Then from (3.9), (3.10) and (3.11) we get

$$R(X, IX, X, JX)=0. \tag{3.12}$$

Now from the sections $H(X, 2\alpha JX+2\alpha\varepsilon KX-Y)$ (where $\varepsilon=1$ if $g(KX, Y)=\alpha$ and $\varepsilon=-1$ if $g(KX, Y)=-\alpha$) and $H(X, 2\alpha IX+2\alpha\varepsilon KX-Y)$ we obtain

$$R(X, JX, X, KX)=R(X, IX, X, KX)=0. \tag{3.13}$$

Then from (3.9), (3.12), (3.13) and Lemma B, M must be a quaternion-space-form. The converse is trivial and finishes the proof.

Note. It is easy to prove similar results in the case of an indefinite quaternion Kaehlerian manifold with the same conditions as above imposed to non-degenerate 0-quaternionic sections. (See [2]).

REFERENCES

1. Ishihara, S. — *Quaternion Kählerian manifolds*, J. Differential Geometry, 9 (1974), 483—500
2. Perez, J.D. and Santos, F.G. — *Indefinite quaternion space forms*, Ann. di Mat., 132 (1982), 383—398
3. Perez, J.D., Santos, F.G. and Urbano, F. — *On the axioms of planes in quaternionic geometry*, Ann. di Mat., 130 (1982), 215—221
4. Salamon, S. — *Quaternionic Kähler manifolds*, Inv. Math., 67 (1982), 143—171
5. Tsuchiya, S. and Kobayashi, M. — *Some conditions for constancy of the holomorphic sectional curvature*, Kodai Math. Sem. Rep., 27 (1976), 379—384
6. Wolf, J.A. — *Complex homogeneous contact manifolds and quaternionic symmetric spaces*, J. Math. Mech., 14 (1965), 1033—1047

Received 1. VII. 1988

Departamento de Geometria y Topologia
Facultad de Ciencias
Universidad de Granada
18071-Granada, Spain

An. şt. Univ. „Al. I. Cuza", Iaşi
Matematică, 36 (1990)

NON-LINEAR CONNECTIONS FOR 2-OSCULATING VECTOR FIELDS

BY

L. DI TERLIZZI and A. FARINOLA

Introduction. The notion of non-linear connection has been studied by many authors. In this paper, we introduce the notion of non-linear connection for 2-osculating vector fields on a manifold M, as covariant derivative of the sections of the vector bundle of 2-osculating vector fields on M and also as a distribution in the vector bundle of non-zero 2-osculating vector fields. We prove that the two definitions are equivalent. Furthermore, we study a particular set of non-linear connections for 2-osculating vector fields, which is bijective to the set of the connections on the principal bundle P of the 2-osculating frames of M. Finally, we study the integrability conditions.

§1. Preliminaries. Let M be a C^∞-differentiable manifold of dimension n. Let us denote with $\mathcal{F}(M)$ the algebra of the C^∞-differentiable real functions on M and, for any $r\in N^*$, with $\Pi_r : D^r(M)\to M$ the r-osculating vector bundle to M. It is well known that, if (U, x^i) $i=1, \dots, n$ is a coordinate neighborhood of M, and $p\in U$, a base of $D^2_p(M)=\Pi_2^{-1}(p)$ is given by $\left\{(e_h)_p, \frac{1}{2}(e_{ij})_p\right\}$, where $e_h=\frac{\partial}{\partial x^h}$ $h=1, \dots, n$, $e_{ij}=\frac{\partial^2}{\partial x^i\partial x^j}$, $1\leqslant i\leqslant j\leqslant n$. Such a base is called the natural 2-osculating frame in p relative to (U, x^i).

We put $I=\{1, \dots, n\}$, $J=\{(i, j)\in I\times I \mid i\leqslant j\}$ and we denote with $(\Pi_2^{-1}(U), x^h, y^h, y^{ij})$, $h\in I$, $(i, j)\in J$ the coordinate neighborhood induced by (U, x^i) on $D^2(M)$. If (U, x^i), $(U', x^{i'})$ are two coordinate neighborhoods on M, with $U\cap U'\neq\emptyset$, we put $\theta^h_{h'}=\frac{\partial x^h}{\partial x^{h'}}=\partial_{h'}x^h$, $\theta^h_{h'l'}=\partial_{l'}\theta^h_{h'}$, $\theta^h_{h'l'm'}=\partial_{m'}\theta^h_{h'l'}$ in $U\cap U'$. Furthermore, we denote with $\Pi : P\to M$ the principal bundle of the 2-osculating frames on M. For more details, we refer the reader to [5], [6].

Let us recall that we can provide the vector space $D(M)=\bigcup_{r\in N^*} D^r(M)$ with the structure of Lie algebra, putting, for any $F, G\in D(M)$, $[F, G]=F\circ G-G\circ F$. It is known that a differentiable transformation Φ of M, induces the automorphism $\tilde{\Phi}$ of $D(M)$, defined by: $(\tilde{\Phi}(F))(f)=F(f\circ\Phi)$, $\forall F\in D(M)$, $\forall f\in\mathcal{F}(M)$.

Def. 1.1. *Let* $X\in D^1(M)$ *be complete,* $\{\Phi_t\}_{t\in\mathbf{R}}$ *be a 1-parameter group of transformations which generates* X. *We call Lie derivative of the r-osculating*

This work was partially supported by M.P.I.

vector field F *in* $p\in M$ *with respect to* X, *the* q*-osculating vector in* p, $1\leqslant q\leqslant r$, *given by*

$$(L_XF)_p=\lim_{t\to 0}\frac{1}{t}\{F_p-(\tilde{\Phi}_t(F))_p\}.$$

The map $L_X: D(M)\to D(M)$ *is called Lie differentiation with respect to* X. *For the local nature of this definition, we can easily modify it when* X *is not complete. In the case* $r=1$, *we obtain the usual definition of Lie derivative* (cfr. [3]).

Prop. 1.1. *For any* $X\in D^1(M)$, *the Lie differentation with respect to* X *verifies the following properties*

(a) L_X *is a derivation of* $D(M)$, *that is* L_X *is* **R**-*linear and for any* $F, G\in D(M)$, $L_X(F\circ G)=(L_XF)\circ G+F\circ(L_XG)$;

(b) $L_XF=[X, F]\quad \forall F\in D(M)$;

(c) $L_X(fF)=fL_XF+X(f)F\quad \forall f\in\mathcal{F}(M),\ \forall F\in D(M)$.

Let $\{\Phi_t\}_{t\in\mathbf{R}}$ be a 1-parameter group of differentiable transformations of M which generates X. For any $F, G\in D(M)$, we have

$$L_X(F\circ G)=\lim_{t\to 0}\frac{1}{t}\{F\circ G-(\tilde{\Phi}_t(F))\circ G\}+\lim_{t\to 0}\frac{1}{t}\{(\tilde{\Phi}_t(F))\circ G-\tilde{\Phi}_t(F)\circ\tilde{\Phi}_t(G)\}=$$

$$=(L_XF)\circ G+F\circ L_XG.$$

The **R**-linearity of L_X is obvious, then we obtain (a). (b) and (c) can be be proved analogously to the proposition 3.2, chap. I in [3].

Remark 1.1. The vector space $\mathcal{L}$ of the derivations of $D(M)$ can be provided by the structure of Lie algebra in the usual way. From (b) and Jacoby identity, it follows that $[L_X, L_Y]=L_{[X,Y]}$, $\forall X, Y\in D^1(M)$ and then, the map $X\to L_X$ is a Lie-algebra homomorphism from $D^1(M)$ to $\mathcal{L}$.

§2. Non-linear connections for 2-osculating vector fields.

Def. 2.1. *A non-linear connection for 2-osculating vector fields (or on* $D^2(M)$*) is any map* $\nabla: D^1(M)\times D^2(M)\to D^2(M)$ *such that*

(1) $\nabla_{fX+gY}F=f\nabla_XF+g\nabla_YF\quad \forall f, g\in\mathcal{F}(M),\quad \forall X, Y\in D^1(M),\quad \forall F\in D^2(M)$;

(2) $(\nabla_XfF)=f(\nabla_XF)+X(f)F\quad \forall f\in\mathcal{F}(M),\quad \forall X\in D^1(M),\quad \forall F\in D^2(M)$;

(3) $(\nabla_XF)_p=(L_XF)_p\quad \forall X\in D^1(M),\quad \forall F\in D^2(M)$ such that $F_p=0$;

(4) $(\nabla_X(F+F'))_p=(\nabla_XF)_p+(\nabla_XF')_p\quad \forall X\in D^1(M),\ \forall F, F'\in D^2(M)$ such that $F_p+F'_p=0$.

If ∇ *verifies the condition*

(2.1) $$\forall X, Y\in D^1(M)\ \ \nabla_XY\in D^1(M)$$

then $(\nabla|_{D^1(M)\times D^1(M)})_{\#}: D^1(M)\times D^1(M)\to D^1(M)$, *is a non-linear connection, in the usual sense* ([1], [7]).

Remark 2.1. Any non-linear connection can be localized.

Prop. 2.1. *Let* ∇ *be a non-linear connection on* $D^2(M)$. *For any open set* U *of* M *there exists a uniquely determined non-linear connection* ∇^U *on* $D^2(U)$ *such that for any* $X\in D^1(M)$, $F\in D^2(M)$

$$\nabla^U_{X|_U}F|_U=(\nabla_XF)|_U.$$

The proof goes on in the usual way.

Remark 2.2. The condition (3) in the definition 2.1 is equivalent to the condition

(3') $(\nabla_XF)_p=(\dot{\nabla}_XF)_p\quad \forall X\in D^1(M),\quad \forall F\in D^2(M)$ such that $F_p=0$,

where $\dot{\nabla}$ is the covariant derivative of any connection on P. Namely, let (U, x^i) be a coordinate neighborhood of M, $p\in U$. If we put $F|_U=F^ie_i+\frac{1}{2}F^{ij}e_{ij}$, $X|_U=X^ie_i$ we obtain

$$(\dot{\nabla}_XF)_p X^i(p)\{(\partial_iF^j)_p(e_j)_p+\frac{1}{2}(\partial_iF^{jk})_p(e_{jk})_p\}=(L_XF)_p=(\nabla_XF)_p\,.$$

Now we shall determine the components of a non-linear connection.

Let ∇ be a non-linear connection on $D^2(M)$, (U, x^i) be a coordinate neighborhood of M, pU and $F\in D^2(M)$, $F_p\neq 0$. If we put $F|_U=F^ie_i+\frac{1}{2}F^{ij}e_{ij}$, we can consider a 2-osculating vector field F', $q\leqslant 2$, such that $F'|_U=F^i(p)e_i+\frac{1}{2}F^{ij}(p)e_{ij}$. By (3) and (4), we obtain

$$(\nabla_{e_i}F)_p-(\nabla_{e_i}F')_p=(\nabla_{e_i}(F-F'))_p=(L_{e_i}(F-F'))_p=(\partial_iF^j)_p(e_j)_p+(\partial_iF^{jk})_p(e_{jk})_p,$$

and then,

$$(\nabla_{e_i}F)_p=(\nabla_{e_i}F')_p+(\partial_iF^j)_p(e_j)_p+(\partial_iF^{jk})_p(e_{jk})_p.$$

If we put $(\nabla_{e_i}F')_p=\Gamma^h_i(p, F_p)(e_h)_p+\frac{1}{2}\Gamma^{hj}_i(p, F_p)(e_{hj})_p$, we have

$$(\nabla_{e_i}F)_p=\{(\partial_iF^h)_p+\Gamma^h_i(p, F_p)\}(e_h)_p+\{(\partial_iF^{hj})_p+\Gamma^{hj}_i(p, F_p)\}(e_{hj})_p$$

so that, if $X\in D^1(M)$, $X|_U=X^ie_i$, the following formula holds:

(2.2) $$(\nabla_XF)|_U=X^i\{\partial_iF^h+\Gamma^h_i(x, y)\}e_h+\frac{1}{2}X^i\{\partial_iF^{hj}+\Gamma^{hj}_i(x, y)\}e_{hj},$$

where $x=(x^i)_{i\in I}$, $y=(y^h, y^{ij})_{h\in I,\ (i,j)\in J}$.

The n^2+n^3 functions Γ^h_i, Γ^{hk}_i are C^∞-differentiable in $\Pi_2^{-1}(U)\cap D'^2(M)$ and can be extended with continuity to $\Pi_2^{-1}(U)$, puting $\Gamma^h_i(x, 0)$, $\Gamma^{hk}_i(x, 0)=0$; this agrees with (3). The functions Γ^h_i, Γ^{hk}_i are called the *components of the non-linear connection* ∇ *with respect to the coordinate neighborhood* (U, x^i). Using (2.1), it can be easely proved that they are homogeneous functions of degree 1 with respect to the variables y^k, y^{jk}.

Prop. 2.2. *Let ∇ be a non-linear connection for 2-osculating vector fields, whose components in two coordinate neighborhoods (U, x^i) and $(U', x^{i'})$ are $\Gamma_i^h(x^j, y^j, y^{jk})$, $\Gamma_i^{hl}(x^j, y^j, y^{jk})$ and $\Gamma_{i'}^{h'}(x^{j'}, y^{j'}, y^{j'k'})$, $\Gamma_{i'}^{h'l'}(x^{j'}, y^{j'}, y^{j'k'})$ respectively. If $U \cap U'$ is not empty, in $\Pi_2^{-1}(U) \cap \Pi_2^{-1}(U')$, we have*

$$\Gamma_i^{hl}(x, y) = \Gamma_{i'}^{h'l'}(x, y)\theta_{h'}^h \theta_{l'}^l \theta_i^{i'} - y^{h'l'}\partial_i(\theta_{h'}^h \theta_{l'}^l) \tag{2.3}$$

$$\Gamma_i^h(x, y) = \Gamma_{i'}^{h'}(x, y)\theta_{h'}^h \theta_i^{i'} + \frac{1}{2}\Gamma_{i'}^{h'l'}(x, y)\theta_{h'l'}^{k'} \theta_i^{i'} y^{h''}\theta_{k'h''}^h \theta_i^{i'} - \frac{1}{2} y^{h'l'}\theta_{i'h'l'}^h \theta_i^{i'}. \tag{2.4}$$

Let us observe that the (2.4) becomes the law of transformation for a non-linear connection when (2.1) holds.

Prop. 2.3. *For each coordinate neighborhood (U, x^i) of M let $\Gamma_i^h(x, y)$, $\Gamma_i^{hl}(x, y)$ be $n^2 + n^3$ functions C^∞-differentiable on $\Pi_2^{-1}(U) \cap D'^2(M)$, continuous on $\Pi_2^{-1}(U)$, homogeneous of degree 1 with respect to the variables y^i, y^{ij}, such that the functions Γ_i^{hl} are symmetric with respect to the pair of indices (h, l). Suppose that the laws of transformation, in the intersection of two coordinate neighborhoods are given by (2.3), (2.4). Then there exists a uniquely determined non-linear connection on $D^2(M)$ whose components with respect to any coordinate neighborhood are Γ_i^h, Γ_i^{hl}.*

§3. Non-linear connection for 2-osculating vector fields determined by a connection for 2-osculating vector fields. It is known that the covariant derivative $\mathring{\nabla}$ determined by a connection $\mathring{\Gamma}$ on the principal bundle P, does not verify, in general, the condition

$$\forall X, Y \in D^1(M) \quad \nabla_X Y \in D^1(M) \tag{3.1}$$

(cfr. [6]). Let us put, for any coordinate neighborhood (U, x^i)

$$\begin{aligned} \mathring{\nabla}_{e_i} e_j &= \Gamma_{ij}^h e_h + \frac{1}{2}\Gamma_{ij}^{hl} e_{hl} \\ \mathring{\nabla}_{e_i} e_{jk} &= \Gamma_{ijk}^h e_h + \frac{1}{2}\Gamma_{ijk}^{hl} e_{hl}, \end{aligned} \qquad i, j, k, l, h \in I \tag{3.2}$$

The functions Γ_{ij}^h, Γ_{ij}^{hl}, Γ_{ijk}^h, Γ_{ijk}^{hl}, symmetric with respect to (h, l), (j, k), are defined on U and are called components of $\mathring{\Gamma}$. Let $(U', x^{i'})$ be another coordinate neighborhood of M. If $U \cap U' \neq \varnothing$, we have, with obvious meaning of symbols

$$\Gamma_{ij}^h = \Gamma_{i'j'}^{h'}\theta_{h'}^h \theta_i^{i'} \theta_j^{j'} + \frac{1}{2}\Gamma_{i'j'}^{h'l'}\theta_{h'l'}^h \theta_i^{i'} \theta_j^{j'} + \theta_{ij}^{h'}\theta_{h'}^h \tag{3.3}$$

$$\Gamma_{ij}^{hl} = \Gamma_{i'j'}^{h'l'}\theta_{h'}^h \theta_{l'}^l \theta_i^{i'} \theta_j^{j'} \tag{3.4}$$

$$\begin{aligned} \Gamma_{ijk}^h = {} & \Gamma_{i'j'k'}^{h'}\theta_{h'}^h \theta_i^{i'} \theta_j^{j'} \theta_k^{k'} + \frac{1}{2}\Gamma_{i'j'k'}^{h'l'}\theta_{h'l'}^h \theta_i^{i'} \theta_j^{j'} \theta_k^{k'} + \Gamma_{i'l'}^{h'}\theta_{jk}^{l'} \theta_i^{i'} \theta_{h'}^h + \\ & + \frac{1}{2}\Gamma_{i'k'}^{h'l'}\theta_k^{k'} \theta_i^{i'} \theta_{h'l'}^h + \theta_{ijk}^{h'}\theta_{h'}^h + \theta_{l'q'}^h \partial_i(\theta_j^{l'} \theta_k^{q'}) \end{aligned} \tag{3.5}$$

$$\Gamma_{ijk}^{hl} = \Gamma_{i'j'k'}^{h'l'}\theta_{h'}^h \theta_{l'}^l \theta_i^{i'} \theta_j^{j'} \theta_k^{k'} + 2\partial_i(\theta_j^{h'} \theta_k^{l'})\theta_{h'}^h \theta_{l'}^l + \Gamma_{i'k'}^{h'l'}\theta_{jk}^{k'} \theta_{h'}^h \theta_{l'}^l \theta_i^{i'}. \tag{3.6}$$

Remark 3.1. The functions Γ_{ij}^{hl} are the components of a tensor field T of type (2,2), symmetric with respect to the pair of indices (h, l). Furthermore, if $T = 0$, then $\mathring{\nabla}$ verifies the condition (3.1). In this case (3.3), (3.5), (3.6) become the laws (3.2), (3.3), (3.4) of [5], and $\nabla = (\mathring{\nabla}_{|D^1(M) \times D^1(M)})_{*}$: $: D^1(M) \times D^1(M) \to D^1(M)$ is a linear connection.

Let us observe that a connection on P can be equivalently determined giving, for each coordinate neighborhood of M, the functions Γ_{ij}^h, Γ_{ij}^{hl}, Γ_{ijk}^h, Γ_{ijk}^{hl}, symmetric with respect to the pairs of indices (h, l), (j, k) and verifying (3.2), (3.3), (3.4), (3.5).

Prop. 3.1. *Any connection on P determines a non-linear connection on $D^2(M)$.*

Let $\mathring{\Gamma}$ be a connection on P. For any coordinate neighborhood (U, x^i) of M, we put

$$\begin{aligned} \Gamma_i^h(x, y) &= \Gamma_{ij}^h(x)y^j + \frac{1}{2}\Gamma_{ijk}^h(x)y^{jk} \\ \Gamma_i^{hl}(x, y) &= \Gamma_{ij}^{hl}(x)y^j + \frac{1}{2}\Gamma_{ijk}^{hl}(x)y^{jk} \end{aligned} \tag{3.7}$$

where $\Gamma_{ij}^h(x)$, $\Gamma_{ijk}^h(x)$, $\Gamma_{ij}^{hl}(x)$, $\Gamma_{ijk}^{hl}(x)$ are the components of $\mathring{\Gamma}$.

The functions defined by (3.7) verify the hypothesis of prop. 2.3 and then they determine a non-linear connection on $D^2(M)$.

Remark 3.2. Let $\Gamma_i^h(x, y)$, $\Gamma_i^{hl}(x, y)$ be the components of a non-linear connection on $D^2(M)$. Then the functions

$$\Gamma_{ij}^h(x, y) = \frac{\partial \Gamma_i^h}{\partial y^j}(x, y), \qquad \Gamma_{ijk}^h(x, y) = 2\frac{\partial \Gamma_i^h}{\partial y^{jk}}(x, y), \qquad \Gamma_{ij}^{hl}(x, y) = \frac{\partial \Gamma_i^{hl}}{\partial y^j}(x, y),$$

$$\Gamma_{ijk}^{hl}(x, y) = 2\frac{\partial \Gamma_i^{hl}}{\partial y^{jk}}(x, y)$$

are homogeneous of degree 0 with respect to the variables y^i, y^{ij} and then they do not depend on y^i, y^{ij} when the functions Γ_i^h, Γ_i^{hl} are C^∞-differentiable. So that in this case, they are the components of a connection $\mathring{\Gamma}$ on P.

Furthermore, the functions Γ_{ij}^{hl} vanish if and only if the covariant derivative determined by Γ (respectively $\mathring{\Gamma}$) verifies (2.1) (respectively (3.1)).

§4. Non-linear connections and distributions on the bundle of non-zero 2-osculating vector fields. We shall indicate with $D'^2(M)$ the bundle of non-zero 2-osculating vector fields, with $V_p = \Pi_2^{-1}(p)$ the fibre of $D'^2(M)$ and, for any $c \in \mathbf{R}$, $c \neq 0$ with R_c the transformation of $D^2(M)$ such that $R_c F = cF$, $F \in D^2(M)$.

Let H be a distribution on $D'^2(M)$ such that

(a) $T_F(D'^2(M)) = T_F(V_{\Pi_2(F)}) \oplus H_F \qquad \forall F \in D'^2(M)$

(b) $dR_c(H_F) = H_{cF} \qquad \forall c \in \mathbf{R}^*, \qquad \forall F \in D'^2(M)$.

Let us observe that if (U, x^i) is a cordinate neighborhood of M, $F \in \Pi_2^{-1}(U)$, then $\left\{\left(\frac{\partial}{\partial y^h}\right)_F, \left(\frac{\partial}{\partial y^{ij}}\right)_F\right\}_{h \in I,\ (i,j) \in J}$ is a base of $T_F(V_{\Pi_2(F)}) = \ker(d_F \Pi_2)$.

Furthermore, since $(d_F\Pi_2)|_{H_F} : H_F \to T_{\Pi_2(F)}(M)$ is an isomorphism, a base of H_F is given by $\left\{\left(\frac{\delta}{\delta x^i}\right)_F\right\}_{i\in I}$, where $\frac{\delta}{\delta x^i} = (d_F\Pi_2^{-1})\left(\frac{\delta}{\delta x^i}\right)$. Then the distribution H is differentiable. If we put

$$\left(\frac{\delta}{\delta x^i}\right)_F = A_i^j\left(\frac{\partial}{\partial x^j}\right)_F - \Gamma_i^j\left(\frac{\partial}{\partial y^j}\right)_F - \left(\frac{2\Gamma_i^{jh}}{1+\delta_h^j}\right)\left(\frac{\partial}{\partial y^{jh}}\right)_F \qquad (j, h)\in J$$

we obtain

$$\left(\frac{\partial'}{\partial x^i}\right)_{\Pi_2(F)} = (d_F\Pi_2)\left(\frac{\delta}{\delta x^i}\right)_F = A_i^j(d_F\Pi_2)\left(\frac{\partial}{\partial x^j}\right)_F = A_i^j\left(\frac{\partial}{\partial x^j}\right)_{\Pi_2(F)}$$

so that, if we put $\Gamma_i^{jh}(x, F) = \Gamma_i^{hj}(x, F)$, for $h \leq j$, we have

$$\left(\frac{\delta}{\delta x^i}\right)_F = \left(\frac{\delta}{\delta x^i}\right)_F - \Gamma_i^j(x, F)\left(\frac{\partial}{\partial y^j}\right)_F - \Gamma_i^{jh}(x, F)\left(\frac{\partial}{\partial y^{jh}}\right)_F, \qquad i, j, h\in I. \tag{4.1}$$

Remark 4.1. The n^2+n^3 functions Γ_i^j, Γ_i^{jh} are defined in $\Pi_2^{-1}(U)\cap D'^2(M)$ and, by (b), are homogeneous of degree 1 with respect to the variables y^i, y^{ij}, so they can be extended with continuity to $\Pi_2^{-1}(U)$, putting $\Gamma_i^j(x, 0)=0$, $\Gamma_i^{jh}(x, 0)=0$. Γ_i^{jh} are symmetric with respect to the pair of indices (j, h).

Prop. 4.1. *A distribution H on $D'^2(M)$ verifying* (a) *and* (b) *defines a non-linear connection ∇ on $D^2(M)$.*

Let (U, x^i) be a coordinate neighborhood of M, Γ_i^j, Γ_i^{jh} the functions on $\Pi_2^{-1}(U)$ determined by the distribution H. It suffices to prove that Γ_i^j, Γ_i^{jh} verify the hypothesis of prop. 2.3. To do it, we can observe that if (U', x'^i) is another coordinate neighnorhood of M, with $U\cap U'\neq\varnothing$, the matrix of the change of base, is given by

$$\left(\begin{array}{c|c|c} \theta_{i'}^i & 0 & 0 \\ \hline y^{h'}\theta_{i'h'}^i + \frac{1}{2}y^{h'l'}\theta_{i'h'l'}^i & \theta_{i'}^i & \frac{1}{2}\theta_{i'j'}^i \\ \hline y^{h'l'}\theta_{i'h'}^i\theta_{l'}^j + y^{h'l'}\theta_{h'}^i\theta_{i'l'}^j & 0 & \theta_{i'}^i\theta_{j'}^j \end{array}\right). \tag{4.2}$$

Since

$$\theta_{i'}^i\frac{\partial}{\partial x^i} - \theta_{i'}^i\Gamma_i^h\frac{\partial}{\partial y^h} - \theta_{i'}^i\Gamma_i^{hl}\frac{\partial}{\partial y^{hl}} = \theta_{i'}^i\frac{\delta}{\delta x^i} = \frac{\delta}{\delta x^{i'}} = \frac{\partial}{\partial x^{i'}} - \Gamma_{i'}^{h'}\frac{\partial}{\partial y^{h'}} - \Gamma_{i'}^{h'l'}\frac{\partial}{\partial y^{h'l'}}$$

using (4.2) and equalizing the components, we obtain (2.3), (2.4).

Prop. 4.2. *A non-linear connection ∇ on $D^2(M)$ determines a distribution H on $D'(M)$ verifyng* (a) *and* (b).

Let (U, x^i) be a coordinate neighborhood of M, Γ_i^j, Γ_i^{jh} the components of ∇ with respect to (U, x^i). For any $F\in\Pi_2^{-1}(U)\cap D'^2(M)$, let H_F be the vector subspace of $T_F(D'^2(M))$ spanned by $\left\{\left(\frac{\delta}{\delta x^i}\right)_F\right\}_{i\in I}$, where $\left(\frac{\delta}{\delta x^i}\right)_F = \left(\frac{\partial}{\partial x^i}\right)_F - \Gamma_i^j(x, F)\left(\frac{\partial}{\partial y^j}\right)_F + \Gamma_i^{jh}(x, F)\left(\frac{\partial}{\partial y^{jh}}\right)_F$.

It is easy to prove that the definition does not depend on (U, x^i) and that the distribution H verifies (a) and (b).

Remark 4.2. The horizontal spaces of the distribution determined by a non-linear connection on $D^2(M)$ are defined by the system of equations

$$\begin{cases} \theta^h = \Gamma_i^h dx^i + dy^h = 0 \\ \theta^{hl} = \Gamma_i^{hl} dx^i + dy^{hl} = 0. \end{cases}$$

§5. Integrability conditions. Let Γ be a non-linear on $D^2(M)$, whose components are Γ_i^h, Γ_i^{hl} with respect to the coordinate neighborhood (U, x^i) of M. Let us consider the following functions on $\Pi_2^{-1}(U)$

$$K_{jik}^l(x, y) = -\frac{\delta\Gamma_{jk}^l}{\delta x^i} + \frac{\delta\Gamma_{ik}^l}{\delta x^j} + \Gamma_{ik}^h\Gamma_{jh}^l - \Gamma_{jk}^h\Gamma_{ih}^l - \frac{1}{2}\Gamma_{jk}^{hm}\Gamma_{ihm}^l + \frac{1}{2}\Gamma_{ik}^{hm}\Gamma_{jhm}^l$$

$$K_{jikh}^l(x, y) = -\frac{\delta\Gamma_{jkh}^l}{\delta x^i} + \frac{\delta\Gamma_{ikh}^l}{\delta x^j} + \Gamma_{ikh}^q\Gamma_{jq}^l - \Gamma_{jkh}^q\Gamma_{iq}^l - \frac{1}{2}\Gamma_{jkh}^{qm}\Gamma_{iqm}^l + \frac{1}{2}\Gamma_{ikh}^{qm}\Gamma_{jqm}^l$$

$$K_{jik}^{lm}(x, y) = -\frac{\delta\Gamma_{jk}^{lm}}{\delta x^i} + \frac{\delta\Gamma_{ik}^{lm}}{\delta x^j} + \Gamma_{ik}^h\Gamma_{jh}^{lm} - \Gamma_{jk}^h\Gamma_{ih}^{lm} + \frac{1}{2}\Gamma_{ik}^{hq}\Gamma_{jhq}^{lm} - \frac{1}{2}\Gamma_{jk}^{hq}\Gamma_{ihq}^{lm}$$

$$K_{jikh}^{lm}(x, y) = -\frac{\delta\Gamma_{jkh}^{lm}}{\delta x^i} + \frac{\delta\Gamma_{ikh}^{lm}}{\delta x^j} + \Gamma_{ikh}^q\Gamma_{jq}^{lm} - \Gamma_{jkh}^q\Gamma_{iq}^{lm} + \frac{1}{2}\Gamma_{ikh}^{pq}\Gamma_{jpq}^{lm} - \frac{1}{2}\Gamma_{jkh}^{pq}\Gamma_{ipq}^{lm}.$$

Remark 5.1. When the components of Γ are C^∞-differentiable, the functions above defined do not depend on y^i, y^{ij} and become the components of the curvature $\mathring{R}$ of the connection $\mathring{\Gamma}$ on P induced by Γ. $\mathring{R}$ is the $\mathcal{F}(M)$-3-linear map

$$\mathring{R} : D^1(M)\times D^1(M)\times D^2(M)\to D^2(M)$$

such that $\mathring{R}(X, Y, F) = [\mathring{\nabla}_X, \mathring{\nabla}_Y](F) - \mathring{\nabla}_{[X,Y]}F$, where $\mathring{\nabla}$ is the covariant derivative of $\mathring{\Gamma}$. $\mathring{R}$ is obviously skewsymmetric with respect to the first two arguments.

Prop. 5.1. *Let Γ be a non-linear connection on $D^2(M)$. The distribution H of Γ is integrable if and only if for any coordinate neighborhood (U, x^i) of M, in $\Pi_2^{-1}(U)\cap D'^2(M)$ we have*

$$\begin{aligned} &K_{jik}^l y^k + \frac{1}{2}K_{jikh}^l y^{kh} = 0 \\ &K_{jik}^{lm} y^k + \frac{1}{2}K_{jikh}^{lm} y^{kh} = 0. \end{aligned} \tag{5.1}$$

Namely, let (U, x^i) be a cordinate neighborhood of M. By a long but easy calculation, we have

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = \left\{K_{jik}^l y^k + \frac{1}{2}K_{jikh}^l y^{kh}\right\}\frac{\partial}{\partial y^l} + \left\{K_{jik}^{lm} y^k + \frac{1}{2}K_{jikh}^{lm} y^{kh}\right\}\frac{\partial}{\partial y^{lm}}$$

and then (5.1) follows.

Remark 5.2. If the covariant derivative ∇ of Γ verifies (2.1), then (5.1) become the integrability condition of the non-linear connection determined by Γ (cf. [7]).

Remark 5.3. Let $\dot\Gamma$ be a connection on the principal bundle P, and ω the 1-form of $\dot\Gamma$. Let (U, x^i) be a coordinate neighborhood of M and $(\Pi^{-1}(U)$, $x^i, y^i, y^i_j, y^{jh}_i, y^i_{jk}, y^{jh}_{ik})$ $i, j \in I$, $(i, h) \in J$, $(j, k) \in J$, the induced local coordinates on P. We can consider the section σ of P, defined on U, such that for any $p \in U$, $\sigma(p)$ is the natural 2-osculating frame $\left\{(e_i)_p, \frac{1}{2}(e_{ij})_p\right\}$; $(i, j) \in J$ at p relative to (U, x^i). Let us put $\omega_U = \sigma^*(\omega)$. If $\{E^i_j, E^{ij}_h, E^i_{hk}, E^{ij}_{hk}\}$ $i \in I$, $j \in I$, $(i, j) \in J$, $(h, k) \in J$ is a base of $gl(l, R)$, $l = \dfrac{n(n+3)}{2}$, we can put: $\omega = \omega^i_j E^j_i + \omega^i_{jh} E^{jh}_i + \omega^{jh}_i E^i_{jh} + \omega^{hk}_{ij} E^{ij}_{hk}$, $\omega_U = (\omega_U)^i_j E^j_i + (\omega_U)^i_{jh} E^{jh}_i + (\omega_U)^{jh}_i E^i_{jh} + (\omega_U)^{hk}_{ij} E^{ij}_{hk}$, $(\omega_U)^i_j = \bar\Gamma^i_{kj} dx^k$ $i \in I$, $j \in I$, $(\omega_U)^i_{jh} = \bar\Gamma^i_{kjh} dx^k$ $i \in I$, $(j, h) \in J$, $(\omega_U)^{hl}_i = \bar\Gamma^{hl}_{ki} dx^k$ $i \in I$, $(h, l) \in J$, $(\omega_U)^{hl}_{ij} = \bar\Gamma^{hl}_{kij} dx^k$ $(h, l) \in J$, $(i, j) \in J$. By direct computation, we have

$$\omega^i_j = \{\bar y^i_s [y^k_j(\bar\Gamma^s_{hk} \circ \Pi) + y^{km}_j(\bar\Gamma^s_{hkm} \circ \Pi)] + \bar y^i_{sq}[y^k_j(\bar\Gamma^{sq}_{hk} \circ \Pi) + y^{km}_j(\bar\Gamma^{sq}_{hkm} \circ \Pi)]\} dy^h + \bar y^i_h dy^h_j + \bar y^i_{rs} dy^{rs}_j \tag{5.2}$$

$$\omega^i_{jh} = \{\bar y^i_t [y^k_{jh}(\bar\Gamma^t_{rk} \circ \Pi) + y^{km}_{jh}(\bar\Gamma^t_{rkm} \circ \Pi)] + \bar y^i_{mt}[y^k_{jh}(\bar\Gamma^{mt}_{rk} \circ \Pi) + y^{kq}_{jh}(\bar\Gamma^{mt}_{rkq} \circ \Pi)]\} dy^r + \bar y^i_t dy^t_{jh} + \bar y^i_{rt} dy^{rt}_{jh} \tag{5.3}$$

$$\omega^{hl}_i = \{\bar y^{hl}_s [y^k_i(\bar\Gamma^s_{rk} \circ \Pi) + y^{km}_i(\bar\Gamma^s_{rkm} \circ \Pi)] + \bar y^{hl}_{sq}[y^k_i(\bar\Gamma^{sq}_{rk} \circ \Pi) + y^{km}_i(\bar\Gamma^{sq}_{rkm} \circ \Pi)]\} dy^r + \bar y^{hl}_s dy^s_i + \bar y^{hl}_{sq} dy^{sq}_i \tag{5.4}$$

$$\omega^{hl}_{ij} = \{\bar y^{hl}_q [y^p_{ij}(\bar\Gamma^q_{rp} \circ \Pi) + y^{ps}_{ij}(\bar\Gamma^q_{rps} \circ \Pi)] + \bar y^{hl}_{qs}[y^p_{ij}(\bar\Gamma^{qs}_{rp} \circ \Pi) + y^{pm}_{ij}(\bar\Gamma^{qs}_{rpm} \circ \Pi)]\} dy^r + \bar y^{hl}_r dy^r_{ij} + \bar y^{hl}_{rs} dy^{rs}_{ij} \tag{5.5}$$

where $\left(\begin{array}{c|c} \bar y^i_h & \bar y^{ij}_h \\ \hline \bar y^h_{ij} & \bar y^{ij}_{hk} \end{array}\right)$ is the inverse matrix of $\left(\begin{array}{c|c} y^i_h & y^{ij}_h \\ \hline y^h_{ij} & y^{ij}_{hk} \end{array}\right)$.

Let us take $u \in \Pi^{-1}(U)$, $X \in T_u(P)$, such that $X = X^i \dfrac{\partial}{\partial y^i} + X^i_j \dfrac{\partial}{\partial y^i_j} + X^i_{rs} \dfrac{\partial}{\partial y^i_{rs}} + X^{rs}_i \dfrac{\partial}{\partial y^{rs}_i} + X^{rs}_{pq} \dfrac{\partial}{\partial y^{rs}_{pq}}$. Using (5.2), (5.3), (5.4), (5.5) we can prove that X is a horizontal vector for the connection $\dot\Gamma$, if and only if

$$X = X^r \left[\frac{\partial}{\partial y^r} + (y^k_j \bar\Gamma^l_{rk} + y^{km}_j \bar\Gamma^l_{rkm}) \frac{\partial}{\partial y^l_j} - (y^k_j \bar\Gamma^{lm}_{rk} + y^{kq}_j \bar\Gamma^{lm}_{rkq}) \frac{\partial}{\partial y^{lm}_j} - (y^k_{jh} \bar\Gamma^l_{rk} + y^{km}_{jh} \bar\Gamma^l_{rkm}) \frac{\partial}{\partial y^l_{jh}} - (y^k_{jh} \bar\Gamma^{lm}_{rk} + y^{ks}_{jh} \bar\Gamma^{lm}_{rks}) \frac{\partial}{\partial y^{lm}_{jh}}\right]. \tag{5.6}$$

Then a base of H_u is given by $\left\{\dfrac{\partial}{\partial y^r}\right\}_{r \in I}$, where $\dfrac{\delta}{\delta y^r} = \dfrac{\partial}{\partial y^r} - (y^k_j \bar\Gamma^l_{rk} + y^{km}_j \bar\Gamma^l_{rkm}) \times \dfrac{\partial}{\partial y^l_j} - (y^k_j \bar\Gamma^{lm}_{rh} + y^{kq}_j \bar\Gamma^{lm}_{rkq}) \dfrac{\partial}{\partial y^{lm}_j} - (y^k_{jh} \bar\Gamma^l_{rk} + y^{km}_{jh} \bar\Gamma^l_{rkm}) \dfrac{\partial}{\partial y^l_{jh}} - (y^k_{jh} \bar\Gamma^l_{rk} + y^{ks}_{jh} \bar\Gamma^{lm}_{rks}) \dfrac{\partial}{\partial y^{lm}_{jh}}$.

Let $\dot\nabla$ be the covariant derivative of $\dot\Gamma$. We have

$$\begin{aligned} \dot\nabla_{e_i} e_j &= \bar\Gamma^p_{ij} e_p + \frac{1}{2} \bar\Gamma^{pk}_{ij} e_{pk} \quad (p, k) \in J \\ \dot\nabla_{e_h}\left(\frac{1}{2} e_{ij}\right) &= \bar\Gamma^p_{hij} e_p + \frac{1}{2} \bar\Gamma^{pk}_{hij} e_{pk} \quad (i, j) \in J,\ (p, k) \in J. \end{aligned} \tag{5.7}$$

Comparing (5.7) and (3.2) we obtain: $\Gamma^h_{ijk} = 2\bar\Gamma^h_{ijk}$, $\Gamma^{pq}_{ijk} = (1 + \delta^p_q)\bar\Gamma^{pq}_{ijk}$, $\Gamma^p_{ij} = \bar\Gamma^p_{ij}$, $\Gamma^{pq}_{ij} = \left(\dfrac{1 + \delta^p_q}{2}\right) \bar\Gamma^{pq}_{ij}$, where any index belongs to I. Then for the components of $\dot R$, we have

$$\dot R^q_{ijk} = \frac{\partial \bar\Gamma^q_{jk}}{\partial x^i} - \frac{\partial \bar\Gamma^q_{ik}}{\partial x^j} + \bar\Gamma^h_{jk} \bar\Gamma^q_{ih} - \bar\Gamma^h_{ik} \bar\Gamma^q_{jh} + \bar\Gamma^{lp}_{jk} \bar\Gamma^q_{ilp} - \bar\Gamma^{lp}_{ik} \bar\Gamma^q_{jlp}$$

$$\dot R^s_{ijkh} = 2\left\{\frac{\partial \bar\Gamma^s_{jkh}}{\partial x^i} - \frac{\partial \bar\Gamma^s_{ikh}}{\partial x^j} + \bar\Gamma^l_{jkh} \bar\Gamma^s_{il} - \bar\Gamma^l_{ikh} \bar\Gamma^s_{jl} + \bar\Gamma^{lp}_{jkh} \bar\Gamma^s_{ilp} - \bar\Gamma^{lp}_{ikh} \bar\Gamma^s_{jlp}\right\}$$

$$\dot R^{sq}_{ijk} = \frac{1 + \delta^s_q}{2}\left\{\frac{\partial \bar\Gamma^{sq}_{jk}}{\partial x^i} - \frac{\partial \bar\Gamma^{sq}_{ik}}{\partial x^j} + \bar\Gamma^h_{jk} \bar\Gamma^{sq}_{ih} - \bar\Gamma^l_{ik} \bar\Gamma^{sq}_{jl} + \bar\Gamma^{lp}_{jk} \bar\Gamma^{sq}_{ilp} - \bar\Gamma^{hp}_{ik} \bar\Gamma^{sq}_{jhp}\right\}$$

$$\dot R^{sq}_{ijkh} = (1 + \delta^s_q)\left\{\frac{\partial \bar\Gamma^{sq}_{jkh}}{\partial x^i} - \frac{\partial \bar\Gamma^{sq}_{ikh}}{\partial x^j} + \bar\Gamma^l_{jkh} \bar\Gamma^{sq}_{il} - \bar\Gamma^l_{ikh} \bar\Gamma^{sq}_{jl} + \bar\Gamma^{lp}_{jkh} \bar\Gamma^{sq}_{ilp} - \bar\Gamma^{lp}_{ikh} \bar\Gamma^{sq}_{jlp}\right\}$$

where any index belongs to I and the pair (l, p) belongs to J.

Prop. 5.2. *Let $\dot\Gamma$ be a connection on P and $\dot H$ its distribution. $\dot H$ is integrabile if and only if its curvature $\dot R$ is zero.*

Let (U, x^i) be a coordinate neighborhood. By a direct computation we have

$$\left[\frac{\delta}{\delta y^r}, \frac{\delta}{\delta y^s}\right] = \left\{y^p_q \dot R^l_{srp} + \frac{1}{2} y^{pm}_q \dot R^l_{srpm}\right\} \frac{\partial}{\partial y^l_q} + \left\{y^p_q \dot R^{lh}_{srp} + \frac{1}{2} y^{pm}_q \dot R^{lh}_{srpm}\right\} \frac{\partial}{\partial y^{lh}_q} + \left\{y^p_{qk} \dot R^l_{srp} + \frac{1}{2} y^{pm}_{qk} \dot R^l_{srpm}\right\} \frac{\partial}{\partial y^l_{qk}} + \left\{y^p_{qk} \dot R^{lh}_{srp} + \frac{1}{2} y^{pm}_{qk} \dot R^{lh}_{srpm}\right\} \frac{\partial}{\partial y^{lh}_{qk}}.$$

Then, $\dot H$ is integrable if and only if

$$y^k_q \dot R^l_{srk} + \frac{1}{2} y^{km}_q \dot R^l_{srkm} = 0 \tag{5.8}$$

$$y^k_q \dot R^{lh}_{srk} + \frac{1}{2} y^{km}_q \dot R^{lh}_{srkm} = 0 \tag{5.9}$$

$$y^k_{pq} \dot R^l_{srk} + \frac{1}{2} y^{km}_{pq} \dot R^l_{srkm} = 0 \tag{5.10}$$

$$g^k_{pq}\dot{R}^{lh}_{sck} + \frac{1}{2}g^{km}_{pq}\dot{R}^{lh}_{sckm} = 0 \tag{5.11}$$

where any index belongs to I and the pairs (p, q), (l, h), (k, m) belong to J. The conditions (5.8), (5.9), (5.10), (5.11) are equivalent to the condition $\dot{R}=0$. Remark 5.1 and prop. 5.2 imply

Corollary. *Let Γ be a non-linear connection on $D^2(M)$. If the components of Γ are C^∞-differentiable, then the distribution H of Γ is integrable if and only if the distribution $\dot{H}$ of the connection $\dot{\Gamma}$ induced on P by Γ is integrable.*

BIBLIOGRAPHY

1. Kandatu, A. — *Tangent bundle of a manifold with a non-linear connection*, Kodai Math. Sem. Rep., 18 (1966), 259–270
2. Kawaguchi, A. — *On the theory of non-linear connection. I.* Tensor, N.S., 2 (1952), 123–142
3. Kobayashy, S.—Nomizu K. — *Foundations of Differential Geometry*, Interscience Publishers, New York, London, 1963, Vol. I
4. Kobayashy S.—Nomizu K. — *Foundations of Differential Geometry*, Interscience Publishers, New York, London, 1969, Vol. II
5. Pastore A.M.— *Connessioni per campi vettoriali 2-osculatori*, Rend. Acc. Naz. dei XL — Serie IV, Vol. XXII, 1972
6. Pastore A.M. — *Connessioni sul fibrato principale dei riferimenti 2-osculatori di una varietà differenziabile*, Rend. Circolo Mat. Palermo, Serie II Tomo XXIX, 1980
7. Yano K. — Ishihara S. — *Tangent and Cotangent Bundles*, Marcel Dekker, Inc. New York, 1973

Received 22, VII, 1988

Dipartimento di Matematica
Università di Bari
Via G. Fortunato
70125 Bari, Italia

An. şt. Univ. „Al. I. Cuza", Iaşi
Matematică, 36 (1990)

ON THE DIFFERENTIAL GEOMETRY OF AN INFINITE DIMENSIONAL LAGRANGE SPACE

BY

DRAGOŞ HRIMIUC

Introduction. The differential geometry of the total space of the tangent bundle was studied by many people. By using some ideas from the Finsler geometry, R. Miron [6], [7] has considerably simplified the theory.

The aim of this paper is to study some aspects of the differential geometry of the tangent bundle to an infinite dimensional manifold and in particular to an infinite dimensional Lagrange space. It seems that this geometry gives the possibilities for studying some important problems of theoretical physics, e.g. hydrodynamics of a perfect fluid and general relativity [1], [4].

Naturally the techniques used in this case differ from those in the finite dimensional case of [6], [7].

Let M be a Banach manifold. If we have a connection N on M, we obtain a splitting of TTM into horizontal and vertical subbundles. This splitting can afford us to define a new atlas of trivializations on TTM and $TTTM$ associated to N, more convenient than the natural atlas. Also we may define the notion of d-connection, a special linear connection on TM which is geometrically associated to N. The projections of a d-connection on the horizontal and vertical subbundles are Otsuki connections on TM [11].

We have two metrical structures on TM, similar to that of Sasaki, induced by that on the vertical or on the horizontal subbundle. We compute the connectors of the Levi-Civita connection and of the metric d-connection, obtaining similar results as those in [6], [7], [9], [10], but in general setting.

If we work in Lagrange spaces, there exists a remarcable nonlinear connection $N : T^2M \to TM$ which depends only on the Lagrangian, with some nice properties.

The Levi-Civita and metric d-connection connectors are expressed in a simple form. Finally we find the Cartan connection for the infinite dimensional Finsler spaces. Only the C^∞ case is presented here although all definitions and results hold in a slightly modified form, if less differentiability is assumed. Also, we suppose that M is a smooth Banach manifold admitting C^∞-partition of unity.

1. Linear connection on TM. Let M be a smooth Banach manifold, modelled on a Banach space $\mathbf{M}$. A chart for M is given by a homeomorphism $\varphi : U \to \varphi(U)$ where U is open in M and $\varphi(U)$ is open in $\mathbf{M}$.

Given another chart for M, (V, ψ), we have the transition map

$$t : \varphi(U \cap V) \to \psi(U \cap V), \quad t(x) = \psi\varphi^{-1}(x). \tag{1.1}$$