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STURMIAN THEOREMS FOR TIME DEGENERATE PARABOLIC INEQUALITIES

BY

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1. **Introduction.** In recent years, numerous results on Sturmian theory have been obtained for partial differential equations of elliptic type while results for parabolic and hyperbolic equations appear to be relatively rare. Picone [11] extended the Sturm comparison result for ordinary differential equation to self-adjoint second order elliptic equations. Further Sturm results for elliptic equations have been established by Hartman and Winter [6], Protter [12], Allegretto and Swanson [1], Swanson [13], Dunninger end Weinacht [5] and Kreith [7]. Furthermore, Kreith [8], Travis and Young [14] established Sturm results for hyperbolic and ultrahyperbolic differential inequalities, while McNabb [9], Dunninger [4], Chan and Young [3] have developed Sturmian theory for parabolic differential inequalities. Picone or Green's identity plays the dominant role in the development of Sturmian theory. More recently Chan [2], Noussair and Swanson [10] obtained Sturm results for elliptic and parabolic differential inequalities using a maximum principle technique. We obtain Sturm separation and comparison results for time degenerate parabolic differential inequalities with mixed boundary conditions, using a maximum principle technique.

The plan of the paper is as follows. In Section 2, we present our results for linear inequalities under mixed boundary conditions by using the maximum principle due to Waid [15]. To illustrate the result we provide an example. The results for quasilinear time degenerate parabolic inequalities are also discussed in Section 3.

2. **Sturm Results for Linear Inequalities.** Let G be a bounded $(n+1)$-dimensional domain in R^{n+1}, where $(t, x)=(t, x_1, x_2, \dots, x_n)$ will represent a variable point in R^{n+1}, with $x=(x_1, x_2, \dots, x_n)\in R^n$. Let $G=(0, T)\times D$ where $D\subset R^n$. Let $\bar{G}=G+\partial G$. We partition the boundary ∂G of G into three subsets $\partial_0 G$, $\partial_1 G$ and $\partial_2 G$, where $\partial_0 G=\{0\}\times \bar{D}$, $\partial_1 G=(0, T]\times\partial D$, $\partial_2 G=\{T\}\times$ $\times D$. Define

$$G_p=G+\partial_2 G,\quad R_p=\partial_0 G+\partial_1 G=\bar{B}+S.$$

Let

$$Lu=a_{ij}(t, x)u_{x_i x_j}+b_i(t, x)u_{x_i}+d(t, x)u-c(t, x)u_t$$

where $u(t, x)$, $a_{ij}(t, x)$, $b_i(t, x)$, $d(t, x)$ and $c(t, x)$ are bounded and real valued functions of t and x on G and the matrix (a_{ij}) is symmetric and positive definite. We shall not make the assumption that $c(t, x)$ is bounded away from zero, since $c(t, x)=0$ for some $(t, x)\in \bar{G}$ is permissible. The operator L may be time degenerate parabolic operator.

Define

$$M(u, G)=\operatorname{Sup}\{u(t, x) : (t, x)\in G\},$$

$$m(u, G)=\operatorname{Inf}\{u(t, x) : (t, x)\in G\}.$$

Let $M(u, G)\geq 0$ and $m(v, G)\geq 0$. Also define the boundary operators

$$B_s u=\partial u/\partial \gamma_i+su,$$

$$B_\sigma v=\partial v/\partial \gamma_i+\sigma v, \text{ on } S.$$

We allow

$-\infty< s(t, x)\leq +\infty$, where $s(p)=+\infty$ and $B_s u(P)=0$ implies $u(P)=0$.

Assume that a sphere K with radius r can be constructed through $(t', x')\in$ $\in R_p$, whose interior lies entirely in G_p; and $c(t', x)>0$ for some $(t', x)\in K$ such that

$$\|(t', x)-(t', x')\|<r.$$

Also suppose that the radial direction from the centre of the sphere to $(t',$ $x')$ is not parallel to the t-axis. A point P on R_p is said to have the *inside strong sphere* property if there exists a closed ball K_1 such that $K_1\subset G_p$ and $K_1\cap R_p=\{P\}$. Unless otherwise stated, we assume that every point of R_p has the inside strong sphere property

$$-\infty\leq \sigma(t, x)\leq s(t, x)\leq +\infty \text{ and } s(t, x)\neq -\infty$$

on the boundary. For $y\in G_p$, the notation H_p will be used to denote the set of all points $z\in G_p$ such that z can be connected to y by a path consisting only of line segments contained in the hyperplane $t=$constant and segments parallel to the positive t-axis.

Theorem 2.1. *If there exists a nonnegative solution $u(t, x)$ of $Lu\geq 0$ in G_p, subject to the boundary conditions $u(0, x)=0$ on $\bar{B}$, $B_s u\leq 0$ on S and $c(t, x)>0$ at some point of each component of each horizontal segment, then no function v, satisfying $Lv\leq 0$ and the boundary condition $B_\sigma v\geq 0$, can be positive throughout $\bar{G}$ unless v is a constant multiple of u in some subdomain H_p of G_p.*

Proof. If $v>0$ throughout $\bar{G}$, then define the function $\omega=u/v$.

We have

$$Lu=L(\omega v)=a_{ij}(\omega v)_{x_i x_j}+b_i(\omega v)_{x_i}+d\omega v-c(\omega v)_t\geq 0 \text{ in } G_p, \tag{2.1}$$

$$v^{-1}L_u=a_{ij}\omega_{x_i x_j}+(2v^{-1}a_{ij}v_{x_i}+b_i)\omega_{x_i}+v^{-1}(Lv)\omega-c\omega_t\geq 0 \text{ in } G_p.$$

Since ω is nonnegative in G_p and ω is continuous on $\bar{G}$, it follows from Waid's maximum principle [15], that either ω attains its positive maximum at a point P on the boundary R_p or ω attains its maximum in a subdomain H_p of G_p. In the latter case, ω is identically constant in some subdomain H_p of G_p, i.e. v is constant multiple of u in some subdomain H_p of G_p.

If v is not a constant multiple of u in some subdomain in H_p of G_p then ω attains its positive maximum at the point P on the boundary R_p. We consider two cases.

i) Suppose $P\in B$, then $\omega(P)=u(P)/v(P)=0$ by hypothesis and ω cannot attain its positive maximum on $\bar{B}$, it may attain on S.

ii) Suppose $P\in S$ and ω attains its positive maximum on S; we have by Waid's maximum principle

$$\partial\omega/\partial\gamma_i>0 \text{ at } P.$$

On the other hand, since $s(P)\neq +\infty$, we have at P

$$\partial\omega/\partial\gamma_i=\partial(u, v)/\partial\gamma_i=(\sigma-s)\omega\leq 0$$

which is a contradiction. Hence the result. This is called the Sturm separation theorem.

As an immediate consequence of Theorem 2.1, we have the following results.

Corollary 2.1. *Under the hypotheses of Theorem 2.1, no function v satisfying $Lv\geq 0$ in G_p and the boundary condition $B_\sigma v\leq 0$ on S can be negative throughout $\bar{G}$ unless v is a constant multiple of u in some subdomain H_p of G_p.*

Proof. If $v<0$ throughout $\bar{G}$, then by setting $v_1=-v$, we have $Lv_1\leq 0$ and $B_\sigma v_1\geq 0$ and $v_1>0$. Now applying Theorem 2.1, we reach at a contradiction. Hence the result.

Corollary 2.2. *Let $K(t, x)$ be a nonnegative function in G_p. If there exists a nonnegative solution $u(t, x)$ of $Lu\geq 0$ in G_p subject to the boundary conditions $B_s u\leq 0$ on S, $u(0, x)=0$ on $\bar{B}$ and $c(t, x)>0$ at some point of each component of each horizontal segment, then no function v satisfying the inequality $(L+K)v\leq$ ≤ 0 (≥ 0) in G_p and the boundary condition $B_\sigma v\geq 0$ (≤ 0), can be positive (negative) throughout $\bar{G}$ unless v is a constant multiple of u in some subdomain H_p of G_p.*

This is called the Sturm comparison result. The following result may be regarded as an extension of Corollary 2, Chan [2] to time degenerate parabolic inequalities.

Corollary 2.3. *Let $g(t, x)$ and $h(t, x)$ be functions defined in G_p where $h(t, x)$ is bounded. If there exists a nonnegative solution $u(t, x)$ of $Lu\geq g$ in G_p, subject to the boundary conditions $B_s u\leq 0$ on S, $u(0, x)=0$ on $\bar{B}$, and $c(t, x)>0$ at some point of each component of each horizontal segment, then no function v satisfying $Lv\leq h$ in G_p and the boundary condition $B_\sigma v\geq 0$ such that $gv-hu\geq 0$ in G_p, can be positive throughout $\bar{G}$ unless v is a constant multiple of u in some subdomain H_p of G_p.*

Proof. If $v>0$ throughout $\bar{G}$, then

$$(L-h/v)u\geq g-hu/v\geq 0 \text{ in } G_p, \quad (L-h/v)v=Lv-h\leq 0 \text{ in } G_p.$$

The corollary then follows from Theorem 2.1.

Remarks. 1. If there exists a nontrivial solution u of $Lu=0$ in G_p, $B_s u=$ $=0$ on S, $u(0, x)=0$ on $\bar{B}$ and $c(t, x)>0$ at some point of each component of each horizontal segment such that either $u>0$ or $u<0$ in G_p, then the function v satisfying $Lv=0$ in G_p, $B_\sigma v=0$ on S must vanish somewhere in $\bar{G}$.

2. If $s\equiv +\infty$ on S, then Theorem 2.1 can be simplified to the following form. If there exists a positive solution $u(t, x)$ of $Lu\geq 0$ in G_p subject to the

boundary condition $u=0$ on R_p and $c(t, x)>0$ at some point of each component of each horizontal segment, then no function v satisfying $Lv\leq 0$ in G_p can be positive throughout $\bar{G}$. If $v>0$ throughout $\bar{G}$, then ω satisfies (2.1) in G_p and $\omega>0$ in G_p. By Waid's maximum principle, ω attains its positive maximum on R_p. This contradicts that $\omega=0$ on R_p. Hence the result.

We consider the following example, to illustrate the result.

Example. Let

$$Lu\equiv u_{xx}+2u-tu_t,\quad (L+3)v\equiv v_{xx}+5v-tv_t.$$

The function $u(t, x)=t\sin x$ satisfies

$$Lu=0,\ 0<x<\pi,\ t>0;\ u(0, x)=0,\ 0<x<\pi,$$

$$u(t, 0)=0=u(t, \pi),\ t>0;\ u>0,\ 0<x<\pi,\ t>0,$$

since the hypotheses of the Theorem 2.1 are satisfied with $s(P)=+\infty$. It follows that a function v, satisfying $(L+3)v=0$, $0<x<\pi$, $t>0$ cannot be positive throughout the region $0\leq x\leq\pi$, $t\geq 0$. For example, $v(t, x)=t\sin 2x$ satisfies $(L+3)v=0$ and vanishes at $x=0$, $\pi/2$, π or at $t=0$.

3. Sturm Results for Quasilinear Inequalities. Here we extend the main result of the preceding section to quasilinear time degenerate parabolic operator defined by

$$Lu\equiv a_{ij}(t, x)u_{x_ix_j}+b_i(t, x)u_{x_i}-c(t, x)u_t\geq d_1(t, x, u, u_x, u_t)u \text{ in } G_p. \tag{3.1}$$

$$Lv\equiv a_{ij}(t, x)v_{x_ix_j}+b_i(t, x)v_{x_i}-c(t, x)v_t\leq d_2(t, x, v, v_x, v_t)v \text{ in } G_p, \tag{3.2}$$

and the boundary conditions

$$B_s u\equiv \partial u/\partial\gamma_i+s(t, x)u\leq 0 \text{ on } S, \tag{3.3}$$

$$B_\sigma v\equiv \partial v/\partial\gamma_i+\sigma(t, x)v\geq 0 \text{ on } S. \tag{3.4}$$

Theorem 3.1. *Let $u(t, x)$ and $v(t, x)$ satisfy the time degenerate parabolic inequalities (3.1) and (3.2) in G_p and the boundary conditions (3.3) and (3.4) respectively; and $u(0, x)=0$ on $\bar{B}$. If*

(i) $u(t, x)$ is nonnegative throughout G_p,

(ii) d_2-d_1 is bounded above by a constant N,

(iii) $c(t, x)>0$ at some point of each component of each horizontal segment,

then no function v can be positive throughout $\bar{G}$ unless v is a constant multiple of u in some subdomain H_p of G_p.

Proof. If $v>0$ throughout $\bar{G}$, then we define $w=u/v$.

Now from (3.1) and (3.2), we have

$$vLu-uLv\geq d_1(t, x, u, u_x, u_t)uv-d_2(t, x, v, v_x, v_t)uv \text{ in } G_p.$$

Then the function w satisfies the differential inequality

$$a_{ij}w_{x_ix_j}+(2v^{-1}a_{ij}v_{x_j}+b_i)w_{x_i}-cw_t+(d_2-d_1)w\geq 0 \text{ in } G_p. \tag{3.5}$$

Now, the argument is similar to that of Theorem 2.1. We omit the details.

Remark. If $s\equiv+\infty$ on S, then theorem 3.1 can be simplified to the following form. If there exists a positive solution $u(t, x)$ of $Lu\geq d_1(t, x, u, u_x, u_t)u$ in G_p, subject to the boundary condition $u=0$ on R_p and if $c(t, x)>0$ at some point of each component of each horizontal segment, then no function v satisfying $Lv\leq d_2(t, x, v, v_x, v_t)v$ in G_p with d_2-d_1 bounded by a constant N cannot be positive throughout $\bar{G}$.

If $v>0$ throughout $\bar{G}$, then w satisfies (3.5) in G_p and $w>0$ in G_p. By Waid's maximum principle, w attains its positive maximum somewhere on R_p. This contradicts that $w=0$ on R_p. Hence the result follows.

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