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ON PERIODIC SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS IN BANACH SPACES

BY

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1. Introduction. We use a fixed point theorem for locally convex vector spaces, due to S a d o v s k i i ([11], Theorem 3.4.3), to establish the existence of periodic solutions on the real line **R** of nonlinear differential equations in Banach spaces. Throughout this paper, **E** denote a Banach space with the norm $\|\cdot\|$, $\mathcal{L}(\mathbf{E})$ the algebra of continuous linear operators from **E** into itself with indouced norm $|\cdot|$, $\mathbf{P}_\omega$ the Banach space of continouos ω-periodic functions from **R** to **E** endowed with the norm $|x(\cdot)|=$ $=\sup\{\|x(t)\|\,;\,t\in\mathbf{R}\}$.

We shall consider the differential equation

$$x'(t)=A(t)x(t)+f(t,x(t)), \tag{1.1}$$

where $t\in\mathbf{R}$, $A(t)\in\mathcal{L}(\mathbf{E})$, and f is a **E**-valued function defined on $\mathbf{R}\times\mathbf{E}$. In addition, we shall assume that $A(t+\omega)=A(t)$ and $f(t+\omega,x)=f(t,x)$, $(t,x)\in$ $\in\mathbf{R}\times\mathbf{E}$, $\omega>0$.

The purpose of this paper is to prove the existence of periodic solutions of the above equation under the assumption that the monodromy operator of $A(t)$ has a logarithm, that its spectrum is contained in the unity disk and f satisfies a regularity condition expressed in terms of the Kuratowski measure of noncompactness α. In case that $\mathbf{E}=\mathbf{R}^n$, the existence of a unique periodic solution of (1.1), using a Floquet representation for $A(t)$ and the dissipativity conditions for $f(t,x)$, has been studied by V. I f t o d e [7]. His proof is based on Leray-Schauder principle and on the method of Lyapunov function. Also, with some similar techniques J.O.C. E z e i l o [5] studies the existence of almost periodic solutions. Other studies on periodic and almost periodic solutions for system (1.1) are P. T a l p a l a r u [12], [13], G. V i l l a r i [14] and many others.

2. Preliminaries. The measure of noncompactness $\alpha(W)$ of a nonempty bounded subset W of **E** is defined by $\alpha(W)=\inf\{\varepsilon>0\,;\,W$ — admits a finite covering by sets of diameter $\leqslant\varepsilon\}$. For the properties of α we refer, e.g., to [4] or [8]. Further, we will use the standard notations. The closure of a set **X**, its diameter and its closed convex hull will be denoted by $\overline{\mathbf{X}}$, diam **X** and $\overline{\mathrm{co}}\,\mathbf{X}$, respectively. The sets $F(\mathbf{X})$ (F mapping on **X**), $t\mathbf{X}$ (t-real) and $\mathbf{X}_1+\mathbf{X}_2$ are defined as usual. Denote by $C(\mathbf{R},\mathbf{E})$ the set of all continuous functions from **R** to **E**. The set $C(\mathbf{R},\mathbf{E})$ will be considered as a vector space endowed with the topology of uniform convergence on compact subsets of $\mathbf{R}$. Let $\mathbf{Y} \subset C(\mathbf{R}, \mathbf{E})$. We will use the notations $\mathbf{Y}(t) = \{y(t) : y \in \mathbf{Y}\}$ and $\int_0^t \mathbf{Y}(s)ds = \{\int_0^t y(s)ds : y \in \mathbf{Y}\}$.

We shall use also the following result.

Lemma 2.1. ([1], [6]). *If $\mathbf{Z}$ is a bounded equicontinuous subset of the usual Banach space of continuous $\mathbf{E}$-valued functions defined on a compact interval $\mathbf{I} \subset \mathbf{R}$, then for $t \in I$ we have*

$$\alpha(\cup\{\mathbf{Z}(t) : t \in \mathbf{I}\}) = \sup\{\alpha(\mathbf{Z}(t)) : t \in \mathbf{I}\}, \tag{2.1}$$

$$\alpha(\int_0^t \mathbf{Z}(s)ds) \leqslant \int_0^t \alpha(\mathbf{Z}(s))ds, \tag{2.2}$$

and, if Q is a continuous mapping from $\mathbf{I}$ to $\mathcal{L}(\mathbf{E})$ and $\mathbf{W}$ is a bounded subset of $\mathbf{E}$, then ([3]),

$$\alpha(\cup\{\mathbf{Q}(t)\mathbf{W} : t \in \mathbf{I}\}) \leqslant \sup\{\|\mathbf{Q}(t)\| : t \in \mathbf{I}\}\alpha(\mathbf{W}). \tag{2.3}$$

Our result will be proved by the following fixed-point theorem of Schauder type due to B.N. S a d o v s k i i [11].

Lemma 2.2. *Let $\mathcal{X}$ be a closed convex subset of $C(\mathbf{R}, \mathbf{E})$ and let Φ be a function which assigns to each subset $\mathbf{Y}$ of $\mathbf{E}$ a real number $\Phi(\mathbf{Y}) \geqslant 0$ having the following properties:* a) $\Phi(\mathbf{Y}_1) \leqslant \Phi(\mathbf{Y}_2)$ *whenever* $\mathbf{Y}_1 \subset \mathbf{Y}_2$; b) $\Phi(\mathbf{Y} \cup \{y\}) = \Phi(\mathbf{Y})$ *for all* $y \in \mathcal{X}$; c). $\Phi(\overline{\mathrm{co}}\, \mathbf{Y}) \leqslant \Phi(\mathbf{Y})$; d) *if* $\Phi(\mathbf{Y}) = 0$, *then* $\bar{\mathbf{Y}}$ *is compact. Assume that* $F : \mathcal{X} \to \mathcal{X}$ *is a continuous mapping satisfying* $\Phi(F(\mathbf{Y})) < \Phi(\mathbf{Y})$ *for arbitrary subset* $\mathbf{Y}$ *of* $\mathcal{X}$ *with* $\Phi(\mathbf{Y}) > 0$. *Then F has at least a fixed point in $\mathcal{X}$.*

We note that M. D a w i d o w s k i and B. R z e p e c k i [3], on the one hand, have applied the Sadovskii fixed-point theorem to study the existence of bounded solutions of (1.1) and B. R z e p e c k i [10], on the other hand, to study the existence of solutions of the Volterra integral equations.

Let us consider the linear differential equation

$$x'(t) = A(t)x(t), \tag{2.4}$$

where $A(t) \in \mathcal{L}(\mathbf{E})$ and $A(t+\omega) = A(t)$, $t \in \mathbf{R}$. We denote by $U(t)$ the Cauchy operator for (2.1), i.e., $U(t) \in \mathcal{L}(\mathbf{E})$ and

$$U'(t) = A(t)U(t), \quad U(0) = \mathcal{I}. \tag{2.5}$$

It is easy to see that (2.5) is also satisfied by $U_1(t) = U(t+\omega)$. $U^{-1}(\omega)$ and from the uniqueness of the solution we have $U_1(t) \equiv U(t)$, from where, $U(t+\omega) = U(t)U(\omega)$. Therefore, the periodicity of $A(t)$ does not imply the periodicity of the Cauchy operator. The operator $U(\omega)$ is called the monodromy operator for (2.1). The spectral radius of $U(\omega)$, i.e., the radius of the smallest circle that contains its spectrum, is given by $r = \lim\limits_{k\to\infty}(\|U^k(\omega)\|)^{1/k}$. In the sequel we shall assume that $U(\omega)$ has a logarithm, and that occurs if, for example, its spectrum $\sigma(U(\omega))$ does not surround the origin. Therefore, there exists $S = \ln U(\omega)$ for which $U(\omega) = e^S$, where $\ln U(\omega) = \frac{-1}{2\pi i}\int_\gamma \ln \lambda . (U(\omega) - \lambda\mathcal{I})^{-1} d\lambda$,

where γ is a contour that encircles $\sigma(U(\omega))$. If we consider the constant operator Γ given by $\Gamma = \omega^{-1}\ln U(\omega)$, it follows $U(\omega) = e^{\omega\Gamma}$ and if we put $Q(t) = U(t)e^{-t\Gamma}$, then $U(t)$ has a Floquet representation

$$U(t) = Q(t)e^{t\Gamma}. \tag{2.6}$$

It is easy to see that on $0 \leqslant t \leqslant \omega$, $Q(t)$ is a continuous differentiable and invertible operator with $Q^{-1}(t)$ continuous and $Q(t+\omega) = Q(t)$, $t \in \mathbf{R}$. Because of $\sigma(U(\omega)) = \sigma(e^{\omega\Gamma}) = e^{\omega\sigma(\Gamma)}$, it follows that for $\mu \in \sigma(\Gamma)$ and $\lambda \in \sigma(U(\omega))$, we have $|\lambda| = e^{\omega \mathrm{Re}\,\mu}$ and therefore $|\lambda| < 1$ if and only if $\mathrm{Re}\,\mu < 0$. If we assume that $\sigma(U(\omega))$ is contained in the unity disk (as in the next Lemmas 2.3 and 2.4), then it follows $\sigma(\Gamma) \subset (\mathrm{Re}\,\lambda < 0)$. For other properties of the monodromy operator in connection with the existence of periodic solutions, the reader is referred to the monograp [2] or [9]. Let us consider the equation

$$x'(t) = A(t)x(t) + g(t), \tag{2.7}$$

where $A(t) \in \mathcal{L}(\mathbf{E})$ and $g(t)$ is an $\mathbf{E}$-valued function defined on $\mathbf{R}$.

Lemma 2.3. *Assume that the following conditions are satisfied*

a) $g \in C(\mathbf{R}, \mathbf{E})$, $g(t+\omega) = g(t)$, $\forall t \in \mathbf{R}$;

b) $A(t) \in \mathcal{L}(\mathbf{E})$, $A(t+\omega) = A(t)$, $\forall t \in \mathbf{R}$;

c) *the Cauchy operator $U(t)$ has a Floquet representation and the spectrum of the monodromy operator, $\sigma(U(\omega))$, is contained in the unity disk. Then the equation* (2.7) *has an unique ω-periodic solution given by*

$$x(t) = \int_{-\infty}^t Q(t)e^{(t-s)\Gamma}Q^{-1}(s)g(s)ds. \tag{2.8}$$

Proof. At first, we note that the hypothesis c) yields $\sigma(\Gamma) \subset (Re\,\lambda < 0)$ and because $\|e^{\tau\Gamma}\| \leqslant \frac{1}{2\pi}\int_\gamma |e^{\lambda\tau}|\,\|R_\lambda\|\,d\lambda \leqslant \bar{N}$, where the contour γ is completely included in $(\mathrm{Re}\,\lambda < 0)$ and $\bar{N} = \frac{\lg(\gamma)}{2\pi}\max\{\|R_\lambda\| : \lambda \in \gamma\}$; $\lg(\gamma)$ is the length of γ and R_λ is the resolvent of Γ. From here, the periodicity of $Q^{-1}(t)$, and the hypothesis a) we conclude that (2.8) has a meaning. Then, we look for $x(t)$ in the form of the variation constants formula, $x(t) = U(t)c(t)$. From (2.7) it follows that $c(t) = c_0 + \int_{-\infty}^t U^{-1}(s)g(d)ds$, and then using (2.6)

$$x(t) = U(t)c(t) = Q(t)e^{t\Gamma}c_0 + \int_{-\infty}^t Q(t)e^{(t-s)\Gamma}Q^{-1}(s)g(s)ds.$$

We remark that the integral is ω-periodic because $Q(t)$ and $g(t)$ are ω-periodic. Obviously, only the choise $c_0 = 0$ ensures $x(t)$ to be ω-periodic.

Before passing to study the existence of periodic solutions for (1.1), we shall consider the equation

$$y'(t) = A(t)y(t) + f(t, x(t)), \tag{2.9}$$

where $x(t) \in \mathbf{P}_\omega$. As a consequence of Lemma 2.3 we have

Lemma 2.4. *Assume that the conditions b) and c) from Lemma 2.3 occur and, in addition, that $f(t,x)\in C(R,E)$, $f(t+\omega,x)=f(t,x)$ for every $(t,x)\in \mathbf{R}\times\mathbf{E}$. Then, the existence of ω-periodic solutions for (2.8) is equivalent to the existence of ω-periodic solutions for the equation*

$$y(t)\equiv(Fx)(t)=\int_{-\infty}^{t} Q(t)e^{(t-s)\Gamma}Q^{-1}(s)f(s,x(s))ds. \tag{2.10}$$

3. Main results. As regards to the existence of ω-periodic solutions for (1.1), we have the following

Theorem 3.1. *Assume that the following conditions are satisfied: a) the Cauchy operator has a Floquet reprezentation and the spectrum of the monodromy operator, $\sigma(U(\omega))$, is contained in the unity disk;*

b) $A(t)\in\mathcal{L}(\mathbf{E})$, $A(t+\omega)=A(t)$ *for every* $t\in\mathbf{R}$;

c) $f\in C(R,E)$, $f(t+\omega,x)=f(t,x)$ *for every* $(t,x)\in\mathbf{R}\times\mathbf{E}$ *and* $\|f(t,x(t))\|\leqslant M=\mathrm{const.}$ *for any* $x(t)\in\mathbf{P}_\omega$;

d) *for any compact subset* $\mathbf{I}\subset\mathbf{R}$ *and each bounded subset* $\mathbf{X}\subset\mathbf{E}$,

$$\alpha(f[\mathbf{I}\times\mathbf{X}])\leqslant \sup\{g(t):t\in\mathbf{I}\}h(\alpha(\mathbf{X})),$$

where $g:\mathbf{R}\to\mathbf{R}_+$, $h:\mathbf{R}_+\to\mathbf{R}_+$, *are such that g is continuous and h nondecreasing;*

e) $$L=\sup\{\int_{-\infty}^{t}\|Q(t)e^{(t-s)\Gamma}Q^{-1}(s)\|g(s)ds:\quad t\in\mathbf{R}\}<\infty$$

and $Lh(t)<t$ *for* $t>0$.

Then there exists at least an ω-periodic solution for the equation

$$x'(t)=A(t)x(t)+f((t,x(t)). \tag{3.1}$$

Proof. We define a mapping F as follows

$$(Fx)(t)=\int_{-\infty}^{t}Q(t)e^{(t-s)\Gamma}Q^{-1}(s)f(s,x(s)),\ \text{for}\ x\in\mathbf{P}_\omega.$$

Taking into account that $\sigma(U(\omega))$ is contained in the unity disk, it follows that $\sigma(\Gamma)\subset(\mathrm{Re}\,\lambda<0)$ and therefore, there exists a positive constant ν such that $\mathrm{Re}\,\lambda<-\nu$ for $\lambda\in\sigma(\Gamma)$. Then, the boundedness of $Q(t)$ and $Q^{-1}(t)$ and the estimate $\|e^{\tau\Gamma}\|\leqslant N_\nu e^{-\nu(t-s)}$ for $\tau>0$ (see the proof of Lemma 2.3) imply

$$\|Q(t)e^{(t-s)\Gamma}Q^{-1}(s)\|\leqslant Ne^{-\nu(t-s)},\ t,\ s\in\mathbf{R},\quad t\geqslant s. \tag{3.2}$$

Denote by $\mathcal{X}$ the set of all $y\in\mathbf{P}_\omega$ such that $\|x(t)\|\leqslant K=NM/\nu$, and $\|x(t_1)-x(t_2)\|\leqslant K\int_{t_1}^{t_2}\|A(s)\|ds+M(t_2-t_1)$, for real t, and t_1, t_2 with $t_1\leqslant t_2$.

It is easily to see that $\mathcal{X}$ is a closed convex bounded subset of $\mathbf{P}_\omega$. We will show that $F:\mathcal{X}\to\mathcal{X}$. For $x\in\mathcal{X}$ we have

$$\|(Fx)(t)\|\leqslant NM\int_{-\infty}^{t}e^{-\nu(t-s)}ds=(NM\int_{0}^{\infty}e^{-\nu\tau}d\tau)=(NM)/\nu=K.$$

By Lemmas 2.3 and 2.4, the function Fx is an ω-periodic solution of the differential equation (2.8) on $\mathbf{R}$. Hence

$$\|(Fx)(t_1)-(Fx)(t_2)\|\leqslant\int_{t_1}^{t_2}\|A(s)(Fx)(s)+f(s,x(s))\|ds\leqslant K\int_{t_1}^{t_2}\|A(s)\|ds+M(t_2-t_1),$$

$$t_1\leqslant t_2.$$

Consequently, $Fx\in\mathcal{X}$. Let now prove the continuity of F. Consider the sequence $\{u_n\}_1^\infty$, $u_n\in\mathcal{X}$ such that $u_n\to u$, as $n\to\infty$, $u\in\mathcal{X}$. Then,

$$\|(Fu_n)(t)-(Fu)(t)\|\leqslant\nu^{-1}\sup\{\|f(s,u_n(s))-f(s,u(s))\|:\ s\in\mathbf{R}\},$$

and because the continuity of $f:\mathbf{R}\times\mathbf{E}\to\mathbf{E}$ implies the continuity of the operator $x(\,.\,)\to f(\,.\,,x(\,.\,))$ from $\mathbf{P}_\omega$ into itself, it follows that F is a continuous map from $\mathcal{X}$ into itself. To define the function from the Sadovskii's theorem, let us put $\Phi(\mathbf{Y})=\sup\{\alpha(\mathbf{Y}(t));\ t\in\mathbf{R}\}$ for a subset $\mathbf{Y}\subset\mathcal{X}$. By the corresponding properties of α, the function Φ satisfies the condition a) — d) listed in Lemma 2.1. Let us assume that $\mathbf{Y}$ is a subset of $\mathcal{X}$ with $\Phi(\mathbf{Y})>0$. Let $t\in\mathbf{R}$ be fixed and let choose a number $a>0$ such that $2Ke^{-\nu a}<\varepsilon$. Then we can write

$$\begin{aligned}F[\mathbf{Y}](t)\subset&\{\int_{-\infty}^{t-a}Q(t)e^{(t-s)\Gamma}Q^{-1}(s)f(s,y(s))ds:\quad y\in\mathbf{Y}\}+\\ &+\{\int_{t-a}^{t}Q(t)e^{(t-s)\Gamma}Q^{-1}(s)f(s,y(s))ds;\quad y\in\mathbf{Y}\}\equiv\mathrm{A}+\mathrm{B}.\end{aligned} \tag{3.3}$$

We have

$$\alpha(\mathrm{A})\leqslant\mathrm{diam}(\mathrm{A})\leqslant 2NM\int_{-\infty}^{t-a}e^{-\nu(t-s)}ds=2Ke^{-\nu a}<\varepsilon. \tag{3.4}$$

Now we shall prove that

$$\alpha(\mathrm{B})\leqslant h(\alpha(\mathbf{Z}))\int_{-\infty}^{t}\|Q(t)e^{(t-s)\Gamma}Q^{-1}(s)\|g(s)ds. \tag{3.5}$$

where $\mathbf{Z}=\cup\{\mathbf{Y}(s);\ t-a\leqslant s\leqslant t\}$. To this end we shall use the argument made in [3]. Indeed, for arbitrary $\varepsilon'>0$ there exists a $\delta>0$ such that $|s'-s''|<\delta$, $s',s''\in[t-a,t]$ implies $\|G(t,s')-G(t,s'')\|<\varepsilon'$, $|g(s')-g(s'')|<\varepsilon'$. Denote by $I_i=[t_{i-1},t_i]$, $i=\overline{1,m}$, where $t_0=t-a<t_1<\ldots<t_m=t$, with $|t_i-t_{i-1}|<\delta$. Let σ_i, $\tau_i\in\mathbf{I}_i$ be points such that $\|G(t,\sigma_i)\|=\sup\{\|G(t,s)\|;\ s\in\mathbf{I}_i\}$, $g(\tau_i)=\sup\{g(s):s\in\mathbf{I}_i\}$, and let $c_1=\sup\{\|G(t,s)\|;\ t-a\leqslant s\leqslant t\}$, $c_2=\sup\{g(s):\ t-a\leqslant s\leqslant t\}$. By the integral mean value theorem,

$$\{\int_{t-a}^{t}Q(t)e^{(t-s)\Gamma}Q^{-1}(s)f(s,y(s))ds;\ y\in\mathbf{Y}\}\subset\sum_{i=1}^{m}(t_i-t_{i-1})\,\overline{\mathrm{conv}}\,(\cup\{G(t,s)f[\mathbf{I}_i\times\mathbf{Z}];\ s\in\mathbf{I}_i\}).$$

By (2.6) we have

$$\alpha(\cup\ G(t,s)f[\mathbf{I}_i\times\mathbf{Z}];\ s\in\mathbf{I}_i)\leqslant\sup\{\|G(t,s)\|;\ s\in\mathbf{I}_i\}\alpha(f[\mathbf{I}_i\times\mathbf{Z}]),$$

so

$$\alpha(\{\int_{t-a}^{t}G(t,s)f(s,y(s))ds:\ y\in\mathbf{Y}\})\leqslant\alpha(\sum_{i=1}^{m}(t_i-t_{i-1})\,\overline{\mathrm{conv}}\,(\cup\{G(t,s)f[I_i\times Z];$$

$$s\in I_i\})\leqslant\sum_{i=1}^{m}(t_i-t_{i-1})\sup\{\|G(t,s)\|;\ s\in I_i\}\alpha(f[\mathbf{I}_i\times\mathbf{Z}])\leqslant\sum_{i=1}^{m}(t_i-t_{i-1})\sup\{\|G(t,s)\|;\ s\in$$

$$\in \mathbf{I}_i \} \sup \{ |g(s)| : s \in \mathbf{I}_i \} \; h(\alpha(\mathbf{Z})) = h(\alpha(\mathbf{Z})) \sum_{i=1}^{m} (t_i - t_{i-1}) |G(t, \sigma_i)| |g(\tau_i)| \leqslant$$

$$\leqslant h(\alpha(\mathbf{Z})) \sum_{i=1}^{m} \int_{\mathbf{I}_i} (|G(t, \sigma_i) - G(t, s)| |g(\tau_i)| + |G(t, s)| \, |g(\tau_i) - g(s)| + |G(t, s)| |g(s)|) \leqslant$$

$$\leqslant h(\alpha(\mathbf{Z})) [\int_{t-a}^{t} |G(t, s)| |g(s)| ds + a(c_1 + c_2)\varepsilon'].$$

and (3.5) follows because ε' is arbitrary. From (3.3) we have

$$\alpha(F[\mathbf{Y}](t)) < \varepsilon + h(\alpha(\mathbf{Z})) \int_{-\infty}^{t} |Q(t) e^{(t-s)\Gamma} Q^{-1}(s)| |g(s)| ds.$$

Since $\mathbf{Y}$ is equicontinuous and bounded, we can apply Ambrosetti's result ([1], Lemma 2.1) to get

$$\alpha(\mathbf{Z}) \sup \{ \{\alpha(\mathbf{Y}(s)) : t - a \leqslant s \leqslant t\} = \sup \{ \alpha(\mathbf{Y}(t)) : t \in \mathbf{R} \} \equiv \Phi(\mathbf{Y}).$$

Hence, by $e)$

$$\alpha(F[\mathbf{Y}](t)) < \varepsilon + Lh(\Phi(\mathbf{Y})), \text{ i.e., } \alpha(F[\mathbf{Y}](t)) \leqslant Lh(\Phi(\mathbf{Y})).$$

Thus, all assumptions of the fixed-point theorem are satisfied. It follows that F has a fixed point in X and this ends the proof.

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ON ISOPTIC CURVES

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1. Introduction. An isoptic curve is a locus of vertices of a constant angle formed by two tangent lines to a given convex curve. Such curves were considered in the classical geometry (see [4], [1], [5]) in various context. For instance, the detailed properties of isoptic curves of algebraic curves are given in [1]. Some problems connected with isoptic curves are considered in the papers of Wunderlich (see e.g. [8]). In the present paper we give some global properties of isoptic curves of ovals. Next, the convexity of isoptic curves of an ellipse is considered. Moreover, using isoptic curves some Crofton-type integral formulas are derived.

2. Preliminaries. In this paper we will consider the family $\mathscr{M}$ of all positively oriented ovals, i.e., C^2, plane closed simple curves with the positive curvature (see [3]). Let us consider an oval $C \in \mathscr{M}$, $s \rightarrow z(s) = x(s) + iy(s)$, $0 \leqslant s \leqslant L$, parametrized by the arc length. The tangent vector to C and the curvature at a point $z(s)$ will be denoted by $T(s)$ and $k(s)$, respectively. For a fixed $a \in (0, L)$ there exists $\alpha \in (0, 2\pi)$ such that $T(a) = e^{i\alpha} T(0)$. The solution of the differential equation

(1) $$\varphi' = \frac{k}{k \circ \varphi}$$

with the initial condition $\varphi(0) = a$ satisfies the condition

(2) $$T \circ \varphi = e^{i\alpha} T,$$

(see [2]); we will say that φ does not change the angle α.

If an oval C is given in an arbitrary parametrization, say $t \rightarrow z(t)$, $0 \leqslant t \leqslant A$, then the function φ satisfies the differential equation

(3) $$\dot{\varphi} = \frac{k}{k \circ \varphi} \cdot \frac{|\dot{z}|}{|\dot{z} \circ \varphi|}.$$

The solution of (3) is given by the formula

(4) $$\varphi(t) = K^{-1}(K(t) + K(\varphi(0))),$$

where

(5) $$K(t) = \int_0^t k(r) |\dot{z}(r)| dr.$$