$$\in \mathbf{I}_i \{\sup\{g(s) :\ s \in \mathbf{I}_i\}\ h(\alpha(\mathbf{Z})) = h(\alpha(\mathbf{Z})) \sum_{i=1}^{m} (t_i - t_{i-1}) |G(t, \sigma_i)| g(\tau_i) \leqslant$$

$$\leqslant h(\alpha(\mathbf{Z})) \sum_{i=1}^{m} \int_{\mathbf{I}_i} (|G(t, \sigma_i) - G(t, s)| g(\tau_i) + |G(t, s)| |g(\tau_i) - g(s)| + |G(t, s)| g(s)) \leqslant$$

$$\leqslant h(\alpha(\mathbf{Z})) [\int_{t-a}^{t} |G(t, s)| g(s) ds + a(c_1 + c_2) \varepsilon'],$$

and (3.5) follows because ε' is arbitrary. From (3.3) we have

$$\alpha(F[\mathbf{Y}](t)) < \varepsilon + h(\alpha(\mathbf{Z})) \int_{0}^{t} |Q(t) e^{(t-s)\Gamma} Q^{-1}(s)| g(s) ds.$$

Since $\mathbf{Y}$ is equicontinuous and bounded, we can apply Ambrosetti's result ([1], Lemma 2.1) to get

$$\alpha(\mathbf{Z}) \sup \{\{\alpha(\mathbf{Y}(s)) ;\ t - a \leqslant s \leqslant t\} = \sup\{\alpha(\mathbf{Y}(t)) ;\ t \in \mathbf{R}\} \equiv \Phi(\mathbf{Y}).$$

Hence, by $e)$

$$\alpha(F[\mathbf{Y}](t)) < \varepsilon + Lh(\Phi(\mathbf{Y})),\ \text{i.e.,}\ \alpha(F[\mathbf{Y}](t)) \leqslant Lh(\Phi(\mathbf{Y})).$$

Thus, all assumptions of the fixed-point theorem are satisfied. It follows that F has a fixed point in X and this ends the proof.

REFERENCES

1. Ambrosetti, A. — *Un theorema di esistenza per le equazioni differenziali negli spazi di Banach*, Rend. Sem. Univ. Padova, 39 (1967), 349—360

2. Daletskii, Yu. L., Krein M.G. — *Stability of solutions of the differential equations in Banach spaces*, Nauka, Moskva, 1970 (Russian)

3. Dawidowski, M., Rzepecki B. — *On bouded solutions of nonlinear differential equations in Banach spaces*, Demonstratio Mathematica, 1, XVIII (1985), 91—102

4. Deimling, D. — *Ordinary differential equations in Banach spaces*, Lecture Notes in Math. 596, Springer. Verlag, Berlin, 1977

5. Ezeilo, J.O.C. — *Almost-periodic solutions of nonlinear systems of differential equations*, Contrib. to Diff. Eqs., III, 3, 1965

6. Goebel, K., Rzymowski, W. — *An existence theorem for equation x′ = f(t, x) in Banach spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 18 (1970), 367—370

7. Iftode, V. — *A-dissipativity for periodic solutions in ordinary differential equations*, Revue Roumaine Math. Pures Appl., 6, XXVIII (1983), 513—521

8. Martin, R.H. — *Nonlinear operators and differential equations in Banach spaces*, New York, 1976

9. Masera, J.L., Schäfefer J.J. — *Linear differential equations and functions spaces*, Academic Press, New York and London, 1966

10. Rzepecki, B. — *Applications of Sadovskii's fixed-point theorem to differential and integral equations in Banach spaces*, Demonstratio Mathematica, 4, XIX (1986), 993—1 009

11. Sadovskii, B.N. — *Limit compact and densifying operators*, Uspehi Math. Nauk, 27 (1972), 81—146 (Russian)

12. Talpalaru, P. — *Solutions périodiques et presque-périodiques des systèmes différentiels*, An. şt. Univ. Iaşi, Secţ. I-Matem., 2, XV (1969), 375—385

13. Talpalaru, P. — *Quelques problèmes qualitatifs de la théorie des équations différentielles dans l'espace de Banach*, An. şt. Univ. Iaşi, Secţ. I a Matem., 2, XVI (1970), 327—337

14. Villari, G. — *Contributi allo studio dell'esistenza di soluzioni periodiche per i sistemi di equazioni differenziali*, Ann. Mat. Pura Appl., (IV), LXIX (1965), 171—190

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Polytechnic Institute of Iassy
Department of Mathematics
6600 Iaşi, Romania

An. şt. Univ. „Al. I. Cuza", Iaşi
Matematică, 36 (1990)

ON ISOPTIC CURVES

BY

KORNEL BENKO*, WALDEMAR CIESLAK, STANISŁAW GÓŹDŹ
and WITOLD MOZGAWA**

1. Introduction. An isoptic curve is a locus of vertices of a constant angle formed by two tangent lines to a given convex curve. Such curves were considered in the classical geometry (see [4], [1], [5]) in various context. For instance, the detailed properties of isoptic curves of algebraic curves are given in [1]. Some problems connected with isoptic curves are considered in the papers of Wunderlich (see e.g. [8]). In the present paper we give some global properties of isoptic curves of ovals. Next, the convexity of isoptic curves of an ellipse is considered. Moreover, using isoptic curves some Crofton-type integral formulas are derived.

2. Preliminaries. In this paper we will consider the family $\mathcal{M}$ of all positively oriented ovals, i.e., C^2, plane closed simple curves with the positive curvature (see [3]). Let us consider an oval $C \in \mathcal{M}$, $s \to z(s) = x(s) + iy(s)$, $0 \leqslant s \leqslant L$, parametrized by the arc length. The tangent vector to C and the curvature at a point $z(s)$ will be denoted by $T(s)$ and $k(s)$, respectively. For a fixed $a \in (0, L)$ there exists $\alpha \in (0, 2\pi)$ such that $T(a) = e^{i\alpha} T(0)$. The solution of the differential equation

$$\varphi' = \frac{k}{k \circ \varphi} \tag{1}$$

with the initial condition $\varphi(0) = a$ satisfies the condition

$$T \circ \varphi = e^{i\alpha} T. \tag{2}$$

(see [2]); we will say that φ does not change the angle α.

If an oval C is given in an arbitrary parametrization, say $t \to z(t)$, $0 \leqslant t \leqslant A$, then the function φ satisfies the differential equation

$$\dot{\varphi} = \frac{k}{k \circ \varphi} \cdot \frac{|\dot{z}|}{|\dot{z} \circ \varphi|}. \tag{3}$$

The solution of (3) is given by the formula

$$\varphi(t) = K^{-1}(K(t) + K(\varphi(0))). \tag{4}$$

where

$$K(t) = \int_{0}^{t} k(r) |\dot{z}(r)| dr. \tag{5}$$

For the further purpose we introduced the following vectors and functions

(6) $$T=\frac{\dot z}{|\dot z|},\quad N=iT,\quad p=z-z\circ\varphi,\quad d=[p,T],\quad D=[p,N],$$

where $[v,s]=ad-bc$ for complex numbers $v=a+bi$, $w=c+di$.

We note that

(7) $$|p|^2=d^2+D^2.$$

3. Convexity of isoptic curves. The locus of points from which one can see an oval C under a fixed angle α will be denoted by C_α. Let φ denote the solution of (1) which does not change the angle $\pi-\alpha$. Then the isoptic curve C_α, $t\to z_\alpha(t)$, has the equation

(8) $$z_\alpha=z-(d\operatorname{ctg}\alpha+D)T.$$

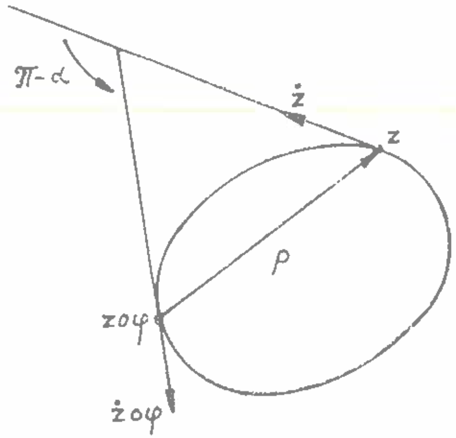


Fig. 1

Making use of (6) and the equality

(9) $$T=ki\dot z=k|\dot z|iT$$

we get the system of differential equations

(10) $$\begin{cases}\dot d=k|\dot z|D+\dot\varphi|\dot z\circ\varphi|\sin\alpha,\\ \dot D=-k|\dot z|d-\dot\varphi|\dot z\circ\varphi|\cos\alpha+|\dot z|.\end{cases}$$

The equalities (8)—(10) imply

(11) $$\frac{1}{k|\dot z|}\dot z_\alpha=(d-D\operatorname{ctg}\alpha)T-(d\operatorname{ctg}\alpha+D)iT$$

and

(12) $$\frac{1}{|\dot z|}\left(\frac{1}{k|\dot z|}\right)^{\cdot}\dot z_\alpha+\frac{1}{k|\dot z|^2}\ddot z_\alpha=(2Dk+2dk\operatorname{ctg}\alpha+\frac{k}{k\circ\varphi}\,\frac{1}{\sin\alpha}-\operatorname{ctg}\alpha)T+$$
$$+(2dk-2Dk\operatorname{ctg}\alpha-1)iT.$$

In view of (11) and (12) we get

$$k_\alpha=\frac{[\dot z_\alpha,\ddot z_\alpha]}{|\dot z_\alpha|^3}=\frac{\sin\alpha}{k|p|^3}\left[2d^2k+2D^2k-d+\frac{k}{k\circ\varphi}(d\cos\alpha+D\sin\alpha)\right].$$

Let us note that

$$D\dot d-\dot Dd=-|\dot z|\left[kd^2+kD^2-d+\frac{k}{k\circ\varphi}(d\cos\alpha+D\sin\alpha)\right].$$

Thus we have

$$k_\alpha=\frac{\sin\alpha}{k|p|^3}\left[kd^2+kD^2-\frac{D\dot d-\dot Dd}{|\dot z|}\right]=\frac{\sin\alpha}{k|p||\dot z|}\left[k|\dot z|-\left(\operatorname{arc\,tg}\frac{D}{d}\right)^{\cdot}\right].$$

This formula implies the following theorem.

Theorem 1. *The isoptic curve C_α of an oval C is an oval iff the inequality*

(13) $$k-\frac{1}{|\dot z|}\left(\operatorname{arc\,tg}\frac{D}{d}\right)^{\cdot}>0$$

holds.

The above theorem suggests that for a fixed oval C there exists an angle $\alpha(C)$ such that for all $\alpha>\alpha(C)$ isoptic curves C_α are ovals (the angle $\alpha(C)$ will be called the limit angle if it exists).

4. Ellipses. We discuss this problem when our oval is an ellipse. Testing the problem on the computer we observed the following statement.

If $1<a/b<\sqrt2$, then all isoptic curves of the ellipse $x^2/a^2+y^2/b^2=1$ are convex. [cf. picture]. This leads us to the following theorem:

Theorem 2. *If C is the ellipse $x^2/a^2+y^2/b^2=1$, then the limit angle $\alpha(C)$ is given by the formula*

(14) $$\frac{\cos\alpha(C)}{1+\cos\alpha(C)}=\frac{b^2}{a^2}.$$

If $1<a/b<\sqrt2$, then all isoptic curves are convex.

Proof. Let us consider the ellipse C

$$x(t)=a\cos t,\quad y(t)=b\sin t,\quad 0\leq t\leq 2\pi.$$

Then its isoptic curve C_α is given by

$$x(t)=a\,\frac{\sin\varphi(t)-\sin t}{\sin(t-\varphi(t))},\qquad y(t)=b\,\frac{\cos t-\cos\varphi(t)}{\sin(t-\varphi(t))},$$

where φ is given by the formula

$$\varphi(t)=\operatorname{arc\,tg}\frac{ab\sin t\cos\alpha+b^2\cos t\sin\alpha}{ab\cos t\cos\alpha-a^2\sin t\sin\alpha}.$$

We will look for $x''(t)$ at a point t_0 such that $\varphi(t_0)=-t_0$. The number $x''(t_0)$ has to be zero for the limit angle $\alpha(C)$. The condition $x''(t_0)=0$ leads to the formula (14). We have

$$x' = \frac{1-\cos(t-\varphi)}{\sin^2(t-\varphi)}\,[\varphi'\sin t + \sin\varphi]$$

and

$$\varphi' = \frac{B^2\cos^2\varphi}{(-\sin t\sin\alpha + B\cos t\cos\alpha)^2},$$

where $B = b/a$. Hence we obtain

$$x'' = \frac{(1-\varphi')}{\sin^4(t-\varphi)}\,[\sin^3(t-\varphi) - (1-\cos(t-\varphi))\sin 2(t-\varphi)]\times$$

$$\times[\varphi'\sin t + \sin\varphi] + \frac{1-\cos(t-\varphi)}{\sin^2(t-\varphi)}\,[\varphi'' + \varphi'\cos t + \varphi'\cos\varphi]$$

and

$$\varphi'' = \varphi'\,\frac{1}{(-\sin t\sin\alpha + B\cos t\cos\alpha)^2}\times$$

$$\times[-B^2\sin^2\alpha - \sin^2\alpha\sin 2t + B\cos 2t\sin 2\alpha + B^2\sin 2t\cos^2\alpha].$$

The condition $\varphi(t_0) = -t_0$ implies equalities

$$\varphi'(t_0) = 1$$

$$\varphi''(t_0) = 2\,\frac{1+\cos\alpha}{B\sin\alpha(1+2\cos\alpha)}\left[B^2 - \sin^2\alpha + B^2\cos^2\alpha + \right.$$

$$\left. + \cos\alpha\,\frac{\sin\alpha - B^2(1+\cos\alpha)^2}{1+\cos\alpha}\right].$$

Thus we have

$$0 = x''(t_0) = \frac{1-\cos 2t_0}{\sin^2 2t_0}\,[\varphi''(t_0)\sin t_0 + 2\varphi'(t_0)\cos t_0] =$$

$$= \sin t_0\,\frac{1-\cos 2t_0}{\sin^2 2t_0}\,[\varphi''(t_0) + 2\operatorname{ctg} t_0]$$

and

$$B^2 = \cos\alpha/(1+\cos\alpha).$$

5. The sine theorem for isoptic curves. Let C, $s\to z(s)$ for $0\leqslant s\leqslant L$ be an oval. Then its isoptic curve C_α is expressed in the following form

(15) $$s\to z_\alpha(s) = z(s) - (d(s)\operatorname{ctg}\alpha + D(s))\,T(s),$$

and equations (10) have the form

(16) $$d' = kD + \varphi'\sin\alpha, \qquad D' = -kD + 1 - \varphi'\cos\alpha.$$

Let us introduce the vectors:

(17) $$q_1 = z_\alpha - z, \quad q_2 = z_\alpha - z\circ\varphi$$

and the angles α_1, α_2 as on the picture (Fig. 2).

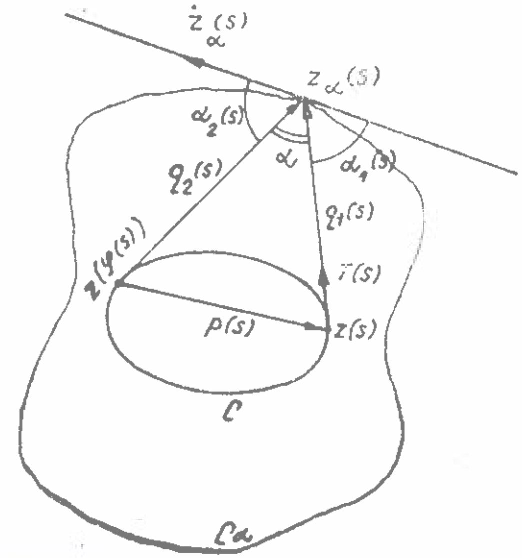


Fig. 2

Now we are in the position to prove the sine theorem for isoptic curves of an oval.

Theorem 3. *For any oval C and its isoptic curve C_α the following formula holds*

(18) $$\frac{|q_1(s)|}{\sin\alpha_1(s)} = \frac{|q_2(s)|}{\sin\alpha_2(s)} = \frac{|p(s)|}{\sin\alpha}$$

for $0\leqslant s\leqslant L$.

Proof. From the considerations concerning equation (8) it follows

(19) $$|q_1| = -(d\operatorname{ctg}\alpha + D), \qquad |q_2| = -\frac{d}{\sin\alpha}.$$

Moreover from (11) we obtain

$$|z'_\alpha| = k[(d - D\operatorname{ctg}\alpha)T - (d\operatorname{ctg}\alpha + D)iT],$$

thus

$$|z'_\alpha| = k|p|/\sin\alpha.$$

From the above considerations we have

$$\sin\alpha_1 = \frac{[T, z'_\alpha]}{|z'_\alpha|} = \frac{-\sin\alpha(d\operatorname{ctg}\alpha + D)}{|p|} = \frac{\sin\alpha|q_1|}{|p|}$$

which means that

$$|q_1|/\sin\alpha_1 = |p|/\sin\alpha.$$

Similarly we obtain

$$|q_2|/\sin\alpha_2 = |p|/\sin\alpha.$$

It is remarkable that the expression $|p(s)|/\sin\alpha$ is equal to the diameter of the circle $c_\alpha(s)$ passing by the points $z(s)$, $z(\varphi(s))$ and $z_\alpha(s)$ (for a fixed s).

Theorem 4. *The tangent line to the isoptic curve at the point $z_\alpha(s)$ is tangent to the circle $c_\alpha(s)$, too.*

Proof. Let us denote by γ the angle formed by the tangent lines to the circle and the isoptic curve at $z_\alpha(s)$ (Fig. 3). By the fact that a circle is an isoptic curve of a segment we have

$$\frac{|q_1|}{\sin(\alpha_1-\gamma)}=\frac{|q_2|}{\sin(\alpha_2+\gamma)}=\frac{|p|}{\sin\alpha}.$$

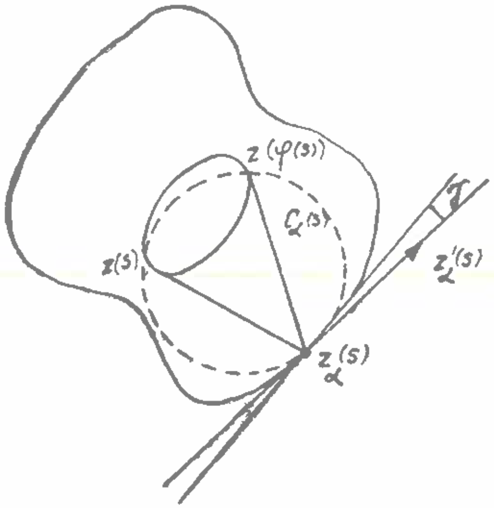


Fig. 3

On the other hand in virtue of the last theorem we have (18), thus $\gamma=0$, which finishes the proof.

6. Crofton-type integral formulas for ovals. Let us fix an oval C, $s\to z(s)$ for $0\leq s\leq L$. We denote by $e(C)$ the exterior of the region bounded by C. It is clear that each point $(x,y)\in e(C)$ belongs to exactly one isoptic curve C_α. Thus we can define a bijective mapping $f: P=\{(s,\alpha): 0\leq s\leq L, 0<\alpha<\pi\}\to e(C)$ by the formula

(20) $$(s,\alpha)\to z_\alpha(s)=z-(d\operatorname{ctg}\alpha+D)T.$$

We note that

(21) $$\frac{\partial}{\partial\alpha}d=\frac{\sin\alpha}{k\circ\varphi},\quad \frac{\partial}{\partial\alpha}D=\frac{-\cos\alpha}{k\circ\varphi},\quad \frac{\partial}{\partial\alpha}z_\alpha(s)=\frac{d}{\sin^2\alpha}T,$$
$$\frac{\partial}{\partial s}z_\alpha(s)=k[(d-D\operatorname{ctg}\alpha)T-(d\operatorname{ctg}\alpha+D)N].$$

Hence we obtain that the jacobian $Jf(s,\alpha)$ of f at (s,α) is equal to $\frac{kd}{\sin^2\alpha}(d\operatorname{ctg}\alpha+D)$, i.e.

(22) $$Jf(s,\alpha)=\frac{kd}{\sin^2\alpha}(d\operatorname{ctg}\alpha+D).$$

The formulas (19) and (22) imply that

(23) $$Jf(s,\alpha)=\frac{k}{\sin\alpha}|q_1||q_2|.$$

Thus f is a diffeomorphism. We recall the well-known two integral formulas (see [6])

$$\int\limits_{e(C)}\frac{\sin\omega}{t_1t_2}\,dxdy=2\pi^2,\qquad \int\limits_{e(C)}\frac{\sin\omega}{t_1t_2}(\rho_1+\rho_2)\,dxdy=2\pi L.$$

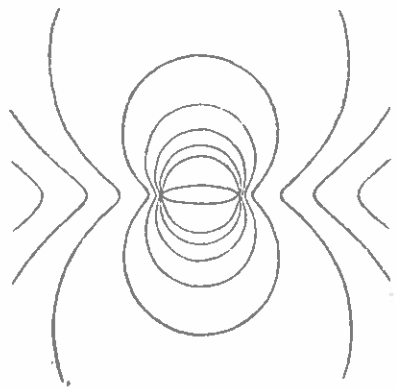
Making use of the diffeomorphism f we can easily obtain these formulas. Indeed, we have

$$\int\limits_{e(C)}\frac{\sin\omega}{t_1t_2}\,dxdy=\int_0^\pi\int_0^L\frac{\sin\alpha}{|q_1||q_2|}\,\frac{k}{\sin\alpha}|q_1||q_2|\,dsd\alpha=\pi\int_0^L k(s)ds=2\pi^2,$$

$$\int\limits_{e(C)}\frac{\sin\omega}{t_1t_2}(\rho_1+\rho_2)\,dxdy=\int_0^\pi\int_0^L\frac{\sin\alpha}{|q_1||q_2|}\left(\frac{1}{k}+\frac{1}{k\circ\varphi}\right)\frac{k}{\sin\alpha}|q_1||q_2|\,dsd\alpha=$$

$$=\int_0^\pi\int_0^L(1+\varphi'(s))\,dsd\alpha=2\pi L.$$

Using the diffeomorphism f we are able to show another integral formulas.



REFERENCES

1. Brocard H., Lemoyne T. — *Courbes géométriques remarquables*, t. II, Blanchard, Paris, 1967
2. Cieslak W., Zajac, J. — *The rosettes*, Math. Scand. 58 (1986), 114–118

3. Laugwitz D. — *Differential and Riemannian Geometry*, Academic Press, New York — London, 1965
4. Loria G. — *Spezielle algebraische und transzendentente ebene Kurven*, B.G. Teubner, Leipzig und Berlin, 1911
5. Savielov A.A. — *Plane Curves*, GIF-ML, Moscow, 1960 (Russian)
6. Santalo L.A. — *Integral geometry and geometric probability*, Encyclopedia of Mathematics and its Applications, Reading, Mass., 1976
7. Wieleitner H. — *Spezielle ebene Kurven*, Göschen, Leipzig, 1908
8. Wunderlich W. — *Kurven mit isoptischer Ellipse*, Monatsch. Math. 75 (1971), 346—362

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(*) *Taxon Computer GmbH*
Schlesische Str. 26
1 000 Berlin 36 (West)
(**) *Instytut Matematyki UMCS*
20—031 Lublin
Pl. M.C. Sklodowskiej 1
Poland

An. şt. Univ. „Al. I. Cuza", Iaşi
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SOME REMARKS ON GENERALIZED COHOMOLOGY THEORIES AND REAL K-THEORY

BY

CRISTIAN N. COSTINESCU

The aim of this paper is to give a solution to the following classical problem: the transition from the graded group associated with the filtration of an abelian group to the group itself. We take the group to be the real K-Theory of a finite CW-complex.

1. Let h be a generalized cohomology theory defined on the category $\mathcal{C}$ of finite based CW-complexes; we shall write an object of $\mathcal{C}$ simply by indicating the space X and very often the base point x_0 of X is naturally given by the context. Now we consider the generalized Atiyah-Hirzebruch spectral sequence associated with $h(X)$ and we recall that the limit term E_∞ is isomorphic to the graded group associated with the filtration:

$$\mathcal{F}_{p+1}(h^q(X)) = \mathrm{Ker}(h^q(X) \to h^q(X_p))$$

where X_p is the p-dimensional skeleton of X (for details see [4]). We also recall the following result

Theorem. (see [4], chapter III). *If h is an arbitrary cohomology theory then $h \otimes Q$ is also a cohomology theory and $E_2 \otimes Q \cong E_\infty \otimes Q$.*

Remark. The above isomorphisms show in fact that there exist the embeddings $\mathrm{Im}\, d_s \subset \mathrm{Tors}\, E_s$ for every s, where we denote by d_s the differential of the previous spectral sequence.

Thus we obtain immediately the following lemma

Lemma 1.1. *There is*

(i) *a natural group epimorphism:* $H^*(\mathrm{Tors}\, E_s) \to \mathrm{Tors}(H^* E_s = E_{s+1})$ *for any* s;

(ii) *a natural group monomorphism:*

$$\mathrm{Gr}^p(\mathrm{Tors}\, h^*(X)) \to \mathrm{Tors}(Gr^p h^*(X) \cong E_\infty^{p,*})$$

for any p.

Proposition 1.2. *Suppose that $X \in \mathcal{C}$ is without torsion (i.e.: $H^*(X; Z)$ has no torsion) and* $2 \cdot \mathrm{Tors}\, h^*(x_0) = 0$. *Then* $\mathrm{Tors}\, E_2(X) \cong (Z_2)^r$, $\mathrm{Tors}\, E_\infty(X) \cong (Z_2)^{r'}$, $\mathrm{Tors}\, h^*(X) \cong (Z_2)^{r''}$ *and* $r'' \leqslant r' \leqslant r$.

Proof. If $H^*(X; Z)$ has no torsion then $\mathrm{Tors}\, E_2(X)$ depends only on the torsion of $h^*(x_0)$. Using the previous lemma, the proposition follows.

Remark. We may replace the second assumption of the proposition 1.2. by $p \cdot \mathrm{Tors}\, h^*(x_0) = 0$, where p is prime.