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SOME REMARKS ON GENERALIZED COHOMOLOGY THEORIES AND REAL K-THEORY

BY

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The aim of this paper is to give a solution to the following classical problem: the transition from the graded group associated with the filtration of an abelian group to the group itself. We take the group to be the real K-Theory of a finite CW-complex.

1. Let h be a generalized cohomology theory defined on the category $\mathcal{C}$ of finite based CW-complexes; we shall write an object of $\mathcal{C}$ simply by indicating the space X and very often the base point x_0 of X is naturally given by the context. Now we consider the generalized Atiyah-Hirzebruch spectral sequence associated with $h(X)$ and we recall that the limit term E_∞ is isomorphic to the graded group associated with the filtration:

$$\mathcal{F}_{p+1}(h^q(X)) = \mathrm{Ker}(h^q(X) \to h^q(X_p))$$

where X_p is the p-dimensional skeleton of X (for details see [4]). We also recall the following result

Theorem. (see [4], chapter III). *If h is an arbitrary cohomology theory then $h \otimes Q$ is also a cohomology theory and $E_2 \otimes Q \cong E_\infty \otimes Q$.*

Remark. The above isomorphisms show in fact that there exist the embeddings $\mathrm{Im}\, d_s \subset \mathrm{Tors}\, E_s$ for every s, where we denote by d_s the differential of the previous spectral sequence.

Thus we obtain immediately the following lemma

Lemma 1.1. *There is*

(i) *a natural group epimorphism:* $H^*(\mathrm{Tors}\, E_s) \to \mathrm{Tors}(H^* E_s = E_{s+1})$ *for any* s;

(ii) *a natural group monomorphism:*

$$\mathrm{Gr}^p(\mathrm{Tors}\, h^*(X)) \to \mathrm{Tors}(Gr^p h^*(X) \cong E_\infty^{p,*})$$

for any p.

Proposition 1.2. *Suppose that $X \in \mathcal{C}$ is without torsion (i.e.: $H^*(X; Z)$ has no torsion) and $2 \cdot \mathrm{Tors}\, h^*(x_0) = 0$. Then $\mathrm{Tors}\, E_2(X) \cong (Z_2)^r$, $\mathrm{Tors}\, E_\infty(X) \cong (Z_2)^{r'}$, $\mathrm{Tors}\, h^*(X) \cong (Z_2)^{r''}$ and $r'' \leqslant r' \leqslant r$.*

Proof. If $H^*(X; Z)$ has no torsion then $\mathrm{Tors}\, E_2(X)$ depends only on the torsion of $h^*(x_0)$. Using the previous lemma, the proposition follows.

Remark. We may replace the second assumption of the proposition 1.2. by $p \cdot \mathrm{Tors}\, h^*(x_0) = 0$, where p is prime.

2. We consider the cohomology theory h to be the real K-Theory.

Proposition 2.1. *If $X \in \mathcal{C}$ is without torsion then:*

(i) $2 \cdot \operatorname{Tors} KO^*(X)=0$;

In the Atiyah-Hirzebruch spectral sequence associated with $KO^*(X)$:

(ii) *the differential $d_s: E_s^{p-s,\, q+s-1} \to E_s^{p,q}$ vanishes for $s \geq 2$ if $q \equiv 0, 4$ (modulo 8);*

(iii) *the differential $d_s: E_s^{p,q} \to E_s^{p+s,\, q-s+1}$ vanishes for $s \geq 2$, if $q \equiv -2, -4$ (modulo 8).*

Sketch of proof (ii) and (iii), simultaneously. If X is a point then $KO^*(X)=0$ and by using the Bott sequence we obtain that r is an epimorphism or c is a monomorphism (where we denote by r and c the Bott morphisms); this implies, because X is without torsion, that $E_2(r)$ is an epimorphism or $E_2(c)$ is a monomorphism. On the other hand, the Atiyah-Hirzebruch spectral sequence for $KU^*(X)$ collapses and so, the proposition follows. For details see [2].

Remark. Suppose $X \in \mathcal{C}$ and $KU^*(X)$ has no torsion (where we denote by KU the complex K-Theory). This is a sufficient condition which implies the first assertion of the previous proposition.

Theorem 2.2. *Suppose, in addition to the conditions in proposition 2.1., that $KO^*(X)$ is a free $KO^*(x_0)$-module. Then*

(i) *The Atiyah-Hirzebruch spectral sequence, associated with the Z_8-graded algebra $KO^*(X)$, collapses;*

(ii) *There exists the group isomorphism*

$$KO^q(X) \cong E_2^q(X) = \bigoplus_{p \geq 0} [H^p(X; Z) \times KO^{q-p}(x_0)]$$

for every q (i.e.: $E_2(X) \cong E_\infty(X) \cong KO^*(X)$ Z_8-graded).

Remark 2.3. Let G be a compact, connected, semisimple Lie group such that $\pi_1(G)=0$; if the conjugation in the reprezentation ring of G is the identity, then $KO^*(G)$ is a free KO (point) — module and his abelian structure is well-determined (see [6]).

Proof of Theorem 2.2. We recall that $\operatorname{Tors} KO^*(x_0)=(Z_2)^2$ (hence the proposition 1.2 is available), rank $KO^*(x_0)=2$ and because $H^*(X; Z)$ has no torsion, we have $E_2^{p,q}=H^p(X; Z) \times KO^q(x_0)$ (1) (see [1] and [5]). Due to a simple problem of the group extension we also have: rank $KO^*(X)=$ rank $[\operatorname{Gr} KO^*(X) \cong E_\infty(X)]$; using again the Hilton's theorem we obtain: rank $KO^*(X)=$ rank $E_2(X)=$ rank $H^*(X; Z) \cdot$ rank $KO^*(x_0)$ and this implies that $KO^*(X) \cong [KO^*(x_0)]$, where we denote by k the rank of $H^*(X; Z)$. The remark above shows that $\operatorname{Tors} KO^*(X) \cong [\operatorname{Tors} KO^*(x_0)]^k=(Z_2)^{2k}$ and on the other hand, via (1), we have $\operatorname{Tors} E_2(X) \cong (Z_2)^{2k}$. Now, by the proposition 1.2, we obtain that the torsion of $E_2(X)$ contains only permanent cycles; because we have too: $\operatorname{Tors} E_2(X)=H^*(X; Z) \otimes KO^{-1(-2)}(x_0)=E_2^{*,-1(-2)}$, this means that the differentials $d_s: E_s^{p,q} \to E_s^{p+s,\, q-s+1}$ and $d_s: E_s^{p-s,\, q+s-1} \to E_s^{p,q}$ vanish for $s \geq 2$, if $q \equiv -1, -2$ (modulo 8). The proposition 2.1 helps to finish the proof of (i).

Using again the lemma 1.1 (ii) we obtain that:

$E_\infty(X) \cong KO^*(X)$ Z_8-graded, which closes the proof of the theorem.

Example. Let X be a simplectic group $\operatorname{Sp}(n)$. Then the Z-cohomology of $\operatorname{Sp}(n)$ is an exterior algebra with generators X_i whose rank $=i$, i.e. $H^*(\operatorname{Sp}(n);$

$Z) \cong \bigwedge_Z(X_3, \ldots, X_{4n-1})$. This implies that $\operatorname{Sp}(n)$ has no torsion; using now the whose rank $=i$, i.e. $H^*(\operatorname{Sp}(n); Z) \cong \bigwedge (X_3, \ldots \varnothing\ X_{4n-1})$. This implies that $\operatorname{Sp}(n)$ remark 2.3. we obtain that $KO^*(\operatorname{Sp}(n))$ is a free $KO^*(x_0)$-module and thus the torsion of $KO^*(\operatorname{Sp}(n))$ is isomorphic to $(Z_2)^{2^n-1}$ (for details see [3]). So, having this information, we may apply the Theorem 2.2.

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