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SOME PROPERTIES OF K-TRIANGULAR MULTIVALUED SET FUNCTIONS

BY

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The concept of k-triangular set function was introduced in [1] by G. I. Areshkin, V. N. Alexiuk and V. M. Klimkin. Some important properties of this kind of non-additive set function are also studied by N. S. Guselnikov [6] and E. Pap [9], [11].

In the present paper we consider the notion of k-triangular multivalued set function and we point out some remarkable properties of this type of multivalued set function.

We also give a Dieudonné type theorem for k-triangular multivalued set functions with (B-M)-regular p-variations.

§ 1. **Basic notions, terminology and notations.** In what follows we shall use the terminology and notations as in [4]. Let Ω be a nonvoid set, $\mathcal{R}$ a ring of subsets of Ω and $\mathcal{P}(\Omega)$ the family of all subsets of Ω.

Let X be a Hausdorff locally convex topological vector space, τ its topology and $\mathcal{V}$ a base of absolutely convex closed neighborhoods of the origin in X.

Let us denote by $\mathcal{A}(X)$ the family of all nonempty subsets of X and by $\mathcal{C}(X)$ the family of all closed subsets of X.

Define on $\mathcal{A}(X)$ an equivalence relation „ρ" by

$$A \rho B \Leftrightarrow \bar{A} = \bar{B}, \tag{1}$$

where $\bar{A}$ is the closure of A in X. We see that the addition $(A, B) \to A + B$ is compatible with ρ. We can also introduce another addition on $\mathcal{A}(X)$, denoted by the symbol $\dot{+}$, defined by

$$A \dot{+} B = \overline{A + B}, \text{ for every } A, B \in \mathcal{A}(X). \tag{2}$$

Thus, $(\mathcal{A}(X), \dot{+})$ admits a factorization by „ρ" and the corresponding quotient semigroup can be identified with $(\mathcal{C}(X), \dot{+})$.

If $\mathcal{U}$ is the invariant uniformity on X compatible with τ, then the uniformity $\tilde{\mathcal{U}}$ on $\mathcal{A}(X)$ defined by the following base of vicinities

$$\mathcal{W}(U) = \{(A, B) \in \mathcal{A}(X) \times \mathcal{A}(X) ;\ A \subset B \dot{+} U,\ B \subset A \dot{+} U\},\ U \in \mathcal{V},$$

is said to be the *exponential extension* of $\mathcal{W}$ to $\mathcal{A}(X)$.

The topology on $\mathcal{A}(X)$ induced by $\tilde{\mathcal{U}}$ will be denoted by $\tilde{\tau}$.

Using the equivalence relation (1), we may identify $(\mathcal{C}(X), \tilde{\tau})$ with $(\mathcal{A}(X), \tilde{\tau})/\rho)$ and $(\mathcal{C}(X), \tilde{\mathcal{U}})$ with the separated uniform space associated with $(\mathcal{A}(X), \tilde{\mathcal{U}})$.

If X is metrizable, then $(\mathcal{A}(X), \tilde{\mathcal{U}})$ becomes a semi-metrizable space for the Hausdorff distance between sets.

If, in addition, X is complete, then so is $\mathcal{C}(X)$. If $(X, \mathcal{U})$ is a complete uniform space then so is $(\mathcal{C}(X), \tilde{\mathcal{U}})$.

We observe that $(\mathcal{C}(X), \dot{+}; \tilde{\mathcal{U}})$ is a Hausdorff uniform commutative semigroup with the unit $\{0\}$.

Furthermore the semigroup $\mathcal{C}(X)$ has the important property that the law of cancellation

$$A \dot{+} C = B \dot{+} C \Rightarrow A = B, \ (\forall) A, B, C \in \mathcal{C}(X), \tag{3}$$

holds in it.

Then, according to a result of Weber [15], there exists a family $\mathcal{D} = \{d\}$, of semi-invariant pseudo-metrics, which generates the uniformity $\tilde{\mathcal{U}}$.

That is for every d holds

$$d(A \dot{+} A_1, B \dot{+} B_1) \leq d(A, B) + d(A_1, B_1) \tag{4}$$

for all $A, A_1, B, B_1 \in \mathcal{C}(X)$.

Without any loss of generality we can suppose that the family $\mathcal{D}$ is directed, that is for every finite family $(d_i)_{i \in I} \subset \mathcal{D}$, there exists a pseudo-metric $d \in \mathcal{D}$ such that

$$\{(x, y) ; d(x, y) \leq \varepsilon\} \subset \bigcap_{i=1}^{n} \{(x, y) ; d_i(x, y) \leq \varepsilon\}.$$

Every pseudometric d induces a triangular functional p in the following way:

$$p(A) = d(A, \{0\}), \ (\forall) A \in \mathcal{C}(X). \tag{5}$$

The functional p has also the properties:

$$\begin{gathered} p(A \dot{+} B) \leq p(A) + p(B), \\ p(A \dot{+} B) \geq |p(A) - p(B)|, \ (\forall) A, B \in \mathcal{C}(X). \end{gathered} \tag{6}$$

Let us denote by $\mathcal{D}$ the family of all triangular functionals induced by the pseudometrics $d \in \mathcal{D}$.

In the following we shall consider correspondences from $\mathcal{R}$ to X, that is set functions μ defined on $\mathcal{R}$ taking values in $(\mathcal{A}(X), \dot{+})$, or, if sets with the same closure are identified, in the semigroup $(\mathcal{C}(X), \dot{+})$ with the property $\mu(\Phi) = 0$.

1.1. Definition. *Let μ be a correspondence from $\mathcal{R}$ to X. The function $|\cdot|_p : \mathcal{R} \to \bar{R}_+$ defined by*

$$|\mu|_p(E) = \sup_{\pi} \{ \sum_{A \in \pi} p(\mu(A)) \}, \ E \in \mathcal{R}, \tag{7}$$

where the supremum is taken over all partitions of E into a finite number of pairwise disjoint members of $\mathcal{R}$, is said to be the p-variation of μ.

It is easy to see that $|\mu|_p$ is a nonnegative, superadditive, non decreasing set function.

In addition, the inequality holds

$$p(\mu(A)) \leq |\mu|_p(A), \ (\forall) A \in \mathcal{R}. \tag{8}$$

1.2. Definition. *A family $\mathcal{M} = (\mu_i)_{i \in I}$ of correspondences from R to X is said to be uniformly exhaustive if for every disjoint sequence $(A_n)_{n \in N} \subset \mathcal{R}$ we have $\lim_{n \to \infty} \mu_i(A_n) = 0$, uniformly with respect to $i \in I$ (Here the limit is considered with respect to the topology $\tilde{\tau}$ induced by the uniformity $\tilde{\mathcal{U}}$).*

In the case where $\mathcal{M} = \{\mu\}$ we simply say that μ is exhaustive.

We remark that a correspondence μ from $\mathcal{R}$ to X (a family $\mathcal{M} = (\mu_i)_{i \in I}$ of correspondences from $\mathcal{R}$ to X) is exhaustive (uniformly exhaustive) if and only if for every disjoint sequence $(A_n)_{n \in N} \subset \mathcal{R}$ holds the equality $\lim_{n \to \infty} p(\mu(A_n)) = 0$ (resp. $\lim_{n \to \infty} p(\mu_i(A_n)) = 0$, $(\forall) i \in I$) for every $p \in \mathcal{D}$.

1.3. Definition. *A correspondence μ from $\mathcal{R}$ to X is said to be order continuous (briefly o-continuous), if $\lim_{n \to \infty} \mu(A_n) = 0$ for every sequence $(A_n)_{n \in N} \subset \mathcal{R}$ such that $A_n \downarrow \Phi$ (that is $A_n \supset A_{n+1}$, $(\forall) n \in N$ and $\lim_{n \to \infty} A_n = \Phi$).*

Also, a correspondence μ from $\mathcal{R}$ to X is order-continuous if and only if $\lim_{n \to \infty} p(\mu(A_n)) = 0$ for every sequence $(A_n)_{n \in N} \subset \mathcal{R}$ such that $A_n \downarrow \Phi$ and for every $p \in \mathcal{D}$.

1.4. Definition *. If μ is a correspondence from $\mathcal{R}$ to X and $p \in \mathcal{D}$ the function $\tilde{\mu}_p : \mathcal{R} \to \bar{R}_+$, defined by*

$$\tilde{\mu}_p(A) = \sup\{p(\mu(T)) ; \ T \subset A, \ T \in \mathcal{R}\}, \ (\forall) A \in \mathcal{R}$$

is said to be the p-semivariation of μ.

It is easy to see that: a) $\tilde{\mu}_p(\Phi) = 0$; b) if $A \subset B$, $A, B \in \mathcal{R} \Rightarrow \tilde{\mu}_p(A) \leq \tilde{\mu}_p(B)$; c) $p(\mu(A)) \leq \tilde{\mu}_p(A)$, $(\forall) A \in \mathcal{R}$.

§ 2. K – Triangular multivalued set functions Let p be a triangular functional from $\mathcal{D}$.

2.1. Definition. *The correspondence μ from $\mathcal{R}$ to X is said to be k-triangular with $k \in (0, \infty)$ if for every $A, B \in \mathcal{R}$ with $A \cap B = \Phi$ we have*

$$p(\mu(A)) - kp(\mu(B)) \leq p(\mu(A \cup B)) \leq p(\mu(A)) + kp(\mu(B)). \tag{9}$$

Examples.

1. Let $\mathcal{K}$ be a family of additive set functions $m : \mathcal{R} \to X$. Then the correspondence μ given by the formula $\mu(A) = \{m(A) ; m \in \mathcal{K}\}$ is k-triangular with $k = 1$.
2. Let μ be a correspondence from $\mathcal{R}$ to X such that
   a) $A, B \in \mathcal{R}$ with $A \subset B \Rightarrow \mu(A) \subset \mu(B)$,
   b) for every $A, B \in \mathcal{R}$ with $A \cap B = \Phi \Rightarrow \mu(A \cup B) \leq \mu(A) + \mu(B)$,
   c) μ is order continuous.

Then μ is a k-triangular multivalued set function.

From the definition of the p-semivariation of a k-triangular multivalued set function μ we easily obtain the following results:

2.2. Proposition. *If μ is a k-triangular multivalued set function from $\mathcal{R}$ to X, then for every $p\in\mathcal{P}$, the p-semivariation of μ has also the properties:*

a) $\tilde{\mu}_p(A\cup B)\le\tilde{\mu}_p(A)+k\tilde{\mu}_p(B)$, $(\forall)A,\ B\in\mathcal{R}$ *with* $A\cap B=\Phi$;

b) if μ is exhaustive, then $\tilde{\mu}_p$ is finite;

c) if μ is exhaustive, then $\tilde{\mu}_p$ is exhaustive;

d) if μ has the property that for every sequence $(A_n)_{n\in N}$ with $A_n\to\Phi$ holds $\lim_{n\to\infty} p(\mu(A_n))=0$, then $\tilde{\mu}_p$ has the same property;

e) if μ is σ-continuous, then for every disjoint sequence $(A_n)_{n\in N}\subset\mathcal{R}$ with $\bigcup_{n=1}^{\infty} A_n\in\mathcal{R}$ holds the inequality

$$\tilde{\mu}_p\Big(\bigcup_{n=1}^{\infty} A_n\Big)\le k\sum_{n=1}^{\infty}\tilde{\mu}_p(A_n),$$

that is $\tilde{\mu}_p$ is k-σ-subadditive.

2.3. Proposition. *I) A family $(\mu_i)_{i\in I}$ of k-triangular multivalued set function is uniformly exhaustive if and only if the family $(\tilde{\mu}_i)_{i\in I}$ of their p-semivariations is uniformly exhaustive.*

II) If the family $(\mu_i)_{i\in I}$ of k-triangular multivalued set functions is uniformly exhaustive, then for every sequence $(E_n\nearrow)$ (resp. $E_n\searrow$) from $\mathcal{R}$ and for every $\varepsilon>0$, there exists $m_0\in N$ such that

$$(\tilde{\mu}_i)_p(E_{m_0}\Delta E_n)<\varepsilon,\quad (\forall)n>m_0,\quad (\forall)\ i\in I.$$

2.4. Theorem. *Let $(\mu_i)_{i\in I}$ be an uniformly exhaustive family of k-triangular multivalued set functions from $\mathcal{R}$ to X.*

If for every $p\in\mathcal{P}$ and every $A\in\mathcal{R}$, there exists $M>0$ such that

(10) $$p(\mu_i(A))\le M,\ (\forall)\ i\in I,$$

then the family of their p-semivariations $\{(\tilde{\mu}_i)_p\}$ is uniformly bounded on $\mathcal{R}$.

Proof. Let us define

(11) $$\mu_0(A)=\sup_{i\in I} p(\mu_i(A)),\quad (\forall)\ A\in\mathcal{R}$$

and

(12) $$\lambda_0(A)=\sup\{p(\mu_0(B))\ ;\ B\subset A,\ B\in\mathcal{R}\},\ A\in\mathcal{R}.$$

We observe that μ_0 is a nonnegative k-triangular exhaustive set function and λ_0 is its semi-variation.

Since

$$p(\mu_i(A))\le(\tilde{\mu}_i)_p(A)\le\lambda_0(A),\quad (\forall)A\in\mathcal{R},\quad (\forall)\ i\in I,$$

it is sufficient to prove that λ_0 is bounded on $\mathcal{R}$.

Suppose that it is not true. Then there exists a sequence $\{A_n\}_{n\in N}\subset\mathcal{R}$ such that

(13) $$\lambda_0(A_j)\ge j+1,\ j=1,2,\ldots$$

Let us take $B_j=\bigcup_{i=1}^{j} A_i$, $j\in N$. We remark that $B_j\subset B_{j+1}$, $(\forall)\ j\in N$, and then by Propossition 2.3 there exists $m_0\in N$ such that

(14) $$\lambda_0(B_j-B_{m_0})<1\ \text{ for } j>m_0.$$

But for $j>m_0$ we have

$$\lambda_0(A_j)-k\lambda_0(B_{m_0})\le\lambda_0(B_j)-k\lambda_0(B_{m_0})\le\lambda_0(B_j-B_{m_0}).$$

Then from (13) and (14) we obtain

$$k\lambda_0(B_{m_0})>j\ \text{ for } j>m_0.$$

Hence $\lambda_0(B_{m_0})=\infty$, which is not possible because, by virtue of proposition 2.2, $\lambda_0(A)<\infty$ for every $A\in\mathcal{R}$.

§3. A Dieudonné type theorem for k-triangular multivalued set functions. In [2] J. M. Belley and P. Morales introduced an abstract type of regularity for measures taking values in a topological group; this type of regularity was also generalized in [7] by J. L. Maria Gonzales for measures taking values in certain topological semigroups.

In [14] we adapted this concept of regularity for multimeasures and we called it the $(B-M)$-regularity.

In this section we shall prove a Dieudonné type theorem for families of k-triangular multivalued set functions with $(B-M)$-regular p-variations. In fact we shall use an easily modified type of $(B-M)$ — regularity, which we shall call $(B-M)^*$-regularity.

In what follows we shall denote by $\mathcal{S}$ a σ-algebra of subsets of Ω, by $\mathcal{O},\mathcal{F}$, two nonvoid classes of subsets of Ω and by $\alpha_1:\mathcal{O}\to\mathcal{P}(\Omega)$, $\alpha_2:\mathcal{F}\to\mathcal{P}(\Omega)$ two functions with the following properties:

a) $\mathcal{F}\subset\mathcal{S}$, $\mathcal{O}\subset\mathcal{S}$;

b) if $F_1,\ F_2\in\mathcal{F}\Rightarrow F_1\cap F_2\in\mathcal{F}$; if $O_1,\ O_2\in\mathcal{O}\Rightarrow O_1\cup O_2\in\mathcal{O}$;

c) for every $O\in\mathcal{O}\Rightarrow O\subset\alpha_1(O)$, for every $F\in\mathcal{F}\Rightarrow\alpha_2(F)\subset F$;

d) if $F_1,\ F_2\in\mathcal{F}\Rightarrow\alpha_2(F_1)\cap\alpha_2(F_2)\subset\alpha_2(F_1\cap F_2)$;

e) for every sequence $(O_n)\subset\mathcal{O}$ with $\bigcap_{n=1}^{\infty}\alpha_1(O_n)=\Phi$ there exists $n_0\in N$ such that $\bigcap_{i=1}^{n_0}\alpha_1(O_i)=\Phi$.

Let us also denote $\mathcal{F}_0=\{F\in\mathcal{F}\ ;\ \alpha_2(F)\subset F\}$.

3.1. Definition. *A k-triangular multivalued set function from $\mathcal{S}$ to X is said to be $(B-M)$-regular if for every $A\in\mathcal{S}$, every $p\in\mathcal{P}$ and every $\varepsilon>0$, there exist $O\in\mathcal{O}$ and $F\in\mathcal{F}$ such that*

$$\alpha_1(O)\subset A\subset\alpha_2(F)\ \textit{ and }\ p(\mu(H))<\varepsilon$$

for every $H\in\mathcal{S}$ with $H\subset F\setminus O$.

We remark that if the p-variation of the multifunction μ is regular in the sense of [2], then μ is $(B-M)$-regular.

3.2. Definition. *A k-triangular multivalued set function from $\mathcal{S}$ to X is said to be $(B-M)^*$-regular if the conditions of the definition 3.1. are satisfied with $F\in\mathcal{F}_0$ instead of $F\in\mathcal{F}$.*

3.3. Theorem. *If $k\geq 1$ and μ is a k-triangular multivalued set function from $\mathcal{S}$ to X with $(B-M)$-regular p-variation then μ is a k-σ-subadditive set function on $\mathcal{F}$, that is for every disjoint sequence $(F_n)_{n\in N}\subset\mathcal{F}$ with $\bigcup_{n=1}^{\infty}F_n\in\mathcal{F}$ holds the inequality*

$$p(\mu(\bigcup_{n=1}^{\infty}F_n))\leq k\sum_{n=1}^{\infty}p(\mu(F_n)). \tag{15}$$

Proof. First let us prove that for every sequence $(F_n)_{n\in N}\subset\mathcal{F}$ such that $F_n\supset F_{n+1}$ $(n\in N)$ with $\bigcap_{n=1}^{\infty}F_n=\Phi$ and for every $p\in\mathcal{P}$ we have

$$\lim_{n\to\infty}p(\mu(F_n))=0.$$

By virtue of $(B-M)$-regularity of the p-variation $|\mu|_p$ of μ, for every $\varepsilon>0$ there exists a sequence $(O_n)_{n\in N}$ such that

$$O_n\subset\alpha_1(O_n)\subset\alpha_2(F_n)\subset F_n \text{ and } |\mu|_p(F_m\setminus O_m)<\varepsilon/2^m\cdot k$$

Then there exists $n_0\in N$ such that for every $n\geq n_0$, $\bigcap_{j=1}^{n}O_j=\Phi$.

Let us take $n\geq n_0$ arbitrarily. Then we have

$$p(\mu(F_n))=p(\mu(F_n\setminus\bigcap_{j=1}^{n}O_j))=p(\mu(\bigcup_{j=1}^{n}(F_n\setminus O_j))\leq|\mu|_p(\bigcup_{r=1}^{n}(F_n\setminus O_j)).$$

But according to the k-triangularity of μ, $|\mu|_p$ is k-subadditive and then for every pair $A_1, A_2\in\mathcal{S}$ holds the inequality

$$|\mu|_p(A_1\cup A_2)\leq|\mu|_p(A_1)+k|\mu|_p(A_2).$$

From here, taking into account that $|\mu|_p$ is also monotonic we obtain

$$p(\mu(F_n))\leq k\sum_{j=1}^{n}|\mu|_p(F_j\setminus O_j)<\varepsilon$$

for all $n\geq n_0$, that is $\lim_{n\to\infty}p(\mu F_n))=0$.

Now let $(F_n)_{n\in N}$ be a sequence of disjoint sets from $\mathcal{F}$ with $\bigcup_{n=1}^{\infty}F_n\in\mathcal{F}$.

We observe that

$$p(\mu(\bigcup_{j=1}^{\infty}F_j))\leq k\sum_{j=1}^{n}p(\mu(F_j))+p(\mu(\bigcup_{j=n+1}^{\infty}F_j)).$$

By virtue of (15), we obtain for $n\to\infty$

$$p(\mu\bigcup_{j=1}^{\infty}F_j))\leq k\sum_{j=1}^{\infty}p(\mu(F_j)).$$

In what follows we shall suppose, in addition, that for every $(F_n)_{n\in N}\subset\mathcal{F}_0$ the set $F=\bigcup_{n=1}^{\infty}F_n\in\mathcal{F}_0$, for every $x\in\Omega$, $\{x\}\in\mathcal{O}$ and for every $F\in\mathcal{F}_0$ such that $\{x\}\subset F$, there exists a set $C\in\mathcal{O}$ such that $\Omega\setminus\alpha_1(C)\in\mathcal{F}_0$ and $\{x\}\subset\alpha_1(C)\subset F$.

We remark that these conditions are not too restrictive because for instance, in the case Ω a Hausdorff locally compact space, α_1 — the adherence operator and α_2 — the interior operator, $\mathcal{S}=\mathcal{B}(\Omega)$, the class of Borel sets of Ω, $\mathcal{F}_0$ the family of open sets and $\mathcal{O}$ — the class of closed sets, all these above conditions are satisfied.

We also admit the following natural condition, for the first time considered in [2].

(β) If $O\in\mathcal{O}$ and $(F_n)_{n\in N}\subset\mathcal{F}$ are such that $\alpha_1(O)\subset\bigcup_{n=1}^{\infty}\alpha_2(F_n)$, then there is a positive integer n_0 for which $\alpha_1(O)\subset\bigcup_{k=1}^{n_0}F_k$.

3.4. Theorem. *Let $\mathcal{M}$ be a family of k-triangular multivalued set functions defined on $\mathcal{S}$ with $(B-M)^*$-regular p-variations.*

If the set $\{p(\mu(F)); \mu\in\mathcal{M}\}$, is bounded for every $F\in\mathcal{F}_0$, then $\{p(\mu A))$; $\mu\in\mathcal{M}$, $A\in\mathcal{S}\}$ is a bounded set.

Proof. First we observe that we can reduce the proof to the case $k=1$. Hence let us prove the result for a family of 1-triangular multivalued set functions.

In fact we shall prove that every point in Ω belongs to a set $F\in\mathcal{F}_0$ for which holds

$$\sup\{p(\mu(A)); A\subset F, A\in\mathcal{S}, \mu\in\mathcal{M}\}<\infty. \tag{16}$$

Suppose that it is not true. Then there exists a point $x\in\Omega$ such that (16) does not hold for every $F\in\mathcal{F}_0$ for which $x\in F_0$.

Hence for $F\in\mathcal{F}_0$ such that $x\in F$ there exist $A\in\mathcal{S}$ with $A\subset F$ and $\mu_1\in\mathcal{M}$ for which the following inequality holds :

$$p(\mu_1(A))>4+\sup_{\mu\in\mathcal{M}}\{p(\mu(\{x\})\}. \tag{17}$$

But, by hypothesis, μ_1 has the p-variation $(B-M)^*$-regular and then there exist $O\in\mathcal{O}$ and $F'\in\mathcal{F}_0$ with $F'\subset F$ such that $\alpha_1(O)\subset A\subset F'$ and $p(\mu_1(L))<1$, for every $L\subset F'\setminus O$.

From the 1-triangularity of μ_1 we obtain

$$p(\mu_1(O))+p(\mu_1(A\setminus O))\geq p(\mu_1(A))$$

which together with (17) and $p(\mu_1(A\setminus O))<1$ implies :

$$p(\mu_1(O))>3+2\sup_{\mu\in\mathcal{M}}\{p(\mu(\{x\}))\}.$$

Let $O_1=O\cup\{x\}$. Then, from the 1-triangularity of μ_1 we have

$$p(\mu O_1))>3+\sup_{\mu\in\mathcal{M}}\{p(\mu(\{x\}))\}. \tag{18}$$

Because $|\mu_1|_p$ is $(B-M)^*$-regular there exists $\tilde{F}\in\mathcal{F}_0$ such that $F\supset\tilde{F}\supset O_1$ and $p(\mu_1(L))<1$ for every $L\subset\tilde{F}\setminus O_1$.

From (18) and

$$p(\mu_1(\tilde{F}))\geq p(\mu_1(O_1))-p(\mu_1(\tilde{F}\setminus O_1))$$

we obtain

(19) $$p(\mu_1(\tilde{F}))>2+\sup_{\mu\in\mathcal{M}}\{p(\mu(\{x\}))\}.$$

Using again the $(B-M)^*$-regularity of $|\mu_1|_p$ we get a set $\tilde{F}_1\in\mathcal{F}_0$ such that $\{x\}\subset\tilde{F}_1\subset\tilde{F}$ and

(20) $$p(\mu_1(L))<1$$

for every $L\subset\tilde{F}_1\setminus\{x\}$.

Now, let $C\in O$ such that $\Omega\setminus\alpha_1(C)\in\mathcal{F}_0$ and $\{x\}\subset\alpha_1(C)\subset\tilde{F}_1$.

Then we have

$$p(\mu_1(\alpha_1(C)))\leq\sup_{\substack{A\subset\alpha_1(C)\setminus\{x\}\\ A\in\mathcal{S}}}p(\mu_1(A))+p(\mu_1(\{x\}))\leq\sup_{\substack{B\subset\tilde{F}_1\setminus\{x\}\\ B\in\mathcal{S}}}p(\mu_1(B))+p(\mu_1(\{x\})).$$

Thus by (20) we get

(21) $$p(\mu_1((\alpha_1(C)))<1+\sup_{\mu\in\mathcal{M}}p(\mu(\{x\})).$$

Let $A_1=\tilde{F}\setminus\alpha_1(C)$. Evidently $A_1\subset F$ and $A_1\cap\alpha_1(C)=\Phi$. But

$$p(\mu_1(A_1))+p(\mu_1(C))\geq p(\mu_1(F))$$

and then, according to (19) and (21) we get

$$p(\mu_1(A_1))>1.$$

Now we shall continue the procedure. Thus, taking in (17) the constant 5 instead of 4 and taking into account that $\{x\}\subset\alpha_1(C)$ and the family $\mathcal{M}$ is not bounded on C we obtain two other disjoint sets $A_2\in\mathcal{F}_0$, $C_1\in O$ and a multifunction $\mu_2\in\mathcal{M}$ such that $x\in\alpha_1(C_1)$ and $\mu_2(A_2)>2$.

In this way, recurently we can construct a sequence $(\mu_i)_{i\in N}\subset\mathcal{M}$ and a sequence $(A_i)_{i\in N}$ of pairwise disjoint sets from $\mathcal{F}_0$ such that

(22) $$p(\mu_i(A_i))>i,\ (\forall)\, i\in N.$$

Let $A=\bigcup_{j=1}^{\infty}\alpha_1(A_j)$. Because $|\mu_i|_p$ is $(B-M)^*$-regular there exist $O\in O$ and $F\in\mathcal{F}_0$ such that

$$O\subset\alpha_1(O)\subset A\subset\alpha_2(F)=F$$

and for every $L\in\mathcal{S}$ with $L\subset F\setminus O$ holds the inequality

(23) $$|\mu_i|_p(L)<\varepsilon.$$

Using the property (β) we obtain $n_0\in N$ such that

$$\alpha_1(O)\subset\bigcup_{j=1}^{n_0}A_j=\bigcup_{j=1}^{n_0}\alpha_2(A_j).$$

Then for $m\geq n_0+1$ we have from (23):

$$p(\mu_i(A_m))\leq\sup_{L\subset A_m\cup(\bigcup_{j=1}^{n_0}A_j)\setminus O}p(\mu_1(L))\leq\sup_{L\subset\bigcup_{j=1}^{\infty}A_j\setminus O}p(\mu_1(L))\leq\sup_{L\subset F\setminus O}p(\mu_i(L))<\varepsilon.$$

This last inequality says that

(24) $$\lim_{n\to\infty}p(\mu_i(A_n))=0,\ (\forall) i\in N.$$

Now we can apply Swartz's lemma [15] for the infinite matrix $[x_{ij}]$, where $x_{ij}=p(\mu(A_j))/i$ and then we obtain a sequence (i_n) from N and a sequence (ε_n) of positive real numbers such that

(25) $$\sum_{k=1}^{n-1}x_{ii}=\left(\frac{1}{2}-\varepsilon_n\right)x_{i_n i_n}\quad\text{and}$$

(26) $$x_{i_n i_n}+b<\frac{\varepsilon_n}{2^i}x_{i_n i_n}\qquad(n\in N).$$

By virtue of theorem 3.3 we have

$$p(\mu_{i_n}(\bigcup_{k=1}^{\infty}A_{i_k}))\geq p(\mu_{i_n}(A_{i_n}))-\sum_{k=1}^{n-1}p(\mu_{i_n}(A_{i_k}))-\sum_{k=n+1}^{\infty}p(\mu_{i_n}(A_{i_k})).$$

Using (25) and (26) we get

$$\frac{p(\mu_{i_n}(\bigcup_{k=1}^{\infty}A_{i_k}))}{i_n}\geq x_{i_n i_n}-\sum_{k=1}^{n-1}x_{i_n i_k}-\sum_{k=n+1}^{\infty}x_{i_n i_k}\geq\frac{x_{i_n i_n}}{2},\ (n\in N).$$

which yields

$$p(\mu_{i_n}(\bigcup_{k=1}^{\infty}A_{i_k}))\geq p(\mu_{i_n}(A_{i_n}))/2,\quad(\forall)\, n\in N.$$

From (22) we obtain now

$$p(\mu_{i_n}(\bigcup_{k=1}^{\infty}A_{i_k}))\geq i_n/2,\quad(\forall)\, n\in N.$$

which contradicts the boundedness of the family $\mathcal{M}$ on $\mathcal{F}_0$; hence (16) is true and the theorem is proved.

Using the results of [12] and [14] we can easily obtain.

3.4. Corollary. *Let Ω be a Hausdorff locally compact space and $\mathcal{M}$ a family of k-triangular multivalued set functions μ from $\mathcal{B}$, the class of all Borel sets of*

Ω, to X. If every $\mu \in \mathcal{M}$ has the regular p-variation (in the sense of [3]) *and the set $\{p(\mu(D))x \,;\, \mu \in \mathcal{M}\}$ is bounded for every open set D, then*

$$\{p(\mu(B)) \,;\, \mu \in \mathcal{M},\ B \in \mathcal{B}\}$$

is a bounded set for every $p \in \mathcal{P}$.

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WEAK PSEUDOUNIFORM STRUCTURES

BY

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1. Introduction. In [6] we define the weak pseudouniform structure on $L^1(X, \mathcal{A}, \mu)$ and we obtain an extension of the classical result of weak compactness due to Dunford and Pettis.

The purpose of this paper is the sequential variant of this extension. In 2. we introduce the weak pseudouniform structure on a normed linear space and we generalize the part "weak compactness $\rightarrow$ weak sequentially compactness" of Eberlein-Šmulian's theorem. In section 3. we use this generalization to $L^1(X, \mathcal{A}, \mu)$.

2. Weak pseudouniform structure. Let $(X, \|.\|)$ be a normed linear space and X^* be the topological dual of X. For every $a \geq 0$, $x^* \in X^*$ and $\varepsilon > 0$, let us denote:

$$U^a_{x^*,\varepsilon} = \{(x, y) \in X \times X : |x^*(x-y)| < a\|x^*\| + \varepsilon\}.$$

For every $a \geq 0$, $\mathcal{S}_a = \{U^a_{x^*,\varepsilon} : x^* \in X^*,\ \varepsilon > 0\}$ is a subbase for a semiuniformity $\mathfrak{U}_a$ on X (see [2]). Let $\mathfrak{U} = \{\mathfrak{U}_a : a \geq 0\}$; then, for every $a \geq 0$ and $U \in \mathfrak{U}_a$, there exists $V \in \mathfrak{U}_{a/2}$ such that $V \circ V \subseteq U$. Furthermore, $a \leq b$ implies $\mathfrak{U}_a \supseteq \mathfrak{U}_b$. It follows that $\mathfrak{U}$ is a pseudouniformity on X with respect to the directed set $(R_+, \leq^{-1})$ (see [6]). We remark that $\mathfrak{U}_0$ is the weak uniformity on X and, for every $a \geq 0$, $\mathfrak{U}_a \subseteq \mathfrak{U}_0$. We say that $\mathfrak{U}$ is *the weak pseudouniformity* on X.

For every set $A \subseteq X$, we denote by $\bar{A}$ the closure of A in the norm topology and by $\bar{A}^a$ the closure of A in the semiuniform space $(X, \mathfrak{U}_a)$.

Proposition 1. *Let A be a subspace of the normed linear space X. Then, for every $a \geq 0$, $\bar{A} \subseteq \bar{A}^a \subseteq T(\bar{A}, a)$,* where $T(\bar{A}, a)$ is the closed sphere which has *$\bar{A}$ for center and a for radius.*

Proof. Obviously, $\bar{A} \subseteq \bar{A}^a$. Now, if $x \in \bar{A}^a - \bar{A}$, then $d(x, \bar{A}) = d > 0$. Hence there is a point $x^* \in X^*$ with $x^*(x) = 1$, $x^*(y) = 0$, for every $y \in \bar{A}$ and $\|x^*\| = 1/d$([5, II.3.12]). For any $\varepsilon > 0$ there is a point $y \in A$ such that $|x^*(x-y)| < a\|x^*\| + \varepsilon$. Therefore, $1 < a.1/d + \varepsilon$, so that $d \leq a$. It follows that $x \in T(\bar{A}, a)$.

Remark. If $a = 0$ than $\bar{A} = \bar{A}^0$, where $\bar{A}^0$ is the weak closure of A. So, we rediscover a classical result.

A set $A \subseteq X$ is **a-relatively compact (a-relatively sequentially compact)** *if every net in A has a cluster point (every sequence in A has a convergent subsequence) in semiuniform space $(X, \mathfrak{U}_a)$.*

We remark that the sequence (x_n) converges to x in semiuniform space $(X, \mathfrak{U}_a)$ if and only if $\overline{\lim\limits_n}\, |x^*(x_n - x)| \leq a.\|x^*\|$, for every $x^* \in X^*$.