Theorem 26. *If there exists a strictly positive potential in S, then the following are equivalent*

1) $\forall p \in T_\theta : \bigwedge \{B^{Y \setminus K_p} K \subset Y$ compact, $\varphi|_K$ continuous$\} = 0$

2) φ *satisfies* (QC).

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Received 13. IX. 1988

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EXTENSIONS OF SOME RESULTS OF M. PAL

BY

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Introduction. In [3] S. Kurepa proved the following theorem.

Theorem K. *If $A \subset \mathbf{R}^n$ (n-dimensional Euclidean space) is Lebesgue measurable and has positive measure and $b_1, b_2, \ldots, b_p$ $(b_k \neq 0)$ are p real numbers then there exists a ball K_r of radius r with center at the origin such that for each $x \in K_r$, there are vectors $a_0(x), a_1(x), \ldots, a_p(x)$ in A such that*

$$x = (a_1(x) - a_0(x))b_1^{-1} = (a_2(x) - a_0(x))b_2^{-1} = \ldots = (a_p(x) - a_0(x))b_p^{-1}.$$

M. Pal [11] extended Kurepa's result to the following.

Theorem P. *Let A be a set of positive Lebesgue measure in $\mathbf{R}^n$. Also let $b_1, b_2, \ldots, b_p (b_k \neq 0)$ be a system of p real numbers and let (c_n) be a sequence of non-zero real numbers such that $\lim_{n\to\infty} c_n = 1$. Then there exists a positive integer N_0 and a ball K with center at the origin such that for every vector x in K and system of p real numbers $c_{n_1}, c_{n_2}, \ldots, c_{n_p}$, of the sequence (c_n) with $n_i > N_0$ theere exist vectors $a_0(x; c_{n_1}, c_{n_2}, \ldots, c_{n_p})$ and $a_k(x; c_{n_1}, c_{n_2}, \ldots, c_{n_p})$ $(k = 1, 2, \ldots, p)$ in A such that $x = [c_{n_k}^{-1} a_k(x; c_{n_1}, c_{n_2}, \ldots, c_{n_p}) - a_0(x; c_{n_1}, c_{n_2}, \ldots, c_{n_p})] b_k^{-1}$, $k = 1, 2, \ldots, p$.*

The mapping T_c, where c is a real number, $T_c : \mathbf{R}^n \to \mathbf{R}^n$, defined by $T_c(x) = cx$, where $cx = (cx_1, cx_2, \ldots, cx_n)$ for each $x = (x_1, x_2, \ldots, x_n)$, is an extremely simple mapping. It is natural to ask if Theorem P can be extended to a theorem about more general mappings than those of type T_c. In addition one might wonder about extending Theorem P in such a way that each $x \in K$ is expressible in infinitely many ways.

Borwein and Ditor [2] have proved the following theorem, answering a question of P. Erdös

Theorem B and D. (i) *If A is a measurable set of real numbers having positive measure and (d_n) is a sequence converging to zero, then, for almost all $x \in A$, $x + d_n \in A$ for infinitely many n; (ii) There is a measurable set A of positive measure and a monotonic sequence (d_n) of positive numbers converging to zero such that, for all x, $x + d_n \notin A$ for infinitely many n.*

Recently the current author has generalized the above result of Borwein and Ditor using general 2-place functions in place of $+$ (see [9]).

In [8] the present author observed that the Baire set analogue of (ii) is not true. In that paper the author also considers a certain property (C)

[1] The work on this paper was supported by the Research Fund of Bosnia and Herzegovina.

for abstract transformations in a general measure space. More exactly (see [8], p. 6):

Suppose $(X, \mathcal{M}, \mu)$ is a measure space, i.e. $\mathcal{M}$ is a σ-algebra of subsets of X and μ is a σ-additive measure on $\mathcal{M}$. Suppose further W is a metric space and $(T_w)_{w\in W}$ is a family of transformations of $\mathcal{M}$ into $\mathcal{M}$. Then $(T_w)_{w\in W}$ is said to satisfy (C), with respect to $w_0 \in W$, if for every $A \in \mathcal{M}$ and (w_n), $w_n \to w_0$ (in the metric space W), $\mu(T_{w_0}(A) \setminus T_{w_n}(A)) \to 0$ as $n \to \infty$.

As shown in [8]: If A is a bounded Lebesgue measurable subset of $\mathbf{R}$, then (a) If $w_n \to 0$, then $\mu(A \setminus (w_n + A)) \to 0$ as $n \to \infty$ and (b) If $w_n \to 1$, then $\mu(A \setminus (w_n A)) \to 0$ as $n \to \infty$, where μ is Lebesgue measure and $w_n + A = (w_n + a : a \in A)$ and $w_n A = (w_n a : a \in A)$.

Clearly, extensions of (a) and (b) hold in $\mathbf{R}^n$. More interesting is the following ([8], p. 7). Let $(X, \mathcal{M}, \mu)$ be the Lebesgue measure space $\mathbf{R}^n$, i.e. let $X = \mathbf{R}^n$, $\mathcal{M}$ be the collection of all Lebesgue measurable subsets of $\mathbf{R}^n$ and μ Lebesgue measure. Let W be the collection of all "n by n" real matrices. The distance between two matrices, (a_{ij}) and (b_{ij}) is given by the formula

$$d((a_{ij}), (b_{ij})) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left| a_{ij} - b_{ij} \right| .$$

Let $w_0 = (a^0_{ij})$ be a fixed non-singular (i.e. $\det(a^0_{ij}) \neq 0$) n by n matrix. If (w_m) is a sequence of n by n matrices converging to w_0 (in the metric d) and $A \in \mathcal{M}$, A bounded, then

$$\mu(T_{w_0}(A) \setminus T_{w_m}(A)) \to 0 \text{ as } m \to \infty .$$

where for each $w \in W$ and every $S \subset \mathbf{R}^n$, $T_w(S)$ is given by the formula

$$T_w(S) = \left\{ (y_1, \ldots, y_n) \in \mathbf{R}^n : y_i = \sum_{j=1}^{n} a_{ij} x_j, \ (x_1, x_2, \ldots, x_n) \in S \right\}, \quad \text{where} \quad w = (a_{ij}).$$

Many papers have been written on transformations of sets in $\mathbf{R}^n$. The articles [1], [5], [6], and [7] are reprezentative of the works in this area. The paper by N e u b r u n n and Š a l á t [10], which initiated this area of research is of particular importance.

In [5], motivated by the above mentioned paper of N e u b r u n n and Š a l á t, the present author considers families $(T_w)_{w\in W}$ of transformations on $\mathcal{M}$ into $\mathcal{M}$ (the set of all Lebesgue measurable subsets of $\mathbf{R}^n$), where W is a metric space, satisfying the following conditions. There exists $w_0 \in W$ and $a, b \in \mathbf{R}^n$ such that for $w_n \to w_0$

(1) $\limsup_{n\to\infty} |a - T_{w_n}(K)| = r$ for each closed ball K with center at b and radius r. Here $|y - C|$ denotes the set of all distances between y and points of C.

(2) $E, F \in \mathcal{M}$, $E \subset F$ implies $T_w(E) \subset T_w(F)$ for each $w \in W$.

(3) If $E \in \mathcal{M}$ and $w_n \to w_0$ (in W), then $\lim_{n\to\infty} m(T_{w_n}(E)) = m(T_{w_0}(E)) = m(E)$.

where m denotes Lebesgue measure in $\mathbf{R}^n$

Theorem 2' in [5] states: *Suppose A and B are two sets of positive measure in $\mathbf{R}^n$ and a is a point of density of A and b is a point of density of B*

and w_0 is a point of W. Suppose $(T_w)_{w\in W}$ is a family of transformations of $\mathcal{M}$ into $\mathcal{M}$ satisfying the conditions (1), (2), and (3) with respect to a, b and w_0. Then if (w_n) is a sequence in W converging to w_0 and p is a positive integer, there exists p strictly increasing integers $n_1, n_2, \ldots, n_p$ such that

$$A \cap T_{w_{n_1}}(B) \cap T_{w_{n_2}}(B) \cap \ldots \cap T_{w_{n_p}}(B)$$

is a set of positive Lebesgue measure.

In [7] it is shown that the last mentioned result can not be extended; namely, the following theorem is proved.

Theorem M and P. *There exist Lebesgue measurable subsets A and B of $\mathbf{R}^2$, each having positive measure, and points a and b of density in A and B, respectively, and a point w_0 in W (a metric space) and a family $(T_w)_{w\in W}$ of transformations of $\mathcal{M}$ into $\mathcal{M}$ (the Lebesgue measurable subsets of $\mathbf{R}^2$) satisfying (1), (2), and (3) with respect to the points a, b, and w_0 such that for every sequence (w_n) in W converging to w_0,*

$$A \cap \bigcap_{k=1}^{\infty} T_{w_{n_k}}(B)$$

is a set of measure zero for each subsequence $(w_{n_k})_{k=1}^{\infty}$ of $(w_n)_{n=1}^{\infty}$.

In light of the last theorem and our desire to represent vectors near the origin in infinitely many ways, we will extend Theorem P, using collections of transformations $(T_w)_{w\in W}$ satisfying condition (C) rather than (1), (2), and (3). It is interesting to note that even though our approach is more general (in two senses) than Pal's approach our proofs are in some sense simpler.

In various papers Baire set analogues of theorems like those mentioned here have been studied. Condition (C) is meaningless for transformations that map Baire sets into Baire sets. Our results, in this paper, about Baire sets will use families of point transformations T, i.e. functions $T : \mathbf{R}^n \to \mathbf{R}^n$ that are differentiable and whose derivatives satisfy certain assumptions, as in [1] and examine Baire set analogues of our present results on measurable sets.

Results. Our first result is an extension of Theorem P mentioned in our introduction.

Theorem 1. *Let A be a set of positive Lebesgue measure in $\mathbf{R}^n$. Suppose that $(b_k)_{k=1}^{\infty}$ is a convergent sequence of non-zero real numbers whose limit is non-zero. Suppose further that W is a metric space, $w_0 \in W$ and $(T_w)_{w\in W}$ is a family of transformations of $\mathbf{R}^n$ into $\mathbf{R}^n$ each of which maps measurable subsets of $\mathbf{R}^n$ into measurable subsets of $\mathbf{R}^n$ and such that*

(i) *$m(T_{w_0}(B)) = 0$ whenever $m(B) = 0$ and*

(ii) *$\lim_{n\to\infty} m(T_{w_0}(A \cap K) \setminus T_{w_n}(A \cap K)) = 0$ for each ball K and sequence (w_n) in W that converges to w_0.*

Suppose that $w_n \to w_0$. Then for each natural number p, there exists a ball K_p with center at the origin, a positive integer N_p such that for each sequence of p integers $N_p < n_1 < n_2 < \ldots < n_p$ there exist two sequences $(n_i)_{i=1}^{\infty}$ (a sub-

sequence of the natural numbers whose first p terms are the prescribed elements $n_1, n_2, \dots, n_p$) and $(b_{k_i})_{i=1}^{\infty}$ (a subsequence of (b_k) whose first p terms are $b_1, \dots, b_p$) such that for each $x \in K_p$, there exists a sequence of points $(a_i)_{i=0}^{\infty}$ all in A, such that

$$x = (T_{w_{n_i}}(a_i) - T_{w_0}(a_0))/b_{k_i} \quad \textit{for each } i = 1, 2, \dots$$

Proof. There exists a ball K such that $m(A \cap K) > 0$. Let $B = T_{w_0}(A \cap K)$. Then, by (i), $m(B) > 0$. There exists a ball K' such that

$$m(B \cap K') > 0.99\, m(K'). \tag{*}$$

Define the set $X(x; n_1, n_2, \dots, n_p)$ by the equation $X(x; n_1, n_2, \dots, n_p) = T_{w_0}(A) \cap (T_{w_{n_1}}(A) - b_1 x) \cap \dots \cap (T_{w_{n_p}}(A) - b_p x)$.

By (*), (ii), the definition of $X(x; n_1, n_2, \dots, n_p)$ and formula (a) in the introduction (applied to $\mathbf{R}^n$) there exists a ball K_p with center at the origin and a natural number N_p, such that if $N_p < n_1 < n_2 < \dots < n_p$ then the measure of the set

$$X(x; x_1, x_2, \dots, n_p) \cap K' \cap (T_{w_0}(A) - bx)$$

is greater than $0.99\, m(K')$ for every $x \in K_p$, where $b = \lim_{k \to \infty} b_k$.

This implies, using (ii) and formula (a), that there exists $n_{p+1} > n_p$ and $k_{p+1} > p$ such that the set

$$X(x; n_1, n_2, \dots, n_p) \cap (T_{w_{n_{p+1}}}(A) - b_{k_{p+1}} x) \cap (T_{w_0}(x) - bx) \cap K'$$

has measure greater than $0.99\, m(K')$ for each $x \in K_p$.

This process can be continued obtaining two sequences $(n_i)_{i=1}^{\infty}$ (a subsequence of the natural numbers whose first p terms are $n_1, n_2, \dots, n_p$) and $(b_{k_i})_{i=1}^{\infty}$ (a subsequence of (b_k) whose first p terms are $b_1, b_2, \dots, b_p$) such that for each $x \in K_p$ the set

$$T_{w_0}(A) \cap \bigcap_{i=1}^{\infty} (T_{w_{n_i}}(A) - b_{k_i} x) \cap (T_{w_0}(A) - bx) \cap K'$$

has measure greater than or equal to $0.99\, m(K')$ and is hence non-empty.

Therefore for each $x \in K_p$ there exists a sequence of points $(a_i)_{i=0}^{\infty}$, all in A, such that $x = (T_{w_{n_i}}(a_i) - T_{w_0}(a_0)) b_{k_i}^{-1}$ for each $i = 1, 2, \dots$, completing the proof.

Our next result, which is an immediate corollary of Theorem 1, clearly implies Theorem P of Pal but does not follow from Pal's theorem.

Corollary 1. *Let A be a set of positive Lebesgue measure in $\mathbf{R}^n$. Suppose that $(b_k)_{k=1}^{\infty}$ is a convergent sequence of non-zero real numbers whose limit is non-zero. Suppose that W is the metric space of all non-singular n by n real matrices (see the introduction) and $w_0 \in W$. If $(w_n)_{n=1}^{\infty}$ is a sequence in W converging to w_0, then for each natural number p, there exists a ball K_p with center at the origin and a positive integer N_p such that for each sequence of p integers $N_p < n_1 < n_2 < \dots < n_p$ there exist two sequences $(n_i)_{i=1}^{\infty}$ (a subsequence of the natural numbers whose first p terms are $n_1, n_2, \dots, n_p$) and $(b_{k_i})_{i=1}^{\infty}$ (a subsequence of (b_k) whose first p terms are $b_1, \dots, b_p$) such that for each $x \in K_p$, there exists a sequence of points $(a_i)_{i=1}^{\infty}$, all in A, such that $x = (T_{w_{n_i}}(a_i) - T_{w_0}(a_0))/b_{k_i}$ for each $i = 1, 2, \dots$.* Here T_w is the linear transformation determined by w (see the introduction).

Remark 1. Although Theorem 1 is a generalization of Theorem P, the proof just presented is simpler than the proof of Theorem P given in Pal's proof. We further remark that in Theorem 1 it is sufficient for the sequence (b_k) to contain a subsequence having non-zero limit.

Our next result is a generalization of Theorem 2 in [11].

Theorem 2. *Suppose that A and B are subsets of $\mathbf{R}^n$, each having positive Lebesgue measure and a is a point of density of A and b is a point of density of B. Suppose that (b_k) is a convergent sequence of non-zero reals whose limit is non-zero. Assume further that W is a metric space, $w_0 \in W$ and $(T_w)_{w \in W}$ is a family of transformations of $\mathbf{R}^n$ into $\mathbf{R}^n$ each of which maps measurable subsets of $\mathbf{R}^n$ into measurable subsets of $\mathbf{R}^n$ and such that:*

(i) *$T_{w_0}(x) = x$ for every $x \in \mathbf{R}^n$,*

(ii) *$\lim_{n \to \infty} m((B \cap K) \setminus T_{w_n}(B \cap K)) = 0$ for each ball K with center at b and having sufficiently small radius and each sequence (w_n) in W converging to w_0 and*

(iii) *there exists a ball $\bar{K}$ with center at c $(c = b - a)$, such that for each sequence (w_n) converging to w_0 and each $z \in \bar{K}$, we have $\lim_{n \to \infty} T_{w_n}(z) = z$.*

Suppose $w_n \to w_0$. Then for each natural number p, there exists a positive integer N_p and balls K with center at the origin and K_1 with center at c such that for each sequence of p integers $N_p < n_1 < n_2 \dots < n_p$ and sequence $(w_m)_{m=1}^{\infty}$ in K_1 converging to c there exist three sequences:

(n_i), a subsequence of the natural numbers whose first p terms are $n_1, n_2, \dots n_p$;

(b_{k_i}), a subsequence of (b_k) whose first p terms are $b_1, b_2, \dots, b_p$;

(z_{m_i}), a subsequence of (z_m) whose first p terms are $z_1, z_2, \dots, z_p$;

satisfying the following property:

For each $x \in K$, there exist vectors $a(x) \in A$ and $b_i(x) \in B$ $(i = 1, 2, \dots)$ such that

$$x = (T_{w_{n_i}}(b_i(x))) b_{k_i}^{-1} - (a(x) + T_{w_{n_i}}(z_{m_i})) b_{k_i}^{-1} \quad \textit{for each } i.$$

Proof. Let p be a natural number. Define the set $X(x; n_1, n_2, \dots, n_p)$ by the equation $X(x; n_1, n_2, \dots, n_p) = A \cap (T_{w_{n_1}}(B) - T_{w_{n_1}}(z_1) - b_1 x) \cap \dots \cap (T_{w_{n_p}}(B) - T_{w_{n_p}}(z_p) - b_p x)$.

Since a is a density point of A and b is a density point of B, there exist a positive number r such that $m(A \cap K_a)$ and $m(B \cap K_b)$ are both greater than $(1 - (100(p+2))^{-1}) m(K_a)$, where K_a is the open ball with center at a and radius r and K_b is the open ball with center at b and radius r.

By (ii) and (iii) there exists balls K with center at the origin and K_1 with center at c and a natural number N_p such that if $N_p < n_1 < n_2 \dots < n_p$;

$z_1, z_2, \ldots, z_p \in K_p$ and $x \in K$. Then the set $X(x : n_1, n_2, \ldots, n_p) \cap K_a \cap (B - c - \bar{b}x)$ has measure greater than $0.99\, m(K_a)$; where $\bar{b} = \lim_{k\to\infty} b_k$.

This implies, using (ii) and (iii), that there exist $n_{p+1} > n_p$ and k_{p+1} with $k_{p+1} > p$ and m_{p+1} with $m_{p+1} > p$ such that the set
$$X(x : n_1, n_2, \ldots, n_p) \cap K_a \cap (B - c - \bar{b}x) \cap (T_{w_{n_{p+1}}}(B) - T_{w_{n_{p+1}}}(z_{m_{p+1}}) - b_{k_{p+1}}x)$$
has measure greater than $0.99\, m(K_a)$ for each $x \in K$.

This process can be continued obtaining three sequences (n_i), a subsequence of the natural numbers whose first p terms are $n_1, n_2, \ldots, n_p$; (b_{k_i}), a subsequence of (b_k) whose first p terms are $b_1, b_2, \ldots, b_p$ and (z_{m_i}), a subsequence of (z_m) whose first p terms are $z_1, z_2, \ldots, z_p$ such that for each $x \in K$, the set
$$A \cap K_a \cap (B - c - \bar{b}x) \cap \bigcap_{i=1}^{\infty} (T_{w_{n_i}}(B) - T_{w_{n_i}}(z_{m_i}) - b_{k_i}x)$$
has measure greater or equal to $0.99\, m(K_a)$ and is hence non-empty. The remainder of the proof follows as in the proof of Theorem 1.

Corollary 2. *If W is taken to be the metric space of all non-singular n by n matrices and w_0 is taken to be the identity matrix then the conclusion of Theorem 2 holds. Here T_w is the linear transformation induced by the matrix w.*

Remark 2. Our Theorem 1 is a generalization of Theorem P which is a generalization of Theorem K. Our Theorem 2 is a generalization of Theorem 2 in [11] which is a generalization of Theorem 1 in [12].

Theorem 3 in [11] can not be extended as the following example shows.

Example. Let A denote the set of irrational numbers in the interval $(-1, 1)$ and for each positive integer n let B_n denote the interval $(-1/n, 1/n)$. Suppose K_0 is an open ball with center at the origin and $K_1, K_2, \ldots$ is a sequence of open intervals. Suppose that there exists a subsequence (n_l) of the natural numbers and a system of numbers $z_l \in K_l$ $(l = 1, 2, \ldots)$ such that for every $x \in K_0$, there exist numbers $a(x) \in A$, $b_{n_i}(x) \in B_{n_i}$ for every i such that
$$x = b_{n_i}(x) - (a(x) + z_i) \text{ for every } i.$$
This would imply that $x + a(x) = -\lim_{i\to\infty} z_i = c$, i.e.
$$x = c - a(x) \text{ and hence}$$
$K_0 \subset c - A$, which is a contradiction since A contains no intervals.

We now move on to Baire set analogues of our results. Our first theorem is a Baire set analogue of Theorem 1.

Theorem 3. *Let A be a second category subset of $\mathbf{R}^n$ having the Baire property, i.e. $A = (G \setminus P) \cup Q$ where G is open and nonempty and P, Q are first category sets. Suppose that $(b_k)_{k=1}^{\infty}$ is a convergent sequence of nonzero real numbers whose limit is nonzero. Suppose further that $(T_i)_{i=0}^{\infty}$ is a sequence of transformations of $\mathbf{R}^n$ into $\mathbf{R}^n$ and $x_0 \in G \setminus P$ satisfying the following properties.*

(1) There exists an $r_0 > 0$ such that $T_i = (f_{i1}, f_{i2}, \ldots, f_{in}) \in C'$ on $K(x_0, r_0)$, the open ball with center at x_0 and radius r_0, for each $i = 0, 1, 2, \ldots$, i.e. *each of the components f_{ik} of T_i has continuous first partial derivatives on $K(x_0, r_0)$.*

(2) For each $i = 0, 1, 2, \ldots$ the real valued function $h_i(z) = \det_{j,k} |D_j f_{ik}(z_k)|$ is nonzero, for each $z_1, z_2, \ldots, z_n \in K(x_0, r_0)$, where $z = (z_1; \ldots; z_n)$ is that point in $\mathbf{R}^{n^2}$ whose first n components are the components of z_1, whose next n components are the components of z_2, and so on (see [1], p. 113) *and D_j denotes the j^{th} partial derivative.*

(3) There exists an r, $0 < r < r_0$ such that $K(x_0, r) \subset G$, the sequence $(T_i)_{i=1}^{\infty}$ converges uniformly to T_0 on the sphere $S(S = x \in \mathbf{R}^n$: the distance between x and x_0 is $r)$ and $\lim_{i\to\infty} T_i(x_0) = T_0(x_0)$.

Then there exists a ball K' with center at the origin and a positive integer N such that for each $x \in K'$, there exists a sequence of points $(a_i)_{i=0}^{\infty}$, all in A, such that
$$x = (T_{N+i}(a_i) - T_0(a_0))/b_{N+i} \text{ for each } i = 1, 2, \ldots .$$

Proof. Define the set $X(x : M)$, where $x \in \mathbf{R}^n$ and M is a positive integer, by the equation
$$X(x : M) = T_0(A) \cap (T_{M+1}(A) - b_{M+1}x) \cap (T_{M+2}(A) - b_{M+2}x) \cap \ldots .$$

We now proceed to use standard Advanced Calculus theorems.

By the proof of Theorem 7—4 ([1], p. 143), each T_i, $i = 0, 1, 2, \ldots$ is one to one on $\bar{K}$, the closure of $K(x_0, r)$, where r is the number refered to in (3). This fact and (2) imply that the conditions of Theorem 7—3 ([1], p. 141) are satisfied for each T_i on $K = K(x_0, r)$, which is contained in G, and hence, as in the proof of Theorem 7—3
$$T_i(K) \supset K(T_i(x_0), m_i/2) \text{ for each } i = 0, 1, 2, \ldots$$
where m_i is the minimum of the function g_i on S (see (3)), where $g_i(x)$ is the distance between the points $T_i(x)$ and $T_i(x_0)$. By our hypotheses $m_i \to m_0$ and $T_i(x_0) \to T_0(x_0)$ as $i \to \infty$. By the proof of Theorem 7—3, $T_0(K) \supset K(T_0(x_0), m_0/2)$. Therefore there exists and N such that
$$T_i(K) \supset K(T_0(x_0), m_0/4) \text{ for all } i > N.$$
Since $(b_k)_{k=1}^{\infty}$ is bounded there exists a ball K' with center at the origin such that $T_i(K) - b_i x \supset K(T_0(x_0), m_0/8)$ for all $i > N$ and all $x \in K'$. Since each transformation T_i is a homeomorphism on K and A contains K less a first category set it follows that the set $T_i(A) - b_i x$ contains $K(T_0(x_0), m_0/8)$ less a first category set for every $x \in K'$ and every $i > N$.

Therefore for every $x \in K'$, $X(x : N)$ is a second category set and hence not empty. The remainder of the proof follows as in the proof of Theorem 1.

Remark 3. Many sequences $(T_i)_{i=0}^{\infty}$ of transformations satisfy conditions (1), (2) and (3) in Theorem 3 including sequences of matrix transformations satisfying obvious conditions (see the introduction). Notice that the conclusion of Theorem 3 is stronger than the conclusion of Theorem 1, i.e. we obtain a stronger conclusion in the Baire set case than we do in the measurable set case.

Our next result is a Baire set analogue of Theorem 2.

Theorem 4. *Suppose that A and B are second category subsets of $\mathbf{R}^n$ having the Baire property:* i.e. *$A = (G_1 \setminus P_1) \cup Q_1$ and $B = (G_2 \setminus P_2) \cup Q_2$ where*

G_1 and G_2 are open and nonempty and P_1, Q_1, P_2 and Q_2 are first category sets. Suppose that $a \in G_1$ and $b \in G_2$. Suppose that $(b_k)_{k=1}^{\infty}$ is a convergent sequence of nonzero reals whose limit is nonzero. Assume further that $(T_i)_{i=1}^{\infty}$ is a sequence of transformations of $\mathbf{R}^n$ into $\mathbf{R}^n$ satisfying the following properties.

(1) *$T_0(x)=x$ for every $x \in \mathbf{R}^n$.*

(2) *There exists $r_0>0$ such that $T_i=(f_{i1}, \ldots, f_{in}) \in C'$ on $K(b, r_0)$, for each $i=1, 2, \ldots$.*

(3) *For each $i=1, 2, \ldots$, the real valued function $h_i(z)=\det\limits_{j,k}[D_j f_{ik}(z_k)]$ is nonzero for each $z_1, \ldots, z_n \in K(b, r_0)$.*

(4) *There exists an r, $0<r<r_0$ such that $K(b, r) \subset G_2$ and the sequence $(T_i)_{i=1}^{\infty}$ converges uniformly to T_0 on the sphere $S=\{x \in \mathbf{R}^n$: the distance between x and b is $r\}$ and $\lim\limits_{i\to\infty} T_i(b)=T_0(b)=b$.*

(5) *There exists a ball $\bar{K}$ with center at c $(c=b-a)$ such that $(T_i)_{i=1}^{\infty}$ converges uniformly to T_0 on $\bar{K}$.*

Then there exists a positive integer N and balls K with center at the origin and K_1 with center at c such that for each $x \in K$ and sequence $(u_i)_{i=1}^{\infty}$ contained in K_1, there exists $a(x) \in A$ and $b_i(x) \in B$ $(i=1, 2, \ldots)$ such that

$$x=T_{N+i}(b_i(x))b_i^{-1}-(a(x)+T_{N+i}(u_i))b_i^{-1}$$

for $i=1, 2, \ldots$.

Proof. The proof of this theorem follows the lines of the last proof completely and hence will be omitted.

Remark 4. Our example, following Remark 2, shows that a Baire set analogue of Theorem 3 in [11] (which is easy to prove) can not be extended.

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Received 10. IX. 1988
Revised 20. I. 1989

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Matematica, 36 (1990)

NEW SUBCLASSES OF MEROMORPHIC CONVEX FUNCTIONS

BY

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1. Introduction. Let Σ denote the family of functions $f(z)=\frac{1}{z}+\sum\limits_{m=0}^{\infty} a_m z^m$ regular in the punctured disk $E=\{z : 0<|z|<1\}$ with a simple pole at $z=0$. The Hadamard product or convolution of two functions $f(z)$ and $g(z)$ in Σ will be denoted by $f * g$. Let

$$D^n f(z)=\frac{1}{z(1-z)^{n+1}} * f(z)=\frac{1}{z}\left(\frac{z^{n+1}f(z)}{n!}\right)^{(n)}=\frac{1}{z}+(n+1)a_0+ \tag{1}$$

$$+\frac{(n+1)(n+2)}{2!}a_1 z+\ldots, \; n=0, 1, 2, \ldots$$

By J_n we mean the subclass of Σ satisfying for every $f \in J_n$ the inequality

$$\operatorname{Re}((D^{n+1}f(z))'/(D^n f(z))'-2)<-n/(n+1) \text{ for } n=0, 1, 2, \ldots \tag{2}$$

and $z \in D=\{z : |z|<1\}$. U r a l e g a d d i and G a n i g i [4] introduced the classes J_n and showed the basic property $J_{n+1} \subset J_n$, $n=0, 1, 2, \ldots$. Thus the elements of J_n are meromorphically convex ($J_0=\Sigma_k$ the class of meromorphically convex functions). We wish to introduce some new classes.

Definitions. *Let $f(z)=\frac{1}{z}+\sum\limits_{n=0}^{\infty} a_n z^n$ be regular in the punctured disk* **E** *with a simple pole at $z=0$. If α and β are fixed real numbers and*

$$\operatorname{Re}\left(\frac{(D^{n+1}f(z))'}{(D^n f(z))'}-\frac{n+2}{n+1}\right)^{\alpha}\left(\frac{(D^{n+2}f(z))'}{(D^{n+1}f(z))'}-\frac{n+3}{n+2}\right)^{\beta}<0 \tag{3}$$

for $z \in D$, where the powers appearing in (3) are meant as principal values, then we say that $f(z) \in \Sigma_k(n, \alpha, \beta)$, $n=0, 1, 2, \ldots$.

This class of functions contains many interesting classes of meromorphic convex functions

$$\Sigma_k(n, 1, 0)=J_n, \; \Sigma_k(n, 0, 1)=J_{n+1} \text{ and } \Sigma_k(0, 1, 0)=J_0=\Sigma_k.$$

In this paper we will show that for certain real values of α and β, $\Sigma_k(n, \alpha, \beta) \subset J_n$. Similar problems were treated in [1] and [3].

2. The classes $\Sigma_k(n, \alpha, \beta)$. Let

$G_1=\{(\alpha, 0) \mid |\alpha|=1\} \cup \{(0, \beta) \mid |\beta|=1\}$,