,IY)-(2\alpha^3+2\alpha)R(X,Y,IX,X)-2\alpha^2R(X,IX,Y,IY)-$$

$$-4\alpha^4\frac{\rho}{m+2}\Big\}=\lambda.$$

From (2.1) and (2.2) (2.3) gives

$$(2.4)\qquad R(X,IX,Y,IY)=2\lambda-(\alpha^2+\tfrac{1}{2})(\sigma(X,IX)+\sigma(Y,IY))-2\alpha^2\frac{\rho}{m+2}.$$

Considering the section $\Pi(\alpha X-\alpha IX+IY,X)$ (respectively, $\Pi(X,\alpha X+\alpha IX+IY)$) we have

$$(2.5)\qquad \lambda=\alpha^2\sigma(X,IX)-2\alpha R(X,IX,IY,X)+R(X,IY,IY,X)$$

respectively,

$$(2.6)\qquad \lambda=\alpha^2\sigma(X,IX)+2\alpha R(X,IX,IY,X)+R(X,IY,IY,X)$$

and substracting (2.6) from (2.5)

$$(2.7)\qquad R(X,IX,IY,X)=0.$$

From (2.5), (2.6) and (2.7) we get as $R(X,IY,IY,X)=R(IX,Y,Y,IX)$ changing X by Y that

$$(2.8)\qquad \sigma(X,IX)=\sigma(Y,IY)$$

that is, (2.4) can be writen as

$$(2.9)\qquad R(X,IX,Y,IY)=2\lambda-(2\alpha^2+1)\sigma(X,IX)-2\alpha^2\frac{\rho}{m+2}.$$

From Lemma A,

$$(2.10)\qquad R(X,Y,IY,IX)=R(X,Y,Y,X)-2\alpha^2\frac{\rho}{m+2}=\lambda-2\alpha^2\frac{\rho}{m+2}$$

and (2.5) and Lemma *A* give

$$(2.11)\qquad R(X,IY,IY,X)=\lambda-\alpha^2\sigma(X,IX)=R(X,IY,IX,Y)+2\alpha^2\frac{\rho}{m+2}.$$

From (2.9), (2.10) and (2.11) and using the identity of Bianchi we get

$$(2.12)\qquad \sigma(X,IX)=\frac{1}{3\alpha^2+1}\left(4\lambda-6\alpha^2\frac{\rho}{m+2}\right).$$

Let us suppose that $g(JX,Y)=\alpha$. Then reasoning as above with the sections $\Pi(X,Y-2\alpha JX)$, $\Pi(Y,-X-2\alpha JY)$, $\Pi\left(\frac{\alpha}{\sqrt{1-\alpha^2}}IX+\frac{1}{\sqrt{1-\alpha^2}}KY,\ \frac{1}{\sqrt{1-\alpha^2}}KX-\frac{\alpha}{\sqrt{1-\alpha^2}}IY\right)$, $\Pi(\alpha X-\alpha JX+JY,X)$ and $\Pi(X,\alpha X+\alpha JX+JY)$ we obtain

$$(2.13)\qquad \sigma(X,JX)=\sigma(Y,JY)=\frac{1}{3\alpha^2+1}\left(4\lambda-6\alpha^2\frac{\rho}{m+2}\right).$$

If $g(KX,Y)=\alpha$ we repeat the reasoning. So let us suppose that $g(KX,Y)=-\alpha$. (The following must be applied if either $g(IX,Y)$ or $g(JX,Y)$ is equal to $-\alpha$ to the corresponding sections). Now $\Pi(X,Y+2\alpha KX)$ is 0-quaternionic and then $\lambda=\sigma(X,Y+2\alpha KX)$, giving

$$(2.14)\qquad R(X,Y,X,KX)=\alpha\sigma(X,KX).$$

From the section $\Pi(Y,-X+2\alpha KY)$ we obtain

$$(2.15)\qquad R(X,Y,Y,KY)=\alpha\sigma(Y,KY).$$

The section $\Pi\left(\frac{\alpha}{\sqrt{1-\alpha^2}}IX+\frac{1}{\sqrt{1-\alpha^2}}JY,\ \frac{1}{\sqrt{1-\alpha^2}}JX-\frac{\alpha}{\sqrt{1-\alpha^2}}IY\right)$ is

0-quaternionic thus $\lambda=\sigma\left(\frac{\alpha}{\sqrt{1-\alpha^2}}IX+\frac{1}{\sqrt{1-\alpha^2}}JY,\ \frac{1}{\sqrt{1-\alpha^2}}JX-\frac{\alpha}{\sqrt{1-\alpha^2}}IY\right)$ which gives

(2.16) $R(Y, KY, KX, X)=-2\lambda+\frac{1}{2}(1+2\alpha^2)(\sigma(X, KX)+\sigma(Y, KY))+2\alpha^2\frac{\rho}{m+2}$.

But the section $\Pi(-\alpha X+\alpha KX+KY, X)$ is 0-quaternionic, obtaining

(2.17) $$\lambda=\alpha^2\sigma(X, KX)+2\alpha R(X, KX, KY, X)+R(X, KY, KY, X).$$

From the section $\Pi(X, \alpha X+\alpha KX-KY)$ we have

(2.18) $$\lambda=\alpha^2\sigma(X, KX)-2\alpha R(X, KX, KY, X)+R(X, KY, KY, X).$$

Now from (2.17) and (2.18) we get

(2.19) $$R(X, KX, KY, X)=0$$

and

(2.20) $$\sigma(X, KX)=\sigma(Y, KY)$$

and as above

(2.21) $$\sigma(X, KX)=\frac{1}{3\alpha^2+1}\left(4\lambda-6\alpha^2\frac{\rho}{m+2}\right).$$

Suppose again that $g(KX, Y)=-\alpha$ and consider the section $\Pi(X, 2\alpha IX+2\alpha JX-Y)$ which is 0-quaternionic. Thus $\lambda=\sigma(X, 2\alpha IX+2\alpha JX-Y)$, giving

(2.22) $$\lambda=4\alpha^2\sigma(X, IX)+8\alpha^2R(X, IX, JX, X)+4\alpha^2\sigma(X, JX)-\\-4\alpha R(X, Y, IX, X)-4\alpha R(X, Y, JX, X)+\lambda.$$

From (2.1), (2.2) and (2.22) we have

(2.23) $$R(X, IX, X, JX)=0.$$

From the section $\Pi(X, -2\alpha JX+2\alpha KX+Y)$ we obtain

(2.24) $$R(X, JX, X, KX)=0$$

and from $\Pi(X, -2\alpha IX+2\alpha KX+Y)$

(2.25) $$R(X, IX, X, KX)=0.$$

If $g(KX, Y)=\alpha$ we apply the same reasoning to the sections $\Pi(IX, -2\alpha KX+2\alpha JX+IY)$, $\Pi(KX, 2\alpha IX-2\alpha JX+KY)$ and $\Pi(X, -2\alpha IX+2\alpha KX+Y)$ and obtain again (2.23), (2.24) and (2.25). Then from (2.12), (2.13), (2.21), (2.22), (2.23) and (2.24) and Lemma B we have that M must be a quaternion-space-form.

The converse is trivial and completes the proof.

3. **Proof of Theorem 2.** Let us suppose that for any 0-quaternionic section $\Pi(X, Y)$ respect to $\{I, J, K\}$, $R(X, \Phi X, \Phi Y, Y)=\mu=$constant for any

$\Phi=I, J, K$ and $g(IX, Y)=\alpha$. Then $\mu=R(X, IX, -IY-2\alpha X, -Y+2\alpha IX)$ which gives

(3.1) $$R(X, Y, IX, X)=\alpha\sigma(X, IX).$$

But also $\mu=R(X, JX, -JY-2\alpha KX, -Y+2\alpha IX)$, giving

(3.2) $$R(X, JX, Y, KX)=\alpha\sigma(X, JX)-\alpha\frac{\rho}{m+2}$$

and from $\mu=R(X, KX, -KY+2\alpha JX, -Y+2\alpha IX)$ we obtain

(3.3) $$R(X, KX, Y, JX)=\alpha\sigma(X, KX)-\alpha\frac{\rho}{m+2}.$$

From (3.1), (3.2), (3.3) and the identity of Bianchi we get

(3.4) $$\sigma(X, IX)=\frac{\rho}{m+2}+\sigma(X, JX)-\sigma(X, KX).$$

Let us suppose that $g(JX, Y)=\alpha$. Then $\Pi(X, -Y+2\alpha JX)$ is 0-quaternionic. (If $g(JX, Y)=-\alpha$ consider $\Pi(X, -Y-2\alpha JX)$). Then $\mu=R(X, IX, -IY+2\alpha KX, -Y+2\alpha JX)$ and we have

(3.5) $$R(X, IX, Y, KX)=-\alpha\sigma(X, IX)+\alpha\frac{\rho}{m+2}.$$

Similarly we obtain

(3.6) $$R(X, JX, Y, X)=\alpha\sigma(X, JX)$$

and

(3.7) $$R(X, KX, Y, IX)=-\alpha\sigma(X, KX)+\alpha\frac{\rho}{m+2}.$$

From (3.5), (3.6) and (3.7) we have

(3.8) $$\alpha(X, JX)=\frac{\rho}{m+2}+\sigma(X, IX)-\sigma(X, KX).$$

From (3.4) and (3.8) we have

(3.9) $$\sigma(X, IX)=\sigma(X, JX)=\sigma(X, KX)=\frac{\rho}{m+2}.$$

Now the section $\Pi(X, 2\alpha IX+2\alpha JX-Y)$ is 0-quaternionic. Thus $\mu==R(X, IX, -2\alpha X+2\alpha KX-IY, 2\alpha IX+2\alpha JX-Y)$. Then we obtain

(3.10) $$R(X, IX, X, JX)=-2\alpha(X, IX)+\frac{\rho}{m+2}.$$

But $\mu=R(X, JX, -2\alpha KX-2\alpha X-JY, 2\alpha IX+2\alpha JX-Y)$ gives

(3.11) $$R(X, JX, X, IX)=\frac{3}{2}\frac{\rho}{m+2}-\frac{1}{2}\sigma(X, KX).$$

Then from (3.9), (3.10) and (3.11) we get

$$R(X, IX, X, JX)=0. \tag{3.12}$$

Now from the sections $H(X, 2\alpha JX+2\alpha\varepsilon KX-Y)$ (where $\varepsilon=1$ if $g(KX, Y)=\alpha$ and $\varepsilon=-1$ if $g(KX, Y)=-\alpha$) and $H(X, 2\alpha IX+2\alpha\varepsilon KX-Y)$ we obtain

$$R(X, JX, X, KX)=R(X, IX, X, KX)=0. \tag{3.13}$$

Then from (3.9), (3.12), (3.13) and Lemma B, M must be a quaternion-space-form. The converse is trivial and finishes the proof.

Note. It is easy to prove similar results in the case of an indefinite quaternion Kaehlerian manifold with the same conditions as above imposed to non-degenerate θ-quaternionic sections. (See [2]).

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NON-LINEAR CONNECTIONS FOR 2-OSCULATING VECTOR FIELDS

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Introduction. The notion of non-linear connection has been studied by many authors. In this paper, we introduce the notion of non-linear connection for 2-osculating vector fields on a manifold M, as covariant derivative of the sections of the vector bundle of 2-osculating vector fields on M and also as a distribution in the vector bundle of non-zero 2-osculating vector fields. We prove that the two definitions are equivalent. Furthermore, we study a particular set of non-linear connections for 2-osculating vector fields, which is bijective to the set of the connections on the principal bundle P of the 2-osculating frames of M. Finally, we study the integrability conditions.

§1. Preliminaries. Let M be a C^∞-differentiable manifold of dimension n. Let us denote with $\mathcal{F}(M)$ the algebra of the C^∞-differentiable real functions on M and, for any $r\in N^*$, with $\Pi_r : D^r(M)\to M$ the r-osculating vector bundle to M. It is well known that, if (U, x^i) $i=1, \dots, n$ is a coordinate neighborhood of M, and $p\in U$, a base of $D^2_p(M)=\Pi_2^{-1}(p)$ is given by $\left\{(e_h)_p, \frac{1}{2}(e_{ij})_p\right\}$, where $e_h=\frac{\partial}{\partial x^h}$ $h=1, \dots, n$, $e_{ij}=\frac{\partial^2}{\partial x^i\partial x^j}$, $1\leqslant i\leqslant j\leqslant n$. Such a base is called the natural 2-osculating frame in p relative to (U, x^i).

We put $I=\{1, \dots, n\}$, $J=\{(i, j)\in I\times I\,|\,i\leqslant j\}$ and we denote with $(\Pi_2^{-1}(U), x^h, y^h, y^{ij})$, $h\in I$, $(i, j)\in J$ the coordinate neighborhood induced by (U, x^i) on $D^2(M)$. If (U, x^i), $(U', x^{i'})$ are two coordinate neighborhoods on M, with $U\cap U'\neq\emptyset$, we put $\theta^h_{h'}=\frac{\partial x^h}{\partial x^{h'}}=\partial_{h'}x^h$, $\theta^h_{h'i'}=\partial_{i'}\theta^h_{h'}$, $\theta^h_{h'i'm'}=\partial_{m'}\theta_{h'i'}$ in $U\cap U'$. Furthermore, we denote with $\Pi : P\to M$ the principal bundle of the 2-osculating frames on M. For more details, we refer the reader to [5], [6].

Let us recall that we can provide the vector space $D(M)=\bigcup\limits_{r\in N^*} D^r(M)$ with the structure of Lie algebra, putting, for any $F, G\in D(M)$, $[F, G]=F\circ G-G\circ F$. It is known that a differentiable transformation Φ of M, induces the automorphism $\tilde\Phi$ of $D(M)$, defined by : $(\tilde\Phi(F))(f)=F(f\circ\Phi)$, $\forall F\in D(M)$, $\forall f\in\mathcal{F}(M)$.

Def. 1.1. *Let* $X\in D^1(M)$ *be complete,* $\{\Phi_t\}_{t\in\mathbf{R}}$ *be a* 1-*parameter group of transformations which generates* X. *We call Lie derivative of the r-osculating*

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