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A NOTE ON CERTAIN GENERALIZATIONS OF OPIAL'S INEQUALITY

BY

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1. Introduction. The classical integral inequality

$$\int_a^b |u(x)u'(x)|\,dx \le \frac{b-a}{4}\int_a^b |u'(x)|^2 dx \tag{1}$$

valid for continuously differentiable function u on $[a, b]$ with $u(a)=u(b)=0$ and $u(x)>0$ for $x\in(a, b)$ was proved in 1960 by Z. Opial [6]. In [5] C. Olech gave a simple proof of inequality (1) and showed that (1) is valid for any absolutely continuous function u on $[a, b]$ with $u(a)=u(b)=0$. In the year 1967, J. Calvert [1] and E.K. Godunova and V.I. Levin [2] proved some interesting generalizations of a slight variant of Opial's inequality (1). For other generalizations of Opial's inequality (1), see [4, pp. 154–162] and the references given therein. In the present note we establish some generalizations of Opial's inequality given in (1) which in turn are the variants of the inequalities established by Calvert in [1, Theorem 1] and Godunova and Levin in [4, Theorems 12 and 13, p. 159]. The method we use to establish our results is elementary and based on the idea used by Olech [5] to prove the Opial's inequality given in (1).

2. Statement of Results. In this section we state our results on the generalizations of Opial's inequality given in (1). For the properties of convex functions involved in the formulation of our Theorems 2 and 3 below, see [3, 4].

Theorem 1. *Let f_r $(r=1, \dots, m)$ be absolutely continuous functions defined on (a, b) with $f_r(a)=f_r(b)=0$ where $-\infty\le a<b\le+\infty$. Let $g_r(u)$ $(r=1, \dots, m)$ be continuous functions defined for all u in the range of f_r and for all real u of the form $u(s)=\int_a^s |f_r'(x)|\,dx$ or $u(s)=\int_s^b |f_r'(x)|\,dx$. Suppose that $|g_r(u)|\le g_r(|u|)$ for all u and $g_r(u_1)\le g_r(u_2)$ for $0\le u_1\le u_2$. If $G_r(u)=\int_0^u g_r(t)dt$, for $u>0$ and $r=1, \dots, m$, then*

$$\int_a^b\left\{\prod_{r=1}^m |g_r(f_r(x))f_r'(x)|\right\}^{1/m} dx\le\frac{1}{m}\sum_{r=1}^m\left[G_r\left(\int_a^c |f_r'(x)|\,dx\right)+G_r\left(\int_c^b |f_r'(x)|\,dx\right)\right], \tag{2}$$

for every $c\in[a, b]$.

Remark 1. It is interesting to note that in the special case when $m=1$, inequality (2) reduces to

$$\int_a^b |g_1(f_1(x))f_1'(x)|\,dx \le G_1\Big(\int_a^c |f_1'(x)|\,dx\Big)+G_1\Big(\int_c^b |f_1'(x)|\,dx\Big), \tag{3}$$

for $c\in[a, b]$. The inequality (3) is a variant of the inequality established by Calvert in [1, Theorem 1]. On taking $g_1(v)=v$ and hence $G_1(u)=\int_0^u v\,dv=u^2/2$ and $c=(a+b)/2$ in (3) and using the Schwarz inequality on the right side, we obtain the Opial's inequality given in (1).

Theorem 2. *Let f_r $(r=1, \ldots, m)$ be absolutely continuous functions on $[a, b]$ with $f_r(a)=f_r(b)=0$. If $h_r(u)$ $(r=1, \ldots, m)$ be nonnegative convex increasing functions for $u>0$ and $h_r(0)=0$, then*

$$\int_a^b \Big\{\prod_{r=1}^m h_r'(|f_r(x)|)|f_r'(x)|\Big\}^{1/m} dx \le \frac{1}{m}\sum_{r=1}^m \Big[h_r\Big(\int_a^c |f_r'(x)|\,dx\Big)+h_r\Big(\int_c^b |f_r'(x)|\,dx\Big)\Big], \tag{4}$$

for $c\in[a, b]$.

Remark 2. In the special case when $m=1$, inequality (4) reduces to

$$\int_a^b h_1'(|f_1(x)|)|f_1'(x)|\,dx \le h_1\Big(\int_a^c |f_1'(x)|\,dx\Big)+h_1\Big(\int_c^b |f_1'(x)|\,dx\Big). \tag{5}$$

We note that the inequality (5) is a slight variant of the inequality established by Godunova and Levin in [4, Theorem 13, p. 159]. By taking $h_1(v)=v^2$ and hence $h_1'(v)=2v$ and $c=(a+b)/2$ in (5) and using the Schwarz inequality on the right side, we obtain the Opial's inequality given in (1).

A slightly different version of inequality (4) is embodied in the following theorem.

Theorem 3. *Let f_r, h_r be as in Theorem 2. Let $p_r(r=1, \ldots, m)$ be positive continuous functions defined on $[a, b]$ and their integrals on $[a, b]$ exist and are positive. If $\psi_r(u)$ $(r=1, \ldots, m)$ are nonnegative convex increasing functions for $u>0$ and $\psi_r(0)=0$, then*

$$\int_a^b \Big\{\prod_{r=1}^m h_r'(|f_r(x)|)|f_r'(x)|\Big\}^{1/m} dx \le$$

$$\le \frac{1}{m}\sum_{r=1}^m \Big[h_r\Big(\Big(\int_a^c p_r(x)dx\Big)\psi_r^{-1}\Big[\int_a^c p_r(x)\psi_r\Big(\frac{|f_r'(x)|}{p_r(x)}\Big)dx\Big/\int_a^c p_r(x)dx\Big]\Big)+ \tag{6}$$

$$+h_r\Big(\Big(\int_c^b p_r(x)dx\Big)\psi_r^{-1}\Big[\int_c^b p_r(x)\psi_r\Big(\frac{|f_r'(x)|}{p_r(x)}\Big)dx\Big/\int_c^b p_r(x)dx\Big]\Big)\Big].$$

for every $c\in[a, b]$.

Remark 3. We note that in the special case when $m=1$, the inequality (6) reduces to a variant of the inequality established by Godunova and Levin in [4, Theorem 12, p. 159]. Further, we note that by specializing the inequality (6) it is easy to obtain the Opial's inequality given in (1).

3. Proofs of theorems 1–3. Let $c\in[a, b]$ and define

$$u_r(x)=\int_a^x |f_r'(t)|\,dt, \tag{7}$$

for $a\le x\le c$ and $r=1, \ldots, m$ and

$$v_r(x)=\int_x^b |f_r'(t)|dt, \tag{8}$$

for $c\le x\le b$ and $r=1, \ldots, m$. From (7) and (8) we have

$$u_r'(x)=|f_r'(x)|, \tag{9}$$

for $a\le x\le c$ and $r=1, \ldots, m$ and

$$v_r'(x)=-|f_r'(x)|, \tag{10}$$

for $c\le x\le b$ and $r=1, \ldots, m$. We note that

$$f_r(x)=\int_a^x f_r'(t)dt, \tag{11}$$

for $a\le x\le c$ and $r=1, \ldots, m$ and

$$f_r(x)=-\int_x^b f_r'(t)dt, \tag{12}$$

for $c\le x\le b$ and $r=1, \ldots, m$.

From the hypotheses of Theorem 1, the elementary inequality

$$\Big\{\prod_{r=1}^m c_r\Big\}^{1/m} \le \frac{1}{m}\sum_{r=1}^m c_r, \tag{13}$$

for $c_1, \ldots, c_m\ge 0$ reals and (11), (9), (7) we observe that

$$\int_a^c \Big\{\prod_{r=1}^m |g_r(f_r(x))f_r'(x)|\Big\}^{1/m} dx \le \frac{1}{m}\sum_{r=1}^m \int_a^c |g_r(f_r(x))f_r'(x)|\,dx \le$$

$$\le \frac{1}{m}\sum_{r=1}^m \int_a^c g_r\Big(\int_a^x |f_r'(t)|\,dt\Big)|f_r'(x)|\,dx = \frac{1}{m}\sum_{r=1}^m \int_a^c g_r(u_r(x))u_r'(x)dx = \tag{14}$$

$$= \frac{1}{m}\sum_{r=1}^m \int_a^{u_r(c)} g_r(t)dt = \frac{1}{m}\sum_{r=1}^m G_r(u_r(c)) = \frac{1}{m}\sum_{r=1}^m G_r\Big(\int_a^c |f_r'(x)|\,dx\Big).$$

Similarly, from the hypotheses of Theorems 1, the inequality (13) and (12), (10), (8) we have

$$(15)\qquad \int_c^b \left\{ \prod_{r=1}^m |g_r(f_r(x)) f_r'(x)| \right\}^{1/m} dx \leq \frac{1}{m} \sum_{r=1}^m G_r \left(\int_c^b |f_r'(x)|\, dx \right).$$

The desired inequality in (2) follows from (14) and (15) and the proof of Theorem 1 is complete.

The proof of Theorem 2 is very close to that of Theorem 1 with suitable modifications and hence we omit the details.

In order to prove Theorem 3, we first observe that

$$(16)\qquad \int_a^c |f_r'(x)|\, dx \Big/ \int_a^c p_r(x) dx = \int_a^c \frac{p_r(x)\, |f_r'(x)|}{p_r(x)}\, dx \Big/ \int_a^c p_r(x) dx,$$

for $c \in [a, b]$ and $r = 1, \ldots, m$. Since ψ_r are convex, we obtain from (16) and using a suitable version of Jensen's inequality (see [3, p. 133])

$$(17)\qquad \psi_r\left(\int_a^c |f_r'(x)|\, dx \Big/ \int_a^c p_r(x) dx \right) \leq \int_a^c p_r(x) \psi_r\left(\frac{|f_r'(x)|}{p_r(x)} \right) dx \Big/ \int_a^c p_r(x) dx,$$

for $c \in [a, b]$ and $r = 1, \ldots, m$. From (17) we observe that

$$(18)\qquad \int_a^c |f_r'(x)|\, dx \leq \left(\int_a^c p_r(x) dx \right) \psi_r^{-1}\left(\int_a^c p_r(x) \psi_r\left(\frac{|f_r'(x)|}{p_r(x)} \right) dx \Big/ \int_a^c p_r(x) dx \right),$$

for $c \in [a, b]$ and $r = 1, \ldots, m$. Similarly we obtain

$$(19)\qquad \int_c^b |f_r'(x)|\, dx \leq \left(\int_c^b p_r(x) dx \right) \psi_r^{-1}\left(\int_c^b p_r(x) \psi_r\left(\frac{|f_r'(x)|}{p_r(x)} \right) dx \Big/ \int_c^b p_r(x) dx \right),$$

for $c \in [a, b]$ and $r = 1, \ldots, m$. Since all the hypotheses of Theorem 2 are satisfied by Theorem 3, we have the inequality (4). Using (18) and (19) in (4) we obtain the required inequality in (6). This completes the proof of Theorem 3.

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NONOSCILLATORY SOLUTIONS OF SECOND ORDER FORCED FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. Introduction. The study of the behaviour of the nonoscillatory solutions of forced functional differential equations begins with the paper of Singh and Dahiya [1] for the equation

$$(1)\qquad (r(t)y'(t))' + a(t)y(t - \tau(t)) = Q(t),\ 0 \leq \tau(t) \leq M,\ M = \text{const.}$$

which is also the object of the investigation of [2]. The nonlinear equation

$$(2)\qquad (r(t)y'(t))' + a(t)h(y(g(t))) = Q(t)$$

is discussed in [3]—[6] and a more general equation

$$(3)\qquad (r(t)y'(t))' + p(t)y'(t) + q(t)y(t) + a(t)h(y(g(t))) = Q(t)$$

in [7], [8]. The paper [9] pertains to the equation

$$(4)\qquad (r(t)y'(t))' + p(t)\varphi(y(t))\psi(y'(t)) + a(t)h(y(g(t))) = Q(t)$$

and [10] to the particular case of (4) when $p(t) = 0$.

In most of papers cited it is assumed that the function containing the deviation is multipled by some positive function of t. This hypothesis will not be required in our paper and the main results will be obtained for a deviating argument which depends on the unknown function and which may be of a retarded, advanced or of mixed type.

This paper is devoted to the study of the equation

$$(5)\qquad (r(t)y'(t))' + f(t, y(\Delta(t, y(t)))) = Q(t) \text{ for } t \geq T > 0$$

in the cases when

$$(6)\qquad \int_T^\infty \frac{dt}{r(t)} < \infty$$

and

$$(7)\qquad \int_T^\infty \frac{dt}{r(t)} = \infty .$$