$$(15)\qquad \int_c^b \left\{\prod_{r=1}^{m} |g_r(f_r(x))f_r'(x)|\right\}^{1/m} dx \le \frac{1}{m}\sum_{r=1}^{m} G_r\left(\int_c^b |f_r'(x)|\,dx\right).$$

The desired inequality in (2) follows from (14) and (15) and the proof of Theorem 1 is complete.

The proof of Theorem 2 is very close to that of Theorem 1 with suitable modifications and hence we omit the details.

In order to prove Theorem 3, we first observe that

$$(16)\qquad \int_a^c |f_r'(x)|\,dx \Big/ \int_a^c p_r(x)dx = \int_a^c \frac{p_r(x)\,|f_r'(x)|}{p_r(x)}\,dx \Big/ \int_a^c p_r(x)dx,$$

for $c \in [a, b]$ and $r = 1, \ldots, m$. Since ψ_r are convex, we obtain from (16) and using a suitable version of Jensen's inequality (see [3, p. 133])

$$(17)\qquad \psi_r\left(\int_a^c |f_r'(x)|dx \Big/ \int_a^c p_r(x)dx\right) \le \int_a^c p_r(x)\psi_r\left(\frac{|f_r'(x)|}{p_r(x)}\right)dx \Big/ \int_a^c p_r(x)dx,$$

for $c \in [a, b]$ and $r = 1, \ldots, m$. From (17) we observe that

$$(18)\qquad \int_a^c |f_r'(x)|\,dx \le \left(\int_a^c p_r(x)dx\right) \psi_r^{-1}\left(\int_a^c p_r(x)\psi_r\left(\frac{|f_r'(x)|}{p_r(x)}\right)dx \Big/ \int_a^c p_r(x)dx\right),$$

for $c \in [a, b]$ and $r = 1, \ldots, m$. Similarly we obtain

$$(19)\qquad \int_c^b |f_r'(x)|\,dx \le \left(\int_c^b p_r(x)dx\right) \psi_r^{-1}\left(\int_c^b p_r(x)\psi_r\left(\frac{|f_r'(x)|}{p_r(x)}\right)dx \Big/ \int_c^b p_r(x)dx\right),$$

for $c \in [a, b]$ and $r = 1, \ldots, m$. Since all the hypotheses of Theorem 2 are satisfied by Theorem 3, we have the inequality (4). Using (18) and (19) in (4) we obtain the required inequality in (6). This completes the proof of Theorem 3.

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NONOSCILLATORY SOLUTIONS OF SECOND ORDER FORCED FUNCTIONAL DIFFERENTIAL EQUATIONS

BY

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1. Introduction. The study of the behaviour of the nonoscillatory solutions of forced functional differential equations begins with the paper of Singh and Dahiya [1] for the equation

$$(1)\qquad (r(t)y'(t))' + a(t)y(t-\tau(t)) = Q(t),\quad 0 \le \tau(t) \le M,\quad M = \text{const}.$$

which is also the object of the investigation of [2]. The nonlinear equation

$$(2)\qquad (r(t)y'(t))' + a(t)h(y(g(t))) = Q(t)$$

is discussed in [3]—[6] and a more general equation

$$(3)\qquad (r(t)y'(t))' + p(t)y'(t) + q(t)y(t) + a(t)h(y(g(t))) = Q(t)$$

in [7], [8]. The paper [9] pertains to the equation

$$(4)\qquad (r(t)y'(t))' + p(t)\varphi(y(t))\psi(y'(t)) + a(t)h(y(g(t))) = Q(t)$$

and [10] to the particular case of (1) when $p(t) = 0$.

In most of papers cited it is assumed that the function containing the deviation is multipled by some positive function of t. This hypothesis will not be required in our paper and the main results will be obtained for a deviating argument which depends on the unknown function and which may be of a retarded, advanced or of mixed type.

This paper is devoted to the study of the equation

$$(5)\qquad (r(t)y'(t))' + f(t, y(\Delta(t, y(t)))) = Q(t)\ \text{ for } t \ge T > 0$$

in the cases when

$$(6)\qquad \int_T^\infty \frac{dt}{r(t)} < \infty$$

and

$$(7)\qquad \int_T^\infty \frac{dt}{r(t)} = \infty.$$

Our main purpose is to establish sufficient conditions for nonoscillation, positiveness (negativeness) and unboundedness above (unboundedness below) of all solutions of (5) and asymptotic decay of bounded nonoscillatory solutions of (5). We shall restrict our attention to solutions $y(t)$ of (5) which exist on some ray $[T_y, \infty)$, and satisfy $\sup_{t \geq T_*} |y(t)| > 0$ for any $T_* \geq T_y \geq T$ where the number T_y depends on the particular solution $y(t)$.

We call a solution $y(t)$ of equation (5) oscillatory if it has arbitrarily large zeros. Otherwise we call it nonoscillatory.

The entire study in this work is object to the assumptions (A)

A1. $r(t) \in C([T, \infty); (0, \infty))$.

A2. $f(t, u) \in C([T, \infty) \times R; R)$, $uf(t, u) > 0$ for $u \neq 0$, $t \geq T$ and $f(t, .)$ is nondecreasing.

A3. $\Delta(t, u) \in C([T, \infty) \times R; R)$, $\Delta(t, u) \to \infty$ as $t \to \infty$ for any fixed $u \in R$ and $\Delta(t, u) \leq \Delta(t, \bar{u})$ for $|u| \leq |\bar{u}|$ and $t \geq T$.

A4. $Q(t) \in C([T, \infty); R)$.

Here $R = (-\infty, \infty)$. Introduce the notation $R(t) = \int_T^t \frac{ds}{r(s)}$, $\rho(t) = \int_t^\infty \frac{ds}{r(s)}$.

We need the following lemma which is a generalization of the lemma of Kusano and Onose [4].

Lemma 1. *Let conditions (A) hold. Then*

1. *All oscillatory solutions of (5) are unbounded above if $Q(t) > 0$ and*

$$\int_T^\infty [Q(s) - f(s, c)]\, ds = \infty \quad \text{for any } c > 0. \tag{8}$$

2. *All oscillatory solutions of (5) are unbounded below if $Q(t) < 0$ and*

$$\int_T^\infty [Q(s) - f(s, -c)]\, ds = -\infty \quad \text{for any } c > 0. \tag{9}$$

Proof. It suffices to prove Statement 1 (Statement 2 proves similarly). Suppose that there exists an oscillatory solution $y(t)$ of (5) such that $y(t) \leq M$ for some $M = \text{const.} > 0$ and $t \geq T_1 \geq T$. Denote by $T_2 \leq T_1$ a zero of $y(t)$ such that $\Delta(t, y(t)) \geq T_1$ for $t \geq T_2$. Since $y(t)$ oscillates, one of the following cases is possible

(i) There exists at least two consecutive zeros τ_ν and $\tau_{\nu+1}$ of $y(t)$ greater than T_2 such that $y(t) > 0$ for $t \in (\tau_\nu, \tau_{\nu+1})$;

(ii) $y(t) \leq 0$ for $t \geq T_3 \geq T_2$.

In case (i) we have

$$y(\Delta(t, y(t))) \leq M \text{ and } f(t, y(\Delta(t, y(t)))) \leq f(t, M) \tag{10}$$

for $t \in (\tau_\nu, \tau_{\nu+1})$. Let $\xi_\nu \in (\tau_\nu, \tau_{\nu+1})$ and $y'(\xi_\nu) = 0$. Integrating (5) from ξ_ν to $t \in [\xi_\nu, \tau_{\nu+1}]$ and using (10) we obtain

$$r(t)y'(t) = \int_{\xi_\nu}^t Q(s)d - \int_{\xi_\nu}^t f(s, y(\Delta(s, y(s))))d \geq \int_{\xi_\nu}^t [Q(s) - f(s, M)]\, ds.$$

Letting $t \to \infty$ and using (8) and A1 we see that $y'(t) > 0$ for $t \geq T_3 \geq \xi_\nu$ which is impossible for the oscillatory function $y(t)$.

In case (ii) using A3 and A2 we obtain that $f(y(\Delta(t, y(t)))) \leq 0$ for $t \geq T_4 \geq T_3$. Then (5) yields $(r(t)y'(t))' \geq 0$ for $t \geq T_4$ since $Q(t) > 0$ for $t \geq T$. Thus, $y'(t)$ has a constant sign wich is a contradiction. Lemma 1 is proved.

2. Main results. Sufficient conditions for nonoscillation of all solutions of retarded equation (2) are obtained in [1], [5] (for $|Q(t)| > 0$) and in [6] ($Q(t)$ is not assumed to be with a constant sign) when (6) holds and in [1] when (7) holds. Our theorem generalizes the results from [1] and [5].

Theorem 1. *Let the following conditions hold*

1. *Conditions (A) are fulfilled, there exists $\lim_{u \to \infty} \frac{f(t, u)}{u} = \varphi(t)$ uniformly on t, $\Delta(t, v) \leq t$ for any fixed $v \in R$ and either $Q(t) > 0$ and (8) holds or $Q(t) < 0$ and (9) holds.*

2. *One of the following conditions is fulfilled*

(i) *(6) holds and* $\int_T^\infty \varphi(t)dt < \infty$;

(ii) *(7) holds and* $\int_T^\infty R(t)\varphi(t)dt < \infty$.

Then all solutions of (5) are nonoscillatory.

Proof. Consider the case when $Q(t) > 0$ and (8) holds. (The case when $Q(t) < 0$ and (9) holds treats similarly). Suppose that there exists an oscillatory solution $y(t)$ of (5). By Lemma 1,

$$\limsup_{t \to \infty} y(t) = \infty.$$

Take $T_1 \geq T$ so large that

$$\int_{T_1}^\infty \frac{dt}{r(t)} < 1, \quad \int_{T_1}^\infty \varphi(t)dt < \frac{1}{2}, \tag{11}$$

when (6) holds, and

$$\int_{T_1}^\infty R(t)\varphi(t)dt < \frac{1}{2} \tag{12}$$

when (7) holds. Let $T_2 \geq T_1$ be a zero of $y(t)$ such that $\Delta(t, y(t)) \geq T_1$ for $t \geq T_2$. In view of the unboundedness above of $y(t)$ we can find two sequences $\{\tau_\nu\}_{\nu=1}^\infty \subset [T_2, \infty)$ and $\{\xi_\nu\}_{\nu=1}^\infty \subset (\tau_1, \infty)$ with the properties: $\tau_\nu \to \infty$, $\xi_\nu \to \infty$, $\tau_\nu < \xi_\nu < \tau_{\nu+1}$, $y(\tau_\nu) = 0$, $y'(\xi_\nu) = 0$ and if $M_\nu = y(\xi_\nu) = \sup_{[T_2, \xi_\nu]} y(t)$ then $M_{\nu+1} \geq M_\nu$ and $M_\nu \to \infty$. It is clear that (10) holds for $t \in [\tau_\nu, \tau_{\nu+1}]$ and $M = M_\nu$.

In view of the positiveness of $Q(t)$ (5) yields

$$(r(t)y'(t))' \geq -f(t, y(\Delta(t, y(t)))) \quad \text{for } t \geq T. \tag{13}$$

Integrating (13) from $t \in (\tau_\nu, \xi_\nu)$ to ξ_ν and using (10) we have

(14) $$r(t)y'(t)\le \int_t^{\xi_\nu} f(s, y(\Delta(s, y(s))))ds \le \int_t^{\xi_\nu} f(s, M_\nu)ds.$$

Dividing (14) by $r(t)$ and integrating from τ_ν to ξ_ν we get

(15) $$M_\nu = y(\xi_\nu) \le \int_{\tau_\nu}^{\xi_\nu} \frac{1}{r(t)} \int_t^{\xi_\nu} f(s, M_\nu)ds dt = \int_{\tau_\nu}^{\xi_\nu}\left(\int_{\tau_\nu}^{t} \frac{ds}{r(s)}\right) f(t, M_\nu)dt \le$$
$$\le \int_{T_1}^{\infty}\left(\int_{T_1}^{t} \frac{ds}{r(s)}\right) f(t, M_\nu)dt.$$

Dividing (15) by M_ν, letting $\nu\to\infty$ and having in mind (11) when (6) holds and (12) when (7) holds we obtain the absurd $1<1/2$. Theorem 1 is proved.

Theorem 2. *In addition to conditions of Theorem 1 suppose either*

(i) (6) *holds*, $\lim_{t\to\infty} r(t)=\infty$ *and* $\lim_{t\to\infty} \frac{\int_T^t |Q(s)|ds}{r(t)} = k = \text{const}>0$

or

(ii) (7) *holds and* $\int_T^\infty |Q(t)|dt=\infty$.

Then $y(t)Q(t)>0$ *for any solution* $y(t)$ *of* (5).

Proof. Let $Q(t)>0$ for $t\ge T$. (The proof is similar when $Q(t)<0$ for $t\ge T$). By Theorem 1 all solutions of (5) are nonoscillatory. Suppose that there exists a negative solution $y(t)$ of (5) for $t\ge T_1\ge T$. In view of A3 and A2 we conclude that $f(t, y(\Delta(t, y(t))))<0$ for $t\ge T_2\ge T_1$. Equation (5) yields the inequality

(16) $$(r(t)y'(t))'\ge Q(t) \text{ for } t\ge T_2.$$

Let (i) be valid. Integrating (16) from T_2 to t and dividing by $r(t)$ we get

$$y'(t)\ge r(T_2)y'(T_2)/r(t)+\int_{T_2}^t Q(s)ds/r(t).$$

Via (i) we can find $T_3\ge T_2$ so large that $\int_{T_2}^t Q(s)ds/r(t)>k-\varepsilon$ and $r(T_2)y'(T_2)/r(t)>-\varepsilon$ for any $\varepsilon\in(0, k/2)$. Then $y'(t)>k-2\varepsilon>0$ for $t\ge T_3$ and hence, $y(t)\xrightarrow[t\to\infty]{}\infty$ which is a contradiction.

Let (ii) be valid. Integrating (16) from T_2 to t and tending $t\to\infty$ we obtain

$$r(t)y'(t)\ge r(T_2)y'(T_2)+\int_{T_2}^t Q(s)ds\xrightarrow[t\to\infty]{}\infty.$$

i.e. $r(t)y'(t)\ge l$ for $t\ge T_3\ge T_2$ and some $l=\text{const}>0$. Dividing the last inequality by $r(t)$, integrating from T_3 to t, tending $t\to\infty$ and using (7) we obtain the contradiction

$$y(t)\ge y(T_3)+\int_{T_3}^t \frac{l}{r(s)}ds\xrightarrow[t\to\infty]{}\infty.$$

Theorem 2 is proved. We note that Theorem 2 generalizes Theorem (4.2) and Corollary (4.2) from [5] when (6) holds. Theorems of this type are not proved in the case when (7) holds.

Theorem 3. *In addition to conditions of Theorem 2 suppose either*

(i) (6) *holds and* $\lim_{t\to\infty}\int_{T_*}^t [Q(s)-f(s, c)]ds/r(t)=k^+>0$ *when* $Q(t)>0$ *and* $\lim_{t\to\infty}\int_{T_*}^t [Q(s)-f(s, -c)]ds/r(t)=k^-<0$ *when* $Q(t)<0$ *for any* $c>0$ *and* $T_*\ge T$ *and some* k^+, $k^-=\text{const}$.

or

(ii) (7) *holds and* $\int_T^\infty [Q(t)-f(t, c)]dt=\infty$ *when* $Q(t)>0$ *and* $\int_T^\infty [Q(t)-f(t, -c)]dt=-\infty$ *when* $Q(t)<0$ *for any* $c>0$.

Then all solutions of (5) *are positive and unbounded above when* $Q(t)>0$ *and negative and unbounded below when* $Q(t)<0$.

Proof. Let $Q(t)>0$ for $t\ge T$ (The proof is similar when $Q(t)<0$ for $t\ge T$). By Theorem 2 all solutions of (5) are positive for $T\ge T_1\ge T$. Suppose that there exists a bounded solution $y(t)$ of (5) and $y(t)\le M$ for some $M=\text{const}>0$ and $t\ge T_2\ge T_1$. From A3 and A2 it follows that $y(\Delta(t, y(t)))\le M$ and (10) holds for $t\ge T_3\ge T_2$.

Let (i) be valid. Integrating (5) from T_3 to t, deviding by $r(t)$ and using (10) we obtain

$$y'(t)\ge r(T_3)y'(T_3)/r(t)+\int_{T_3}^t [Q(s)-f(s, M)]ds/r(t).$$

Via (i) we can find $T_4\ge T_3$ so large that $\int_{T_3}^t [Q(s)-f(s, M)]ds/r(t)>k^+-\varepsilon$ and $r(T_3)y'(T_3)/r(t)>-\varepsilon$ for any $\varepsilon\in(0, k^+/2)$ and $t\ge T_4$. Then $y'(t)>k^+-2\varepsilon>0$ for $t\ge T_4$ and hence $y(t)\xrightarrow[t\to\infty]{}\infty$ which is a contradiction.

Let (ii) be valid. Integrating (5) from T_3 to t, using (10) and tending $t\to\infty$ we obtain

$$r(t)y'(t)\ge r(T_3)y'(T_3)+\int_T^t [Q(s)-f(s, M)]ds\xrightarrow[t\to\infty]{}\infty,$$

i.e. $r(t)y'(t)\ge l$ for $t\ge T_4\ge T_3$. As in the end of the proof of Theorem 2 we obtain the contradiction $y(t)\xrightarrow[t\to\infty]{}\infty$. Theorem 3 is proved.

Analogous results have been obtained for (2) in [4] and [5] when (6) hold and in [4] when (7) holds.

The following theorem pertains to the case when $Q(t)$ is not assumed to be with a constant sign.

Theorem 4. *In addition to conditions* (A) *suppose*

1. *The derivatives of* $\Delta(t, u)$ *exist and they are bounded.*
2. $\inf\limits_{\substack{t \geq T \\ u \neq 0}} |f(t, u)| > 0$ *and* $r'(t) \geq 0$.
3. $\int\limits_T^\infty |Q(s)|\, ds < \infty$ *and* $\sup\limits_{t \geq T} \int\limits_T^t |f(s, c)|\, ds / r(t) < \infty$ *for any* $c \neq 0$.
4. *One of the following conditions holds*

(i) (6) *holds and* $\int\limits_T^\infty \rho(t) |f(t, c)|\, dt = \infty$ *for any* $c \neq 0$;

(ii) (7) *holds and* $\sup\limits_{t \geq T} (\int\limits_T^t |Q(s)|\, ds / r(t) < \infty$.

Then all bounded nonoscilatory solutions of (5) *tend to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a bounded nonoscillatory solution of (5). Without any loss of generality we can assume that $0 < y(t) \leq M$ for $t \geq T_1 \geq T$ and some $M = \text{const} > 0$. (The proof is similar when $-M \leq y(t) < 0$ for $t \geq T_1 \geq T$). In view of A3 and A2 we have (10) for $t \geq T_2 \geq T_1$.

First we will prove that $\liminf\limits_{t \to \infty} y(t) = 0$. Suppose the contrary and let $m > 0$ and $T_3 \geq T_2$ be such that $y(t) \geq m$ for $t \geq T_3$. Via A3 and A2 we get $f(t, y(\Delta(t, y(t)))) \geq f(t, m)$ for $t \geq T_4 \geq T_3$. Then integrating (5) from T_4 to t we obtain

$$r(t)y'(t) \leq r(T_4)y'(T_4) + \int\limits_{T_4}^t |Q(s)|\, ds - \int\limits_{T_4}^t f(s, m)\, ds. \tag{17}$$

Let (i) be valid. Dividing (17) by $r(t)$, integrating from T_4 to t and tending $t \to \infty$ we obtain the contradiction

$$m \leq y(t) \leq y(T_4) + r(T_4)y'(T_4) \int\limits_{T_4}^t \frac{ds}{r(s)} + \int\limits_{T_4}^t \frac{1}{r(\tau)} \int\limits_{T_4}^\tau |Q(s)|\, ds\, d\tau - \int\limits_{T_4}^t \frac{1}{r(\tau)} \int\limits_{T_4}^\tau f(s, m)\, ds\, d\tau =$$

$$= y(T_4) + r(T_4)\, y'(T_4) \int\limits_{T_4}^t \frac{ds}{r(s)} + \int\limits_{T_4}^t \left(\int\limits_\tau^t \frac{ds}{r(s)} \right) |Q(\tau)|\, d\tau + \int\limits_{T_4}^t \left(\int\limits_\tau^t \frac{ds}{r(s)} \right) f(\tau, m)\, d\tau \underset{t \to \infty}{\longrightarrow} -\infty.$$

Let (ii) be valid. Tending $t \to \infty$ we obtain $r(t)y'(t) \leq -k$ for $t \geq T_5 \geq T_4$ and some $k = \text{const} > 0$. Dividing the last inequality by $r(t)$ integrating from T_5 to t and having in mind (7) we get the contradiction

$$m \leq y(t) \leq y(T_5) - k \int\limits_{T_5}^t \frac{ds}{r(s)} \underset{t \to \infty}{\longrightarrow} -\infty.$$

Thus,

$$\liminf_{t \to \infty} y(t) = \liminf_{t \to \infty} y(\Delta(t, y(t))) = 0 \text{ for } t \geq T_1. \tag{18}$$

Now we will prove that $\limsup\limits_{t \to \infty} y(t) = 0$. Suppose to the contrary that

$$\limsup_{t \to \infty} y(t) = \limsup_{t \to \infty} y(\Delta(t, y(t))) = 2q \leq M \tag{19}$$

for some positive constant q and $t \geq T_1$.

Integrating (5) from T_2 to t, dividing by $r(t)$ and using conditions 2 and 3 of Theorem 4 and (10) we obtain

$$y'(t) = \frac{r(T_2)y'(T_2)}{r(t)} + \frac{1}{r(t)} \int\limits_{T_2}^t Q(s)ds - \frac{1}{r(t)} \int\limits_{T_2}^t f(s, y(\Delta(s, y(s))))ds \leq$$

$$\leq \frac{r(T_2)|y'(T_2)|}{r(T_2)} + \frac{1}{r(T_2)} \int\limits_{T_2}^\infty |Q(s)|ds + \frac{1}{r(t)} \int\limits_{T_2}^t f(s, M)ds.$$

i.e. $y'(t)$ is bounded for $t \geq T_2$ and hence $y'(\Delta(t, y(t)))$ is bounded for $t \geq T_3 \geq T_2$.

In view of (18) and (19) we can find a sequence $\{\delta_n\}_{n=1}^\infty$ such that $T_3 \leq \leq \sigma_1 < \sigma_2 < \ldots, \sigma_n \underset{n \to \infty}{\longrightarrow} \infty$ and $y(\Delta(\sigma_n, y(\sigma_n))) > q$.

Let $[\alpha_n, \beta_n]$ be the largest interval around σ_n such that $y(\Delta(\alpha_n, y(\alpha_n))) = y(\Delta(\beta_n, y(\beta_n))) = 2q/3$ and $y(\Delta(t, y(t))) > 2q/3$ for $t \in (\alpha_n, \beta_n)$ $(n \geq 1)$. By mean value theorem there exists $\gamma_n \in (\alpha_n, \beta_n)$ such that

$$y'(\Delta(\gamma_n, y(\gamma_n))) \left[\frac{\partial \Delta}{\partial t} (\gamma_n, y(\gamma_n)) + \frac{\partial \Delta}{\partial v} (\gamma_n, y(\gamma_n)) y'(\gamma_n) \right] =$$

$$= \frac{y(\Delta(\sigma_n, y(\sigma_n))) - y(\Delta(\alpha_n, y(\alpha_n)))}{\sigma_n - \alpha_n} > \frac{q - 2q/3}{\sigma_n - \alpha_n} = \frac{q}{3(\sigma_n - \alpha_n)} > \frac{q}{3(\beta_n - \alpha_n)}.$$

In view of condition 1 of Theorem 4 and the boundedness of $y'(t)$ and $y'(\Delta(t, y(t)))$ we conclude that

$$\beta_n - \alpha_n \geq \eta \tag{20}$$

for some $\eta = \text{const} > 0$ and $n \geq 1$. Denote $\mathcal{U} = \bigcup\limits_{n=1}^\infty [\alpha_n, \beta_n]$.

Let us write equation (5) in the form

$$y'(t) = \frac{1}{r(t)} \left[x(t) + \int\limits_{T_1}^t Q(s)\, ds \right], \quad x'(t) = -f(t, y(\Delta(t, y(t)))).$$

If we suppose that $\int\limits_{T_1}^\infty f(s, y(\Delta(s, y))))ds = \infty$ then we get $x(t) \underset{t \to \infty}{\longrightarrow} -\infty$ and hence, $y'(t) \leq 0$ for $t \geq T_6 \geq T_1$, which is impossible. Thus, $\int\limits_{T_1}^\infty f(s, y(\Delta)(s, y(s))))ds < \infty$.

Since $f(t, y(\Delta(t, y(t)))) > f(t, 2q/3) > F$ for $t \in \mathcal{U}$ and $F = \text{const} > 0$ we obtain the contradiction

$$\infty > \int_{T_1}^{\infty} f(s,\ y(\Delta(s,\ y(s))))ds > \int_{\overline{u}}^{\infty} f(s,\ y(\Delta(s,\ y(s))))ds \geqq \int_{\overline{u}}^{\infty} f\left(s,\ \frac{2y}{3}\right)ds =$$

$$= \sum_{n=1}^{\infty} \int_{\alpha_n}^{\beta_n} f\left(s,\ \frac{2y}{3}\right)ds > \sum_{n=1}^{\infty} \int_{\alpha_n}^{\beta_n} F ds = \sum_{n=1}^{\infty} F(\beta_n - \alpha_n) \geqq \sum_{n=1}^{\infty} F\varepsilon = \infty$$

which proves Theorem 4.

We note that theorems of this type are proved in [2] for equation (1) when (6) holds with a technique, based on the techniques used by H a m m e t t [11] and S i n g h [12]. In the case when (7) holds similar theorems are proved in [2, 3, 8–10]. While [2] and [8] treat retarded equations (1) and (3), and [3, 9, 10] — equations (2) and (3), with deviating argument $g(t)\underset{t\to\infty}{\longrightarrow}\infty$ such that $0 < g'(t) \leqq N$ for some $N = \text{const} > 0$, here we investigate equation (5) without the above assumptions. Our proof is based on the technique from [7].

3. **Examples.** 1) Consider the equation

$$(20)\qquad (t^2y'(t))' + [ty(\sqrt{t}+y^2(t)) + \sqrt[3]{y(\sqrt{t}+y^2(t))}]t^{-7} = (6t^9 + t(\sqrt{t}+t^4)^2 + (\sqrt{t}+t^4)^{2/3})t^{-7}$$

for $t \geqq T_v = (1+2v^2+\sqrt{1+4v^2})/2$, $v \in R$.

Here $r(t) = t^2$, $f(t,\ u) = u/t^6 + u^{1/3}/t^7$, $\Delta(t,\ v) = \sqrt{t}+v^2 \leqq t$ for $t \geqq T_v$ and $Q(t) = t^{-7}[6t^9 + t(\sqrt{t}+t^4) + (\sqrt{t}+t^4)^{2/3}]$. Conditions (A) are fulfilled. Since $\int_{T_v}^{\infty} \frac{dt}{r(t)} < \infty$, $\lim_{|u|\to\infty} \frac{f(t,\ u)}{u} = \varphi(t) = \frac{1}{t^6}$, $\int_{T_v}^{\infty} \varphi(t)dt < \infty$, $Q(t) > 0$ and $\int_{T_v}^{\infty} (Q(s) - f(s,\ c)]ds = \int_{T_v}^{\infty} (6s^2 - c/s^6 - c^{1/3}s^{-7})ds = \infty$ by Theorem 1 all solutions of (20) are nonoscillatory.

But $r(t) = t^2$ and $\int_{T_v}^{t} Q(s)ds > \int_{T_v}^{t} 6s^2 ds = 2(t^3 - T_v^3)$. Then $(\int_{T_v}^{t} Q(s)ds/r(t) > 2(t^3 - T_v^3)/t^2 \to \infty$ and by Theorem 2 all solutions of (20) are positive.

Since $\int_{T_v}^{t} [Q(s) - f(s,\ c)]ds/r(t) > \left(\int_{T_v}^{t} \left[6s^2 - \frac{c}{s^6} - \frac{c^{1/3}}{s^5}\right]ds\right)\Big/ t^2 = \left(2t^3 + \frac{c}{5t^5} + \frac{c^{1/3}}{6t^6} - 2T_v^3 - \frac{c}{5T_v^5} - \frac{c^{1/3}}{6T_v^6}\right)\Big/ t^2 \to \infty$ then by Theorem 3 all solutions of (20) are unbounded above.

In fact, $y(t) = t^2$ is such a solution of (20).

2) Consider the equation

$$(21)\qquad (t^{-1}y'(t))' + \frac{y(\sqrt{t}+y^2(t)) + t\sqrt[3]{y(\sqrt{t}+y^2(t))}}{t^4} = \frac{(1+t^{15/2})^4}{t^2} + \frac{(1+t^{15/2})^{4/3}}{t^{7/3}} + 8t$$

for $t \geqq T_v = \frac{1}{2}(1+2v^2+\sqrt{1+4v^2})$.

It is easy to verify that conditions of Theorems 1–3 are satisfied in case (ii). According to these theorems all solutions of (21) are nonoscillatory, positive and unbounded above. Indeed, $y(t) = t^4$ is such a solution of (21).

3) Consider the equation

$$(22)\qquad (t^2y'(t))' + 1/t^2 y(\sqrt{t}+y^2(t)) + ty^3(\sqrt{t}+y^2(t)) = \frac{t^6}{(t^{9/2}+1)^2} + \frac{t^{25}}{(t^{9/2}+1)^6} + \frac{2}{t^2}$$

for $t \geqq T > 0$. Here $r(t) = t^2$, $f(t,\ u) = u/t^2 + u^3$, $\Delta(t,\ v) = \sqrt{t}+v^2$, $Q(t) = t^6/(t^{9/2}+1)^2 + t^{25}/(t^{9/2}+1)^6 + 2/t^2$, $\rho(t) = 1/t$ and conditions (A) are satisfied. Since conditions 1–4 of Theorem 4 are fulfilled in case (i), by this theorem we conclude that all bounded nonoscillatory solutions of (22) tend to zero as $t\to\infty$. Indeed, $y(t) = 1/t^2$ is such one solution of (22).

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