$$g^k_{pq} R^{lh}_{srk} + \frac{1}{2} g^{km}_{pq} \mathring{R}^{lh}_{srkm} = 0 \tag{5.11}$$

where any index belongs to I and the pairs (p, q), (l, h), (k, m) belong to J. The conditions (5.8), (5.9), (5.10), (5.11) are equivalent to the condition $\mathring{R}=0$. Remark 5.1 and prop. 5.2 imply

Corollary. *Let Γ be a non-linear connection on $D^2(M)$. If the components of Γ are C^∞-differentiable, then the distribution H of Γ is integrable if and only if the distribution $\mathring{H}$ of the connection $\mathring{\Gamma}$ induced on P by Γ is integrable.*

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ON THE DIFFERENTIAL GEOMETRY OF AN INFINITE DIMENSIONAL LAGRANGE SPACE

BY

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Introduction. The differential geometry of the total space of the tangent bundle was studied by many people. By using some ideas from the Finsler geometry, R. Miron [6], [7] has considerably simplified the theory.

The aim of this paper is to study some aspects of the differential geometry of the tangent bundle to an infinite dimensional manifold and in particular to an infinite dimensional Lagrange space. It seems that this geometry gives the possibilities for studying some important problems of theoretical physics, e.g. hydrodynamics of a perfect fluid and general relativity [1], [4].

Naturally the techniques used in this case differ from those in the finite dimensional case of [6], [7].

Let M be a Banach manifold. If we have a connection N on M, we obtain a splitting of TTM into horizontal and vertical subbundles. This splitting can afford us to define a new atlas of trivializations on TTM and $TTTM$ associated to N, more convenient than the natural atlas. Also we may define the notion of d-connection, a special linear connection on TM which is geometrically associated to N. The projections of a d-connection on the horizontal and vertical subbundles are Otsuki connections on TM [11].

We have two metrical structures on TM, similar to that of Sasaki, induced by that on the vertical or on the horizontal subbundle. We compute the connectors of the Levi-Civita connection and of the metric d-connection, obtaining similar results as those in [6], [7], [9], [10], but in general setting.

If we work in Lagrange spaces, there exists a remarcable nonlinear connection $N : T^2M \to TM$ which depends only on the Lagrangian, with some nice properties.

The Levi-Civita and metric d-connection connectors are expressed in a simple form. Finally we find the Cartan connection for the infinite dimensional Finsler spaces. Only the C^∞ case is presented here although all definitions and results hold in a slightly modified form, if less differentiability is assumed. Also, we suppose that M is a smooth Banach manifold admitting C^∞-partition of unity.

1. Linear connection on TM. Let M be a smooth Banach manifold, modelled on a Banach space $\mathbf{M}$. A chart for M is given by a homeomorphism $\varphi : U \to \varphi(U)$ where U is open in M and $\varphi(U)$ is open in $\mathbf{M}$.

Given another chart for M, (V, ψ), we have the transition map

$$t : \varphi(U \cap V) \to \psi(U \cap V), \; t(x) = \psi\varphi^{-1}(x). \tag{1.1}$$

Let $\tau : TM \to M$ be the tangent bundle. If $(T\varphi, \varphi, U)$ and $(T\psi, \psi, V)$ are two trivializations on TM and $U \cap V \neq \emptyset$, then the transition map is

(1.2) $\quad {}^{\prime}H(x, a) = (l(x), l'(x)a)$ (l' denotes the Fréchet differential of l)

For the tangent bundle of TM, $\sigma : TTM \to TM$, the transmition map is

$$(1.3) \qquad {}^{\prime\prime}H(x, a, \lambda, b) = (l(x), l'(x)a, l'(x)\lambda, l''(x)\lambda a + l'(x)b).$$

Definition 1.1. *A smooth connection (nonlinear) on M is a map N : $TTM \to TM$ such that for any local trivializations $(T\varphi, \varphi, U)$ of TM and $(T^2\varphi, T\varphi, TU)$ of TTM there is a smooth map $\omega_\varphi : \varphi(U) \times \mathbf{M} \to L(\mathbf{M}, \mathbf{M})$ giving the local representative of N by the formula*

$$(1.4) \qquad N_\varphi(x, a, \lambda, b) = (x, b + \omega_\varphi(x, a)\lambda).$$

The local components of N are related by

$$(1.5) \qquad l'(x)\omega_\varphi(x, a)\lambda = l''(x)\lambda a + \omega_\psi(l(x), l'(x)a)l'(x)\lambda.$$

If M is a smooth Banach manifold admitting C^∞-partitions of unity then there exists connections on M.

In the sequel we suppose that there exists a nonlinear connection N : : $TTM \to TM$. Then we have the splitting

$$TTM = TTM \oplus VTM \quad (HTM = \mathrm{Ker}\, N, \; VTM = \mathrm{Ker}\, \tau_*)$$

and a vector bundle isomorphism

$B : TTM \to \tau^*(TM) + \tau^*(TM)$ ($\tau^*(TM)$ is the pull back of TM by τ)

locally given by

$$B_\varphi(x, a, \lambda, b) = (x, a, \lambda, b + \omega_\varphi(x, a)\lambda).$$

It induces on TTM a new C^∞-atlas of trivialization (called the adapted atlas) [3]

$$\mathcal{A}^0_* = \{(B_\varphi T^2\varphi, T\varphi, TU), \; (T^2\varphi, T\varphi, TU) \in \mathcal{A}_*\}$$

equivalent to the natural atlas $\mathcal{A}_*$. The change of trivializations is given by

$$(1.6) \qquad H^0_*(x, a, \lambda, b) = (l(x), l'(x)a, l'(x)\lambda, l'(x)b).$$

If we consider the maps

$$\tilde{B}_\varphi : \varphi(U) \times \mathbf{M}^7 \to \varphi(U) \times \mathbf{M}^7$$

$$\tilde{B}_\varphi(x, a, \lambda, b ; \mu, c, \nu, d) = (x, a, \lambda, b ; \mu, c + \omega_\varphi(x, a)\mu, \nu, d)$$

then

$$\mathcal{A}^0_{**} = \{(\tilde{B}_\varphi T(B_\varphi T^2\varphi), B_\varphi T^2\varphi, T^2U), \; (T^2\varphi, T\varphi, TU) \in \mathcal{A}_*\}$$

is a vector bundle atlas of $TTTM$ (equivalent to the natural atlas). The transition map is given by

$$(1.7) \qquad \begin{aligned} H^0_{**}(x, a, \lambda b ; \mu, c, \nu, d) = {} & (l(x), l'(x)a, l'(x)\lambda, l'(x)b; l'(x)\mu, l'(x)c, l''(x)\mu\lambda + \\ & + l'(x)\nu, l''(x)\mu + l'(x)d). \end{aligned}$$

Remark. In the new atlases the change of the charts is simpler than that in the natural atlas.

Definition 1.2. *A linear connection on TM is a smooth map K : $TTTM \to$ $\to TTM$ which have the following local forms in the adapted trivializations:*

$$K_\varphi(x, a, \lambda, b ; \mu, a, \nu, d) = (x, a, \nu + \Omega^h_\varphi(x, a)(\lambda, b)(\mu, c), d + \Omega^v_\varphi(x, a)(\lambda, b)(\mu, c)),$$

where

$$\Omega^h_\varphi : \varphi(U) \times \mathbf{M} \to L(\mathbf{M} \times \mathbf{M}, \mathbf{M} \times \mathbf{M} ; \mathbf{M}), \; \Omega^v_\varphi : \varphi(U) \times \mathbf{M} \to L(\mathbf{M} \times \mathbf{M}, \mathbf{M} \times \mathbf{M} ; \mathbf{M})$$

are smooth maps. We denote

$${}^{11}\Omega^h_\varphi(x, a)(\lambda, \mu) = \Omega^h_\varphi(x, a)(\lambda, 0)(\mu, 0), \; {}^{12}\Omega^h_\varphi(x, a)(\lambda, c) = \Omega^h_\varphi(x, a)(\lambda, 0)(0, c),$$

$${}^{21}\Omega^h_\varphi(x, a)(b, \mu) = \Omega^h_\varphi(x, a)(0, b)(\mu, 0)$$

and

$${}^{22}\Omega^h_\varphi(x, a)(b, c) = \Omega^h_\varphi(x, a)(0, b)(0, c).$$

Analogously for Ω^h_φ. We get the following transformation formulas for the local connectors (in the adapted trivializations):

$$(1.8) \qquad \begin{aligned} & l'(x)\Omega^h_\varphi(x, a)(\lambda, b)(\mu, c) = l''(x)\mu\lambda + \Omega^h_\psi(\bar{x}, \bar{a})(\bar{\lambda}, \bar{b})(\bar{\mu}, \bar{c}) \\ & l'(x)\Omega^v_\varphi(x, a)(\lambda, b)(\mu, c) = l''(x)\mu b + \Omega^v_\psi(\bar{x}, \bar{a})(\bar{\lambda}, \bar{b})(\bar{\mu}, \bar{c}), \end{aligned}$$

where $\bar{x} = l(x)$, $\bar{a} = l'(x)a$, $\bar{b} = l'(x)b$, $\bar{\lambda} = l'(x)\lambda$, $\bar{c} = l'(x)c$, $\bar{\mu} = l'(x)\mu$.

Definition 1.3. *A linear connection K on TM is called a d-connection, if ${}^{21}\Omega^h_\varphi = {}^{22}\Omega^h_\varphi = {}^{11}\Omega^v_\varphi = {}^{12}\Omega^v_\varphi = 0$ for every local trivialization.*

We denote

$$L^h_\varphi = {}^{11}\Omega^h_\varphi, \quad L^v_\varphi = {}^{21}\Omega^v_\varphi, \quad C^h_\varphi = {}^{12}\Omega^h_\varphi, \quad C^v_\varphi = {}^{22}\Omega^v_\varphi$$

Let $X \in \mathcal{X}(TM)$ be a vector field. In the natural trivializations, X has the following local form

$$X_\varphi(x, a) = (x, a, X^1_\varphi(x, a), X^2_\varphi(x, a)),$$

and in the adapted trivializations

$$X_\varphi(x, a) = (x, a, X^h_\varphi(x, a), X^v_\varphi(x, a)),$$

we have

$$X^h_\varphi(x, a) = X^1_\varphi(x, a), \quad X^v_\varphi(x, a) = X^2_\varphi(x, a) + \omega_\varphi(x, a)X^1_\varphi(x, a).$$

If $X \in \mathcal{X}(TM)$, we have

$X = X^H + X^V$ (X^H, X^V are sections in the horizontal and vertical bundles) When $X^H = 0$, we say that X is vertical and if $X^V = 0$, then X is horizontal. In the natural trivializations, a vertical field has the following local form

$$X_\varphi(x, a) = (x, a, 0, X^2_\varphi(x, a))$$

and a horizontal field

$$X_\varphi(x, a) = (x, a, X^1_\varphi(x, a), -\omega_\varphi(x, a)X^1_\varphi(x, a)).$$

If $X, Y \in \mathcal{X}(TM)$, in the adapted trivializations generated by (U, φ), the bracket $[X, Y]$ has the following local form

$$(1.9)\qquad \begin{aligned}[X, Y](x, a) = (x, a, \ &\mathcal{D}Y^h(X^h, X^v) - \mathcal{D}X^h(Y^h, Y^v),\\ &\mathcal{D}Y^v(X^h, X^v) - \mathcal{D}Y^v(Y^h, Y^v) + \mathcal{D}\omega(Y^h, Y^v)X^h - \mathcal{D}\omega(X^h, X^v)Y^h),\end{aligned}$$

where we denote

$$\mathcal{D}Y^h(x, a)(X^h, X^v) = DY^h(x, a)(X^h, X^v - \omega(x, a)X^h)$$

(D is the Fréchet derivative) and everywhere the Frechet derivative, as well as the principal parts of X, Y, are calculated in (x, a). Sometimes, it will be notationally convenient to omit the index which denotes the used trivialization and the point (x, a) in which we calculate the Fréchet derivative, connectors, principal parts of fields.

It is very important the bilinear continuous map which appears in the local expression of the bracket

$$(1.10)\qquad R(x, a)(u, v) = D\omega(x, a)(u, -\omega(x, a)u)v - D\omega(x, a)(v, -\omega(x, a)v)u.$$

We remark that these maps define a global antisymmetric section R of $L^2(TTM, VTM) \to TM$. It will be called the torsion of the given connection N.

Let $X, Y \in \mathcal{X}(TM)$ and $K : TTTM \to TTM$ a linear connection.

We define the covariant derivative by

$$(1.11)\qquad \begin{gathered}\nabla : \mathcal{X}(TM) \times \mathcal{X}(TM) \to \mathcal{X}(TM),\ (X, Y) \to \nabla_X Y = \nabla(X, Y)\\ \nabla_X Y = KY_* X.\end{gathered}$$

In the adapted trivializations we have locally (for the principal part)

$$(1.12)\qquad \begin{aligned}\nabla_X Y = (\mathcal{D}Y^h(X^h, X^v) + \Omega^h(Y^h, Y^v)(X^h, X^v),\ &\mathcal{D}Y^v(X^h, X^v) +\\ &+ \Omega^v(Y^h, Y^v)(X^h, X^v)).\end{aligned}$$

Proposition 1.1. *A linear connection on TM is a d-connection if and only if*

$$(\nabla_X Y^H)^V = (\nabla_X Y^V)^H = 0.$$

(Thus, ∇ preserves by parallelism the vertical and the horizontal distributions).

The torsion is defined by the usual formulas

$$T(X, Y) =) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Theorem 1.1. *The connection is torsion-free if and only if the following conditions hold*

(i) ${}^{11}\Omega^h(x, a)$, ${}^{22}\Omega^h(x, a)$, ${}^{22}\Omega^v(x, a)$ *are continuous symmetric bilinear maps*

(ii) ${}^{21}\Omega^h(x, a)(b, \lambda) = {}^{12}\Omega^h(x, a)(\lambda, b)$

(iii) ${}^{11}\Omega^v(x, a)(\lambda, \mu) - {}^{11}\Omega^v(x, a)(\mu, \lambda) = R(x, a)(\lambda, \mu)$

(iv) ${}^{21}\Omega^v(x, a)(b, \lambda) - {}^{12}\Omega^v(x, a)(\lambda, b) = D_2\omega(x, a)b\lambda$

for every trivializations and base point (x, a). (D_2 denotes the partial Fréchet derivative with respect to a.)

Remark. A *d*-connection is torsion-free if and only if

(i) $L^h(x, a)$ and $C^v(x, a)$ are continuous symmetric bilinear maps,

$$\text{(ii)}\qquad C^h = 0,\ R = 0,\ L^v(x, a)(b, \lambda) = D_2\omega(x, a)b\lambda.$$

2. **Metrical structure on TM.** Let $\check{\sigma} : L_s^2(\sigma) \to TM$ be the vector bundle of symmetric bilinear continuous sections of σ. If we consider the adapted atlas on TTM, we obtain a new atlas of trivializations on $\check{\sigma}$. Let s be a section of $\check{\sigma}$ and s_φ, s_ψ the principal parts of s in the trivializations generated by (U, φ) respectively (V, ψ).

Then, in the adapted trivializations, we have

$$(2.1)\qquad s_\varphi(x, a)(\lambda, b)(\mu, c) = s_\psi(t(x), t'(x)a)(t'(x)\lambda, t'(x)b)(t'(x)\mu, t'(x)c).$$

Definition 2.1. *A metrical structure on TM is a smooth section of $\check{\sigma}$, with nondegenerate principal part.*

Remark. 1. The existence of the metrical structure implies that $\mathbf{M}$ is a reflexive Banach space, therefore we may suppose that our manifolds are modelled on reflexive Banach spaces.

2. If for every trivialization there is a smooth map $s_\varphi : \varphi(U) \times \mathbf{M} \to L_s^2(\mathbf{M} \times \mathbf{M})$ which verifies (2.1), then there exists a global section on $\check{\sigma}$.

Let g be a metrical structure in the horizontal bundle (or in the vertical bundle). We consider now (in adapted trivilizations)

$$(2.2)\qquad G_\varphi(x, a)(\lambda, b)(\mu, c) = <g_\varphi(x, a)\lambda, \mu> + <g_\varphi(x, a)b, c>$$

and

$$(2.3)\qquad \tilde{G}_\varphi(x, a)(\lambda, b)(\mu, c) = <g_\varphi(x, a)\lambda, c> + <g_\varphi(x, a)\mu, b>.$$

which defines two metrical structures on TM similar to that of Sasaki. (The brackets denotes the duality of $\mathbf{M}$ and $\mathbf{M}^*$.)

We shall try to obtain the Levi-Civita connection associated to each of these metrics.

Theorem 2.1. *The connectors of the Levi-Civita connection associated with G are given by the relations (in the adapted atlas)*

$$\text{(i)}\quad <{}^{11}\Omega^h(\lambda, \mu), \nu^*> = \frac{1}{2}(<\mathcal{D}g(\lambda, 0)\mu, \nu> + <\mathcal{D}g(\mu, 0)\lambda, \nu> - <\mathcal{D}g(\nu, 0)\lambda, \mu>;$$

$$\text{(ii)}\quad <{}^{12}\Omega^h(\lambda, b), \mu^*> = \frac{1}{2}(<D_2 gb\lambda, \mu> + <R(\lambda, \mu), b^*>);$$

$$\text{(iii)}\quad <{}^{21}\Omega^h(b, \lambda), \nu^*> = \frac{1}{2}(<D_2 gb\lambda, \mu> + <R(\lambda, \mu), b^*>);$$

$$\text{(iv)}\quad <{}^{22}\Omega^h(b, c), \lambda^*> = -\frac{1}{2}<Dg(\lambda, 0)b, c> + \frac{1}{2}(<D_2\omega c\lambda, b^*> + <D_2\omega b\lambda, c^*>);$$

(v) $<{}^{11}\Omega^v(\lambda,\mu), b^*> = -\frac{1}{2}(<D_2 gb\lambda, \mu> + <R(\mu,\lambda), b^*>)$;

(vi) $<{}^{12}\Omega^v(\lambda, b), c^*> = -\frac{1}{2}<D_2\omega b\lambda, c^*> + \frac{1}{2}(<Dg(\lambda, 0)b, c> -$

$- <D_2\omega c\lambda, b^*>)$;

(vii) $<{}^{21}\Omega^v(b,\lambda), c^*> = \frac{1}{2}<D_2\omega b\lambda, c^*> + \frac{1}{2}(<Dg(\lambda, 0)b, c> -$

$- <D_2\omega c\lambda, b^*>)$;

(viii) $<{}^{22}\Omega^v(b, c), d^*> = \frac{1}{2}(<D_2 gbc, d> + <D_2 gcb, d> - <D_2 gdb, c>)$,

where $\lambda^* = g(x, a)\lambda$, *etc. and Fréchet derivatives are calculated in* (x, a).

Proof. We have

$$XG(Y, Z) = G(\nabla_X Y, Z) + G(Y, \nabla_X Z),$$

which has the following local form in the adapted trivializations

$$\mathcal{D}g(X^h, X^v)(Y^h, Z^h) + \mathcal{D}h(X^h, X^v)(Y^v, Z^v) = g(\Omega^h(Y)(X), Z^h) +$$
$$+ g(\Omega^h(Z)(X), Y^h) + h(\Omega^v(Y)(X), Z^v) + h(\Omega^v(Z)(X), Y^v).$$

By using Theorem 1.1 and by a straightforward computation, we obtain (i)–(viii).

Theorem 2.2. *The connectors of the Levi-Civita connection associated with* $\bar{G}$ *are given by the relations*

(i) $<{}^{11}\Omega^h(\lambda,\mu), b^*> = \frac{1}{2}(<\mathcal{D}g(\lambda, 0)\mu, b> + <\mathcal{D}g(\mu, 0)\lambda, b> -$

$- <D_2\omega b\lambda, \mu^*> - <D_2\omega b\mu, \lambda^*>)$.

(ii) $<{}^{12}\Omega^h(\lambda, b), d^*> = \frac{1}{2}(<D_2 gb\lambda, d> - <D_2 gdb, \lambda>)$.

(iii) ${}^{21}\Omega^h(b,\lambda) = {}^{12}\Omega^h(\lambda, b)$.

(iv) ${}^{22}\Omega^h = 0$.

(v) $<{}^{11}\Omega^v(\lambda,\mu), \nu^*> = \frac{1}{2}(<R(\lambda,\mu), \nu^*> + <R(\mu,\nu), \lambda^*> -$

$- <R(\nu,\lambda), \mu^*>)$.

(vi) $<{}^{21}\Omega^v(b,\lambda), \mu^*> = \frac{1}{2}(<\mathcal{D}g(\lambda, 0)\mu, b> - <\mathcal{D}g(\mu, 0)\lambda, b> +$

$+ <D_2\omega b\lambda, \mu^*> + <D_2\omega b\mu, \lambda^*>)$.

(vii) $<{}^{12}\Omega^v(\lambda, b), \mu^*> = \frac{1}{2}(<\mathcal{D}g(\lambda, 0)\mu, b> - <\mathcal{D}g(\mu, 0)\lambda, b> +$

$+ <D_2\omega b\mu, \lambda^*> - <D_2\omega b\lambda, \mu^*>)$.

(viii) $<{}^{22}\Omega^v(b, c), \lambda^*> = \frac{1}{2}(<D_2 gbc, \lambda> + <D_2 gcb, \lambda>)$.

Proof. (i)–(viii) are obtained by solving the system

$$<{}^{22}\Omega^h(b, c), d^*> + <{}^{22}\Omega^h(d, c), b^*> = 0$$
$$<{}^{21}\Omega^h(b,\lambda), c^*> + <{}^{21}\Omega^v(c,\lambda), b^*> = 0$$
$$<{}^{11}\Omega^v(\lambda,\mu), \nu^*> + <{}^{11}\Omega^v(\nu,\mu), \lambda^*> = 0$$
$$<{}^{12}\Omega^v(\lambda, b), \mu^*> + <{}^{12}\Omega^v(\mu, b), \lambda^* = 0$$
$$<{}^{22}\Omega^v(b, c), \lambda^*> + <{}^{12}\Omega^h(\lambda, c), b^*> = D_2 gcb\lambda$$
$$<{}^{11}\Omega^v(\lambda,\mu), b^*> + <{}^{21}\Omega^v(b,\mu), \lambda^*> = Dg(\mu, 0)\lambda b$$

and using Theorem 1.1.

In particular we obtain some results of V. Oproiu [9], [10].

Definition 2.2. *A d-connection on* TM *is metrical with respect to* G, *if*

$$XG(Y, Z) = G(\nabla_X Y, Z) + G(Y, \nabla_X Z) \qquad X, Y \in \mathcal{X}(TM).$$

Definition 2.3. *A d-connection on* TM *is called a normal connection, if* $L^h = L^v$ *and* $C^h = C^v$.

We denote $L = L^h = L^v$ and $C = C^h = C^v$.

Let $\hat{\sigma} : L(\sigma) \to TM$ be the vector bundle of linear continuous maps of σ.

Let K be a linear connection on TM and s a section of $\hat{\sigma}$. We say that s is compatible with K, if

$$(\nabla_X s)(Y) = \nabla_X s(Y) - s(\nabla_X Y) \qquad X, Y \in \mathcal{X}(TM) \text{ [2]}.$$

We have some important sections of $\hat{\sigma}$ geometrically associated to N

(i) an almost product structure P, $P(x, a)(\lambda, b) = (\lambda, -b)$,

(ii) an almost tangent structure J, $J(x, a)(\lambda, b) = (0, \lambda)$,

(iii) an almost complex structure F, $F(x, a)(\lambda, b) = (b, -\lambda)$

(the local representatives are given in the adapted trivializations).

Proposition 2.1. *Let* K *be a linear connection on* TM. *Then* K *is a d-connection if and only if it is compatible with* P.

Proposition 2.2. *If* K *is a d-connection on* TM, *then the following statements are equivalent*

(i) K *is normal*,

(ii) K *is compatible with* J,

(iii) K *is compatible with* F.

Theorem 2.3. *There is a unique normal d-connection, metrical with respect to* $\bar{G}$ *and* G, *with* $L(x; a)$, $C(x, a)$ *continuous symmetric bilinear maps. The connectors have the expressions*:

$$<L(\lambda,\mu), \nu^*> = \frac{1}{2}(<\mathcal{D}g(\lambda, 0)\mu, \nu> + <\mathcal{D}g(\mu, 0)\lambda, \nu> -$$
$$- <\mathcal{D}g(\nu, 0)\lambda, \mu>),$$

$$<C(\lambda, b), \mu^*> = \frac{1}{2}(<D_2 g\lambda b, \mu> + <D_2 gb\mu, \lambda)> - <D_2 g\mu\lambda, b>),$$

where $\mu^* = g(x, a)\mu$ *etc.*

3. **Infinite dimensional Lagrange spaces.** The connections on M are connected with the second order equations on M.

Definition 3.1. *A second-order equation on a manifold M is a vector field X on TM such that $\tau_* \circ x = 1_{TM}$.*

A second-order equation has the following local form

$$X_\varphi(x, a) = (x, a, a, X_\varphi^2(x, a)) \text{ (in the natural trivializations).} \tag{3.1}$$

Proposition 3.1. *Let $N : T^2M \to TM$ be a connection on M, locally given by $N_\varphi(x, a, \lambda, b) = (x, b + \omega_\varphi(x, a)\lambda)$. Then it induces a second-order equation on M such that the local representative has the form*

$$X_\varphi(x, a) = (x, a, a, -\omega_\varphi(x, a)a).$$

Conversely, we have

Proposition 3.2. *Let X be a second-order equation on M locally given by (3.1). Then the maps*

$$\omega_\varphi : \varphi(U) \times \mathbf{M} \to L(\mathbf{M}, \mathbf{M}), \quad \omega_\varphi(x, a) = -\frac{1}{2} D_2 X_\varphi^2(x, a)$$

are the local components of a connection on the tangent bundle.

Definition 3.2. *Let M be a Banach manifold and $L \in \mathcal{F}(TM)$. We call L a regular Lagrangian if $D_2D_2L(x, a) : \mathbf{M} \to \mathbf{M}^*$ is a isomorphism. A Lagrange space is a pair (M, L), where M is a Banach manifold and L is a regular Lagrangian.*

Proposition 3.3. *Let (M, L) be a Lagrange space and $G_\varphi(x, a) = [D_2D_2L(x, a)]^{-1}(D_1D_2L(x, a)a - D_1L(x, a))$. Then $\omega_\varphi(x, a) = \frac{1}{2} D_2G(x, a)$ are the connectors of a connection $N : T^2M \to TM$.*

Proof. Let $A : TM \to \mathbf{R}$, locally given by $A(x, a) = D_2L(x, a)a$ and $E = A - L$. Let X_E be the Lagrange vector field defined by: $i_{X_E}\Omega_L = dE$ ([1], pag. 213), where Ω_L is the canonical Lagrange 2-form. X_E is a second-order equation on M. Locally,

$$X_E^2(x, a) = [D_2D_2L(x, a)]^{-1}(D_1L(x, a) - D_1D_2L(x, a)a).$$

According to Proposition 3.2, we could conclude that $\omega(x, a)\lambda = -\frac{1}{2}D_2X_E^2(x, a)\lambda$ are the connectors of a connection on M.

The connection given by Proposition 3.3 is called the canonical connection of the Lagrange space.

In the sequel we use only this connection.

We have the metrical structure on TM given by (2.2) and (2.3), where

$$<g(x, a)\lambda, b> = <D_2D_2L(x, a)\lambda, b>.$$

We can define an almost-symplectic structure on TM locally (in the adapted trivializations) given by

$$\Omega(x, a)(\lambda, b)(\mu, c) = <g(x, a)\lambda, c> - <g(x, a)b, \mu> \tag{3.2}$$

We obtain that Ω is globally defined on TM.

Lemma 3.1. *The canonical connection of a Lagrange space satisfies*

$$<\omega(x, a)\lambda, \mu^*> - <\omega(x, a)\mu, \lambda^*> = <D_1D_2L(x, a)\lambda, \mu> - <D_1D_2L(x, a)\mu, \lambda>,$$

where $\mu^* = g(x, a)\mu$, $\lambda^* = g(x, a)\lambda$.

We know [1] that

$$\theta(x, a)(\lambda, b) = D_2L(x, a)\lambda$$

is a globally defined *l*-form on TM.

Theorem 3.1. $-d\theta = \Omega$, *thus Ω is a symplectic structure on TM.*

Proof. We have $d\theta(X, Y) = X\theta(Y) - Y\theta(X) - \theta[X, Y]$ $X, Y \in \mathcal{X}(M)$ and locally in the adapted trivializations:

$$d\theta(x, a)(\lambda, b)(\mu, c) = \mathcal{D}\theta(x, a)(\lambda, b)(\mu, c) - \mathcal{D}\theta(x, a)(\mu, c)(\lambda, b). \tag{3.3}$$

Finally we use the Lemma 3.1.

Remark. The symplectic structure Ω is the canonical Lagrange 2-form. But here it has a simple local form with a geometrical significance.

Proposition 3.4. *The canonical connection of a Lagrange space has the following properties:*

(i) $D_2\omega(x, a)\mu\lambda = D_2\omega(x, a)\lambda\mu$,

(ii) $<R(\lambda, \mu), \nu^*> + <R(\mu, \nu), \lambda^*> + <R(\nu, \lambda), \mu^*> = 0$,

(iii) $<D_2\omega\nu\mu, \lambda^*> - <D_2\omega\nu\lambda, \mu^*> = <\mathcal{D}g(\mu, 0)\lambda, \nu> - <\mathcal{D}g(\lambda, 0)\mu, \lambda>$

(in the adapted trivializations).

Proof. (ii) and (iii) are obtained by differentiating the relation (3.3) and by using the adapted trivializations.

For the Lagrange spaces the Levi-Civita connection are expressed in a simple form.

Theorem 3.2. *If (M, L) is a Lagrange space, then the local connectors of the Levi-Civita connection associated with G are given by*

(i) $<{}^{11}\Omega^h(\lambda, \mu), \nu^*> = \frac{1}{2}(<\mathcal{D}g(\lambda, 0)\mu, \nu> + <\mathcal{D}g(\mu, 0)\lambda, \nu> - <\mathcal{D}g(\nu, 0)\lambda, \mu>)$.

(ii) ${}^{21}\Omega^v = {}^{11}\Omega^h$, ${}^{21}\Omega^h(b, \lambda) = {}^{12}\Omega^h(\lambda, b)$.

(iii) $<{}^{22}\Omega^v(\lambda, \mu), \nu^*> = \frac{1}{2}<D_2g\lambda\mu, \nu>$.

(iv) ${}^{12}\Omega^v = -{}^{22}\Omega^h$, ${}^{22}\Omega^h(\lambda, \mu) = D_2\omega\lambda\mu - {}^{11}\Omega^h(\lambda, \mu)$.

(v) ${}^{11}\Omega^v(\lambda, \mu) + {}^{22}\Omega^v(\lambda, \mu) = \frac{1}{2}R(\lambda, \mu)$.

(vi) $<{}^{12}\Omega^h(\lambda, \mu), \nu^*> = <{}^{22}\Omega^v(\lambda, \mu), \nu^*> + \frac{1}{2}<R(\lambda, \nu), \mu^*>$.

Proof. To prove these relations, we use Theorem 2.1, Proposition 3.4 and Lemma 3.1.

Remark. Only two connectors of the Levi-Civita connection are essential, ${}^{11}\Omega^h$ and ${}^{22}\Omega^v$, the others are expressed by using these connectors.

Theorem 3.3. *If (M, L) is a Lagrange space, then the local connectors of the Levi-Civita connection associated with $\tilde{G}$ are given by*

(i) ${}^{12}\Omega^h = {}^{21}\Omega^h = {}^{22}\Omega^h = {}^{12}\Omega^v = 0$,

(ii) $<{}^{11}\Omega^h(\lambda, \mu), b^* > = <\mathcal{D}g(\lambda, 0)\mu, b> - < D_2\omega b\lambda, \mu^* >$,

(iii) $<{}^{11}\Omega^v(\lambda, \mu), \nu^* > = <R(\lambda; \nu), \mu^*>$,

(iv) ${}^{21}\Omega^v(b, \lambda) = D_2\omega b\lambda$,

(v) ${}^{22}\Omega^v(b, c) = D_2 gbc$.

Remark. In a Lagrange space, for the normal connection given by Theorem 2.3, we have

$$<C(\lambda, b), \mu^* > = \frac{1}{2} <D_2 g\lambda b, \mu>.$$

4. Infinite dimensional Finsler spaces

Definition 4.1. *A Lagrange space (M, L) with $L(x, a)$ 2-homogeneous with respect to $a \neq 0$ is called a Finsler space.*

From the equality $L(x, ra) = r^2 L(x, a)$ with r a real number, we obtain

$$L(x, a) = \frac{1}{2} <g(x, a)a, a>, \tag{4.1}$$

$$<D_2 g(x, a)cb, a> = <D_2 g(x, a)ab, c> = <D_2 g(x, a)ba, c> = 0. \tag{4.2}$$

If we denote by $\Gamma(x, a)$ the Christoffel symbols of $g(x, a)$:

$$<\Gamma(x, a)(b, c), d^*> = \frac{1}{2}(<D_1 g(x, a)bc, d> + <D_1 g(x, a)cb, d> - <D_1 g(x, a)db, c>), \tag{4.3}$$

then we have

Theorem 4.1. *The canonical nonlinear connection of a Finsler space is locally given by the equality:*

$$\omega(x, a)\lambda = \frac{1}{2} D_2 \Gamma^0(x, a)\lambda, \tag{4.4}$$

where $\Gamma^0(x, a) = \Gamma(x, a)(a, a)$.

Proof. One uses Proposition 3.3.

Theorem 4.2. *Let (M, L) be a Finsler space and N a nonlinear connection on M. Let K be the normal metrical connection given by Theorem 2.3. Then N is the canonical connection (4.4) if and only if in any trivialization*

$$L(x, a)(a, \lambda) = \omega(x, a)\lambda, \qquad \forall (x, a) \in \varphi(U) \times \mathbf{M},\ \lambda \in \mathbf{M}.$$

Proof. Let N be a connection on M, therefore

$$<L(\lambda, \mu), \nu^*> = \frac{1}{2}(<\mathcal{D}g(\lambda, 0)\mu, \nu> + <\mathcal{D}g(\mu, 0)\lambda, \mu> - <\mathcal{D}g(\nu, 0)\lambda, \mu),$$

and with (4.2) we get

$$<L(\lambda, \mu), \nu^*> = \frac{1}{2}(<D_1 ga\lambda, \mu> + <D_1 g\lambda a, \mu> - <D_1 g\mu a, \lambda>) - \frac{1}{2} <D_2 g\omega a\lambda, \mu)>.$$

But $L(x, a)(a, \lambda) = \omega(x, a)\lambda$, thus

$$<\omega(x, a)\lambda, \mu^*> = <\Gamma(x, a)(a, \lambda), \mu^*> - \frac{1}{2} <D_2 g\lambda\mu, \omega(x, a)a>.$$

Then $\omega(x, a)a = \Gamma(x, a)(a, a)$.

Therefore

$$<\omega(x, a)\lambda, \mu^*> = <\Gamma(x, a)(\lambda, a), \mu^*> - \frac{1}{2} <D_2 g\lambda\mu, \Gamma(x, a)(a, a)>. \tag{4.5}$$

From (4.3)

$$<g(x, a)\Gamma(x, a)(a, a), b> = <D_1 g(x, a)aa, b> - \frac{1}{2} <D_1 g(x, a)ba, a>$$

and, by differentiating partially with respect to a in the direction λ, we get

$$<D_2\Gamma(x, a)\lambda(a, a), g(x, a)b> + 2<\Gamma(x, a)(\lambda, a), g(x, a)b> + <\Gamma(x, a)(a, a), D_2 g(x, a)\lambda b> = <D_1 g(x, a)\lambda a, b> + <D_1 g(x, a)a\lambda, b> + <D_1 g(x, a)b\lambda, a>.$$

Thus, $<D_2\Gamma(x, a)\lambda(a, a), g(x, a)b> = -<\Gamma(x, a)(a, a), D_2 g(x, a)\lambda b>$ and from (4.5) we obtain (4.4). Conversely, if N is the canonical connection (4.4), then $L(x, a)(a, \lambda) = \omega(x, a)\lambda$.

Remark. The d-connection given by Theorem 2.3 for a Finsler space is just the Cartan connection.

Theorem 4.1 generalizes a result of Matsumoto [5] to infinite dimensional Finsler spaces.

5. **Examples.** Let (M, g) be an infinite dimensional pseudo-riemannian manifold modelled on a reflexive Banach space $\mathbf{M}$, and $\Gamma_\varphi : \varphi(U) \to L(\mathbf{M}, \mathbf{M})$ the connectors of the Levi-Civita connection associated with g. We have

$$<g(x)\Gamma(x)(a, b), c> = \frac{1}{2}(<g'(x)ab, c> + <g'(x)ba, c> - <g'(x)ca, b>).$$

We consider the Legrangian

$$L(x, a) = \frac{1}{2} <g(x)a, a> - V(x), \tag{5.1}$$

where $V \in \mathcal{F}(M)$ is the potential (see [1]).

Then, for the Lagrange space (M, L) with L given by (5.1), we have

(i) $\omega(x, a)\lambda = \Gamma(x)(a, \lambda)$,

(ii) $G(x, a)(\lambda, b)(\mu, c) = <g(x)\lambda, \mu> + <g(x)b, c>$.

(iii) $L(x, a) = \Gamma(x)$, $C = 0$.

If we consider the Lagrangian

$$L(x, a) = \frac{1}{2} < g(x)a, a > + < A(x), a > - V(x) \tag{5.2}$$

with $V \in \mathcal{F}(M)$ and A the principal part of a 1-form, we have

(i) $\omega(x, a)\lambda = \Gamma(x)(a, \lambda) + \frac{1}{2} g^{-1}(x)(A'(x) - A'^{*}(x)\lambda)$, where $A'^{*}(x)$ is the dual of $A'(x)$.

(ii) $L(x, a) = \Gamma(x)$, $C = 0$.

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