G_1 and G_2 are open and nonempty and P_1, Q_1, P_2 and Q_2 are first category sets. Suppose that $a \in G_1$ and $b \in G_2$. Suppose that $(b_k)_{k=1}^{\infty}$ is a convergent sequence of nonzero reals whose limit is nonzero. Assume further that $(T_i)_{i=1}^{\infty}$ is a sequence of transformations of $\mathbf{R}^n$ into $\mathbf{R}^n$ satisfying the following properties.

(1) *$T_0(x) = x$ for every $x \in \mathbf{R}^n$.*

(2) *There exists $r_0 > 0$ such that $T_i = (f_{i1}, \ldots, f_{in}) \in C$ on $K(b, r_0)$, for each $i = 1, 2, \ldots$.*

(3) *For each $i = 1, 2, \ldots$, the real valued function $h_i(z) = \det_{j,k}[D_j f_{ik}(z_k)]$ is nonzero for each $z_1, \ldots, z_n \in K(b, r_0)$.*

(4) *There exists an r, $0 < r < r_0$ such that $K(b, r) \subset G_2$ and the sequence $(T_i)_{i=1}^{\infty}$ converges uniformly to T_0 on the sphere $S = \{x \in \mathbf{R}^n$: the distance between x and b is $r\}$ and $\lim_{i \to \infty} T_i(b) = T_0(b) = b$.*

(5) *There exists a ball $\bar{K}$ with center at c ($c = b - a$) such that $(T_i)_{i=1}^{\infty}$ converges uniformly to T_0 on $\bar{K}$.*

Then there exists a positive integer N and balls K with center at the origin and K_1 with center at c such that for each $x \in K$ and sequence $(u_i)_{i=1}^{\infty}$ contained in K_1, there exists $a(x) \in A$ and $b_i(x) \in B$ ($i = 1, 2, \ldots$) such that

$$x = T_{N+i}(b_i(x))b_i^{-1} - (a(x) + T_{N+1}(u_i))b_i^{-1}$$

for $i = 1, 2, \ldots$.

Proof. The proof of this theorem follows the lines of the last proof completely and hence will be omitted.

Remark 4. Our example, following Remark 2, shows that a Baire set analogue of Theorem 3 in [11] (which is easy to prove) can not be extended.

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NEW SUBCLASSES OF MEROMORPHIC CONVEX FUNCTIONS

BY

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1. Introduction. Let Σ denote the family of functions $f(z) = \frac{1}{z} + \sum_{m=0}^{\infty} a_m z^m$ regular in the punctured disk $E = \{z : 0 < |z| < 1\}$ with a simple pole at $z = 0$. The Hadamard product or convolution of two functions $f(z)$ and $g(z)$ in Σ will be denoted by $f * g$. Let

$$D^n f(z) = \frac{1}{z(1-z)^{n+1}} * f(z) = \frac{1}{z}\left(\frac{z^{n+1} f(z)}{n!}\right)^{(n)} = \frac{1}{z} + (n+1)a_0 + \tag{1}$$
$$+ \frac{(n+1)(n+2)}{2!} a_1 z + \ldots, \quad n = 0, 1, 2, \ldots$$

By J_n we mean the subclass of Σ satisfying for every $f \in J_n$ the inequality

$$\mathrm{Re}((D^{n+1} f(z))' / (D^n f(z))' - 2) < -n(n+1) \quad \text{for } n = 0, 1, 2, \ldots \tag{2}$$

and $z \in D = \{z : |z| < 1\}$. Uralegaddi and Ganigi [4] introduced the classes J_n and showed the basic property $J_{n+1} \subset J_n$, $n = 0, 1, 2, \ldots$. Thus the elements of J_n are meromorphically convex ($J_0 = \Sigma_k$ the class of meromorphically convex functions). We wish to introduce some new classes.

Definitions. *Let $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ be regular in the punctured disk* **E** *with a simple pole at $z = 0$. If α and β are fixed real numbers and*

$$\mathrm{Re}\left(\frac{(D^{n+1} f(z))'}{(D^n f(z))'} - \frac{n+2}{n+1}\right)^{\alpha} \left(\frac{(D^{n+2} f(z))'}{(D^{n+1} f(z))'} - \frac{n+3}{n+2}\right)^{\beta} < 0 \tag{3}$$

for $z \in D$, where the powers appearing in (3) are meant as principal values, then we say that $f(z) \in \Sigma_k(n, \alpha, \beta)$, $n = 0, 1, 2, \ldots$.

This class of functions contains many interesting classes of meromorphic convex functions

$$\Sigma_k(n, 1, 0) = J_n, \quad \Sigma_k(n, 0, 1) = J_{n+1} \quad \text{and} \quad \Sigma_k(0, 1, 0) = J_0 = \Sigma_k.$$

In this paper we will show that for certain real values of α and β, $\Sigma_k(n, \alpha, \beta) \subset J_n$. Similar problems were treated in [1] and [3].

2. The classes $\Sigma_k(n, \alpha, \beta)$. Let

$G_1 = \{(\alpha, 0) \mid |\alpha| = 1\} \cup \{(0, \beta) \mid |\beta| = 1\}$,

$$G_2=\{(\alpha,\beta)\mid(4k-1\le\alpha)\cap(\alpha+\beta\le 4k+1),\ \beta\ge 0,\ k\in I\},$$

$$G_3=\{(\alpha,\beta)\mid(4k-1\le\alpha+\beta)\cap(\alpha\le 4k+1),\ \beta\le 0,\ k\in I\},$$

$$G_4=\{(\alpha,\beta)\mid(4k-1\le\alpha-\beta)\cap(\alpha\le 4k+1),\ \beta\ge 0,\ k\in I\},$$

$$G_5=\{(\alpha,\beta)\mid(4k-1\le\alpha)\cap(\alpha-\beta\le 4k+1),\ \beta\le 0,\ k\in I\}$$

and

$G=G_1\cup(G_2\cap G_4)\cup(G_3\cap G_5)$, where I is the set of integers and α, β are real numbers. We now show that if $(\alpha,\beta)\in G$ and $f(z)\in\Sigma_k(n,\alpha,\beta)$ then $f(z)\in J_n$. We need the following lemma due to I.S. Jack [2].

Lemma. *Let w be non constant regular in $|z|<1$, $w(0)=0$. If $|w|$ attains its maximum value on the circle $|z|=r<1$ at z_0, we have $z_0w'(z_0)=kw(z_0)$ where $k\ge 1$.*

Theorem. *For $(\alpha,\beta)\in G$, $\Sigma_k(n,\alpha,\beta)\subset J_n$, $n=0,1,2,\ldots$.*

Proof. The case $(\alpha,\beta)\in G_1$ is trivial.

Suppose $f(z)\in\Sigma_k(n,\alpha,\beta)$. Define $w(z)$ in D by

$$(D^{n+1}f(z))'/(D^nf(z))'-2=-(1-(1-2n/(n+1))w(z))/(1+w(z)). \tag{4}$$

Clearly $w(z)$ is regular and $w(0)=0$, $w(z)\neq\pm 1$. Equation (4) may be written as

$$(D^{n+1}f(z))'/(D^nf(z))'=((n+1)+(n+3)w(z))/((n+1)(1+w(z))). \tag{5}$$

To complete the proof it suffices to show that $|w(z)|<1$ in D and $(\alpha,\beta)\in G$. Differentiating (5) logarithmically and using the identity

$$z(D^nf(z))'=(n+1)D^{n+1}f(z)-(n+2)D^nf(z) \tag{6}$$

we obtain

$$\frac{(D^{n+2}f(z))'}{(D^{n+1}f(z))'}-\frac{n+3}{n+2}=\frac{1}{n+2}\left(\frac{w(z)-1}{w(z)+1}+\frac{2zw'(z)}{(1+w(z))(n+1+(n+3)w(z))}\right). \tag{7}$$

Substituting (5) and (7) in (3) we obtain

$$\begin{aligned}\operatorname{Re}P_n(f(z),\alpha,\beta)&=\operatorname{Re}\left\{\left(\frac{(D^{n+1}f(z))'}{(D^nf(z))'}-\frac{n+2}{n+1}\right)^{\alpha}\left(\frac{(D^{n+2}f(z))'}{(D^{n+1}f(z))'}-\frac{n+3}{n+2}\right)^{\beta}\right\}=\\&=(n+1)^{-\alpha}(n+2)^{-\beta}\operatorname{Re}\left(\frac{w(z)-1}{w(z)+1}\right)^{\alpha}\left(\frac{w(z)-1}{w(z)+1}+\frac{2zw'(z)}{(1+w(z))(n+1+(n+3)w(z))}\right)^{\beta}.\end{aligned} \tag{8}$$

Now suppose on the contrary that there exists $z_0\in D$ such that $\max_{|z|\le|z_0|}|w(z)|=|w(z_0)|=1$, $w(z_0)\neq\pm 1$.

Then by the above lemma, $z_0w'(z_0)/(1+w(z_0))=kw(z_0)/(1+w(z_0))$.

Let $w(z_0)=e^{i\theta_0}$. Then $(w(z_0)-1)/(w(z_0)+1)=i\sin\theta_0(1+\cos\theta_0)=\lambda i$

and (8) becomes

$$\begin{aligned}&P_n(f(z_0),\alpha,\beta)=C(\lambda)^{\alpha}\ (x+i\lambda y)^{\beta},\ \text{where } C=(n+1)^{-\alpha}(n+2)^{-\beta}>0,\\&x=k(n+2)(1+\lambda^2)/(2((n+2)^2+\lambda^2))\end{aligned} \tag{9}$$

and

$$y=(2(n+2)^2+\lambda^2)-k(1+\lambda^2))/(2((n+2)^2+\lambda^2)),$$

$$\operatorname{Re}P_n(f(z_0),\alpha,\beta)=C\lambda^{\alpha}(x^2+y^2\lambda^2)^{\beta/2}\cos(\alpha\theta_1+\beta\theta_2),$$

where $\theta_1=\arg\lambda i$, $\theta_2=\arg(x+i\lambda y)$.

Case 1 : $\lambda>0$. (a). Let $y>0$, $\theta_1=\pi/2$, $0\le\theta_2\le\pi/2$; if $(\alpha,\beta)\in G_2$, then

$$(4k-1)\pi/2\le\alpha\pi/2\le\alpha\theta_1+\beta\theta_2\le(\alpha+\beta)\pi/2\le(4k+1)\pi/2$$

and if $(\alpha,\beta)\in G_3$, then

$$(4k-1)\pi/2\le(\alpha+\beta)\pi/2\le\alpha\theta_1+\beta\theta_2\le\alpha\pi/2\le(4k+1)\pi/2.$$

Thus if $(\alpha,\beta)\in G_2$ or G_3 then $\cos(\alpha\theta_1+\beta\theta_2)\ge 0$.

This shows that $\operatorname{Re}P_n(f(z_0),\alpha,\beta)\ge 0$ which contradicts $f\in\Sigma_k(n,\alpha,\beta)$.

(b) Let $y<0$, $\theta_1=\pi/2$, $-\pi/2\le\theta_2\le 0$; if $(\alpha,\beta)\in G_4$, then

$$(4k-1)\pi/2\le(\alpha-\beta)\pi/2\le\alpha\theta_1+\beta\theta_2\le\pi/2\le(4k+1)\pi/2$$

and if $(\alpha,\beta)\in G_5$, then

$$(4k-1)\pi/2\le\alpha\pi/2\le\alpha\theta_1+\beta\theta_2\le(\alpha-\beta)\pi/2\le(4k+1)\pi/2,$$

which leads to the same contradiction.

Case 2. $\lambda<0$. Let $\theta_3=\arg\lambda i$ and $\theta_4=\arg(x+i\lambda y)$. Then $\theta_3=-\theta_1$ and $\theta_4=-\theta_2$, $\cos(\alpha\theta_1+\beta\theta_2)\ge 0$, contradiction. This completes the proof of the theorem.

Remark. Since $\Sigma_k(n,0,1)=J_{n+1}$ the above theorem shows the basic inclusion relationship of Uralegaddi and Ganigi [4].

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