5. N. Bokan — *Curvature tensors of Riemannian manifolds with connection without torsion*. Matematički Vesnik, 37 (1985), 356—364.
6. N. Bokan — *On the complete decomposition of curvature tensors of Riemannian manifolds with symmetric connection*, preprint 1988.
7. D. Chinea, C. Gonzalez — *A classification of almost contact metric manifolds*, preprint.
8. J. Dieudonné, J. Carell — *Invariant theory, old and new*. Adv. in Math. 4 (1970), 1—80.
9. A. Gray and L. M. Hervella — *The sixteen classes of almost Hermitian manifolds*, Ann. Mat. Pura Appl. 123 (1980), 35—58.
10. N. Iwahori — *Some remarks on tensor invariants of O(n), U(n), Sp(n)*. J. Math. Soc. Japan 10 (1958), 145—160.
11. D. Janssens and L. Vanhecke — *Almost contact structures and curvature tensors*, Kodai Math. J., 4, 1 (1981) 1—27.
12. D. L. Johson C — *Sectional curvature and curvature normal forms*. Michigan Math. J., 27 (1980), 275—291
13. P. Matzeu — *Su una caratterizzazione delle varietà normali*, Boll. U.M.I., VI, 1 (1985) 153—160.
14. P. Matzeu — *Decomposition of curvature tensors on normal almost contact manifolds*. Rend. Sem. Mat. Univers. Politecn. Torino 47(2) (1989), 179—210.
15. P. Matzeu, V. Oproiu — *C-projective curvature of normal almost contact manifolds*. Rend. Sem. Mat. Univers. Politecn. Torino, 45, 2 (1987), 41—58.
16. H. Mori — *On the decomposition of generalized K-curvature tensor fields*, Tohoku Math. J., 25 (1973), 225—235.
17. A. M. Naveira — *A classification of Riemannian almost product manifolds*. Rend. di. Mat., 3, VII, (1983), 577—592.
18. S. Nikčević — *On the decomposition of curvature tensor fields on Hermitian manifolds*. Coll. Diff. Geom., August 20—26, 1989, Eger (Hungary), Abstracts, p. 39, preprint 1989.
19. S. Nikcevic — *Induced representation of unitary group in curvature tensor fields on Hermitian manifolds*, to appear.
20. K. Nomizu — *On the decomposition of generalized curvature tensor fields*, Differential Geometry (in honor of K. Yano), Kinokuniya, Tokyo, 1972, pp 335—345.
21. I. M. Singer, J. A. Thorpe — *The curvature of 4-dimensional Einstein spaces*, Global Analysis (papers in honor of Kodaira), Univ. of Tokyo Press, Tokyo, 1969, p. 133—365.
22. M. Sitaramayya — *Curvature tensors in Kähler manifolds*, Trans. Amer. Math. Soc., 183 (1973), 341—351.
23. R. S. Strichartz — *Linear algebra of curvature tensors and their covariant derivatives*, Cana J. Math 40, 5 (1988), 110 —1143.
24. F. Tricerri, L. Vanhecke — *Decomposition of a space of curvature tensors on a quaternionic-Kahler manifold and spectrum theory*. Simon Stevin 53 (1979), 163—173.
25. F. Tricerri, L. Vanhecke — *Curvature tensors on almost Hermitian manifolds*, Trans. Amer. Math. Soc., 267, 2. (1981), 365—398
26. I. Vaisman — *Symplectic curvature tensors*. Mh. Math. 100, 299—327 (1985)
27. K. Yano — *Differential Geometry on Complex and Almost Complex Spaces*, Pergamon Press, New York 1965.

Received 8. I 1990

Departimento di Matematica.
Università di Cagliari
Via Ospedale 72, Cagliari Italia

University of Belgrade
Sindjelićeva 25, Belgrade
Yugoslavia

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA"
Tomul XXXVII, s. I a, Matematică, 1991, f. 1.

REGULAR VECTOR FIELDS AND CONNECTIONS ON COTANGENT BUNDLES

BY

V. OPROIU

The tangent and cotangent bundles of a smooth manifold are isomorphic vector bundles when they are studied from the differential-topological point of view. However, there are some essential differences when one considers their differential geometry. The tangent bundle has a naturally defined integrable almost tangent structure and one can define the notion of a semispray (second order differential equation vector field) on it. From a semispray on the tangent bundle one can derive easily a (nonlinear) connection on this bundle. On the contrary, the cotangent bundle has a naturally defined symplectic structure but one cannot define naturally something similar to the notions of almost tangent structure and semispray.

In this paper we study some geometric properties of the cotangent bundle T^*M of the smooth manifold M, related to the notions of adapted almost tangent structure and regular vector fields. An adapted almost tangent structure on T^*M is essentialy determined by a nondegenerate M-tensor field of type (0, 2) (see [9] for the definition of M-tensor fields). From the integrability condition for an adapted almost tangent structure on T^*M we get a regular M-vector field on T^*M. This M-vector field defines naturally an M-bundle (local) diffeomorphism) Φ from T^*M to the tangent bundle TM of M. Using this M-bundle diffeomorphism it is possible to transfer many results from the differential geometry of the tangent bundle to that of the cotangent bundle. A semispray on TM is transformed into a regular vector field on T^*M with the associated M-vector field defined by the components of the M-bundle diffeomorphism Φ. The natural almost tangent structure of TM is transformed into an integrable adapted almost tangent structure on T^*M. The connection derived from a semispray on TM is transformed into a connection on T^*M with many interesting geometric properties. In the case where the M-bundle diffeomorphism Φ is obtained as the (inverse of the) Legendre transformation defined by a regular Hamiltonian on T^*M i. e. a smooth function on T^*M with nondegenerate Hessian with respect to the cotangential coordinates) the regular vector field on T^*M corresponding to a semispray on TM defines a mechanical structure on T^*M. In this case there are another two symmetric connections on T^*M. If the regular vector field on T^*M is semi-Hamiltonian these three connections do coincide. Henceforth there are studied some geometric problems related to the connections defined on T^*M. It is shown that the Helmholtz conditions from the inverse problem of Lagrangian dynamics may be related to the problem for the image of a semispray on TM by Φ^{-1} to be a Hamilton vector field on T^*M. Further, there are studied the conditions under which a given regular vector field on T^*M is horizontal with respect to the corresponding connection or it is tangent to the images of M by a section in T^*M (i. e. by a 1-form on M).

Throughout this paper the summation convention is used and the indices h, i, j, k, l, m run over the range $\{1, ..., n\}$.

1. **Adapted almost tangent structures on T*M.** We begin by describing the geometric structures of the tangent and cotangent bundles TM, T^*M of a smooth (i. e. C^∞) n-dimensional manifold $M \cdot TM$, T^*M are $2n$-dimensional smooth manifolds and some local charts induced naturally from local charts on M may be used. Denote by $\tau: TM \to M$, $\pi: T^*M \to M$ the natural projections and let (U, x) be a local chart on M with the domain U and the coordinate function $x = [x^i]$; $i = 1, ..., n$. Then the local chart $(\tau^{-1}(U), (x, y))$ is induced on TM where $(x, y) = [x^i, y^i]$; $i = 1, ..., n$, with $x^i = x^i \circ \tau$ (by abuse of notation) and y^i are the vector space coordinates with respect to the natural local frame $(\partial/\partial x^1, ..., \partial/\partial x^n)$ in TM defined by (U, x). Similarly the local chart $(\pi^{-1}(u), q, p)$ is induced on T^*M where $(q, p) = [q^i, p_i]$; $i = 1, ..., n$, with $q^i = x^i \circ \pi$ and p_i are the vector space coordinates with respect to the natural local frame $(dx^1, ..., dx^n)$ in T^*M defined by (U, x). Throughout this paper we shall use only induced local charts on TM and T^*M. Due to the special from of the change rules of the induced local charts on TM, T^*M, it is possible to define the notion of M-tensor fields on these bundles (see [9]). An M-tensor field of type (k, l) on TM (on T^*M) is defined by sets of local coordinate components $T^{i_1 \dots i_k}_{j_1 \dots j_l}$; $i_1, ..., i_k, j_1, ..., j_l = 1, ..., n$, assigned to every induced local chart on TM (on T^*M) such that the change rule when a change of induced local charts on TM, (on T^*M) is performed, is the same with the change rule of the local coordinate components of a tensor field of type (k, l) on M. Every tensor field T on M may be thought of as an M-tensor field on TM or on T^*M with the same local coordinate components (constant on the fibres) in induced local charts as those of T in the original local charts on M. Remark that in the case of a tensor field T of type $(0, l)$ on M the corresponding M-tensor field on TM (on T^*M) may be identified with the pull-back of T by τ (by π).

Consider the natural almost tangent structure on TM defined by the tensor field J of type $(1, 1)$ on TM such that

$$J^2 = 0, \ \operatorname{Ker} J = \operatorname{Im} J = \operatorname{Ker} \tau_*, \ N_J = 0, \tag{1}$$

where N_J is the Nijenhuis tensor field of J. Recall that $\operatorname{Ker} \tau_* = VTM =$ the integrable vertical distribution on TM. The local coordinate expression of J in an induced local chart $(\tau^{-1}(U), (x, y))$ on TM is

$$J = \partial/\partial y^i \otimes dx^i. \tag{2}$$

In the case of the cotangent bundle T^*M we have the integrable vertical distribution $VT^*M = \operatorname{Ker} \pi_*$ with many properties similar to those of VTM but we have not a naturally defined almost tangent structure.

Definition. An adapted almost tangent structure on T^*M is an almost tangent structure J on T^*M such that

$$\operatorname{Ker} J = \operatorname{Im} J = VT^*M.$$

It follows easily that the local coordinate expression of an adapted almost tangent structure on T^*M is

$$J = g_{ij} \frac{\partial}{\partial p_i} \otimes dq^j, \tag{3}$$

where the components $g_{ij}(q, p)$ define an M-tensor field of type $(0, 2)$ on T_*M. Moreover, this M-tensor field is nondegenerate i. e. $\det(g_{ij}) \neq 0$.

Proposition 1. *The adapted almost tangent structure J on T^*M is integrable if and only if the M-tensor field g_{ij} satisfies the condition*

$$g_{hi} \frac{\partial g_{kj}}{\partial p_h} - g_{hj} \frac{\partial g_{ki}}{\partial p_h} = 0. \tag{4}$$

Proof. From the definition of the Nijenhuis tensor field N_J we get $N_J(\partial/\partial p_i, \partial/\partial p_j) = 0$, $N_J(\partial/\partial p_i, \partial/\partial q^j) = 0$ and the condition $N_J(\partial/\partial q^i, \partial/\partial q^j) = 0$ is equivalent to (4).

Consider now the local coordinate components $g^{ij}(q, p)$; $i, j = 1, ..., n$ defined as the entries of the inverse of the matrix $(g_{ij}(q, p))^{-1}$ i. e.

$$g_{ih} g^{jh} = g_{hi} g^{hj} = \delta^j_i. \tag{5}$$

The components g^{ij} define an M-tensor field of type $(2, 0)$ on T^*M. Remark that there is not a naturally defined tensor field on T^*M whose nontrivial local coordinate chmponents are just g^{ij}.

Proposition 2. *The adapted almost tangent structure J on T^*M is integrable if and only if there exists an M-vector field on T^*M defined by the components $\xi^i(q, p)$, such that*

$$g^{ij} = \frac{\partial \xi^j}{\partial p_i}. \tag{6}$$

Proof. We get easily that the integrability condition (4) is equivalent to

$$\frac{\partial g^{jk}}{\partial p_i} = \frac{\partial g^{ik}}{\partial p_j}. \tag{4'}$$

Then the components g^{ij} may be expressed by (6) on the domain $\pi^{-1}(U) \subset T^*M$. Using a smooth partition of unity on M we may get the desired M-vector field ξ^i, globally defined on T^*M.

The components $\xi^i(q, p)$ of an M-vector field on T^*M may be thought of as the vector space coordinates with respect to the natural local frame $(\partial/\partial x^1, ..., \partial/\partial x^n)$ of TM of a certain vector field on M depending on the parametrs p. Equivalently, $\xi^i(q, p)$ may be thought of as the last n components of an M-bundle map $\Phi: T^*M \to TM$ (i. e. $\tau \circ \Phi = \pi$; however Φ is not necessarily a vector bundle morphism i. e. it is not necessarily linear when restricted to the fibres). The local coordinate expression of Φ is

(7) $$x^i = q^i, \quad y^i = \xi^i(q, p).$$

The special local coordinate expression of Φ as well as the nondegeneracy condition in the definition of an integrable adpted almost tangent structure J on T^*M assures that Φ associated with the M-vector field corresponding to J is a local diffeomorphism defined, e. g. on a tubular neighborhood of the null section in T^*M.

An M-vector field on T^*M defined by the components $\xi^i(q, p)$ such that the matrix $[g^{ij}] = (\partial \xi^j / \partial p_i)$ is nondegeralerate is called regular. Every integrable adapted almost tangent structure on T^*M is defined by a regular M-vector field on T^*M and there exists a local M-bundle diffeomorphism Φ from T^*M to TM associated with this regular M-vector field.

Conversely, every M-bundle diffeomorphism $\Phi : T^*M \to TM$ defines a regular N-vector field on T^*M, hence an integrable adapted almost tangent structure on T^*M.

Remark. An integrable adpted almost tangent structure on T^*M (or, equivalently, a regular M-vector field on T^*M) there always exists. Consider a Riemannian metric on M expressed locally by $g_{ij} dx^i dx^j$. Consider the tensor field of type (2, 0) on M defined by the entries g^{ij} of the inverse matrix of (g_{ij}) and think of this tensor field as an M-tensor field on T^*M. Then the components $\xi^i = g^{ji} p_j$ define a regular M-vector field on T^*M. More generally, a regular Hamiltonian h on T^*M defines a regular M-vector field with the components $\xi^i = \partial h / \partial p_i$. A quite general procedure to get regular M-vector fields on T^*M can be described as follows. Consider a smooth diffeomorphism $F : \mathbf{R}^n \to \mathbf{R}^n$ such that its Jacobian matrix is symmetric and positive. Denote by $f^i(z)$ the components of F. Consider a local M-vector field on every domain $\pi^{-1}(U)$ of an induced local chart, defined by the components $\xi^i = f^i(p)$. Using a partition of unity on M we get a globally defined regular M-vector field on T^*M. The described procedure is closely related to the locally conformal Minkowski spaces in Finsler geometry.

2. Connections on T*M A (nonlinear) connection on T^*M is defined by an almost product structure Γ on T^*M (i. e. $\Gamma^2 = I$) such that the eigendistribution on T^*M corresponding to the eigenvalue -1 of Γ is exactly VT^*M (see [6], [7], [8], [2]). The eigendistribution on T^*M corresponding to the eigenvalue $+1$ of Γ is denoted by HT^*M and is called the horizontal distribution defined by Γ. The local coordinate expression of Γ is

(8) $$\Gamma = \frac{\partial}{\partial q_i} \otimes dq^i - \frac{\partial}{\partial p_i} \otimes dp_i - 2\Gamma_{ij} \frac{\partial}{\partial p_i} \otimes dq^j.$$

The components Γ_{ij} are called the coefficients of the connection Γ in the induced local chart $(\pi^{-1}(U), (q, p))$. A connection Γ on T^*M such that $\Gamma_{ij} = \Gamma_{ji}$ is called a symmetric connection. It follows easily that the property of a connection Γ on T^*M to be symmetric is independent of the induced local charts on T^*M.

The local vector fields $\partial / \partial p_i$; $i = 1, \ldots, n$ define a local frame in VT^*M while the local vector fields

(9) $$\frac{\delta}{\delta q_i} = \frac{\partial}{\partial q^i} - \Gamma_{ji} \frac{\partial}{\partial p_j}; \quad i = 1, \ldots, n$$

define a local frame in HT^*M. We have

(10) $$TT^*M = VT^*M \oplus HT^*M$$

and the vector fields $\partial / \partial p_i$, $\delta / \delta q^i$; $i = 1, \ldots, n$, define a local frame on T^*M adapted to the direct sum decomposition (10). The corresponding local dual frame in T^*T^*M is defined by δp_i, dq^i; $i = 1, \ldots, n$ where

(11) $$\delta p_i = dp_i + \Gamma_{ij} dq^j.$$

By using these notations the expression (8) of Γ becomes

(8′) $$\Gamma = \frac{\delta}{\delta q^i} \otimes dq^i - \frac{\partial}{\partial p_i} \otimes \delta p_i.$$

Remark. The conditions from the definition of a connection Γ on T^*M are equivalent to the conditions

(12) $$\Gamma \cdot J = -J, \quad J \cdot \Gamma = J$$

where J is an adapted (not necessarilly integrable) almost tangent structure on T^*M.

Let K be a vector field on T^*M with the local coordinate expression

(13) $$K(q, p) = \xi^i(q, p) \frac{\partial}{\partial q^i} + \varkappa_i(q, p) \frac{\partial}{\partial p_i}.$$

The components ξ^i define an M-vector field on T^*M, called the associated M-vector field of K. The vector field K is called regular if its associated M-vector field is regular. Assume that K is a regular vector field on T^*M and denote by J_ξ the integrable adapted almost tangent structure on T^*M defined by (3) by the M-vector field associated with K. There is a coordinate free description of J_ξ.

Proposition 3. *The integrable adapted almost tangent structure J_ξ on T^*M is characterized by the property*

(14) $$J_\xi [V, K] = V$$

for every vertical vector field V on T^*M (i. e. $V(q, p) \in VT^*M$ for every $(q, p) \in T^*M$).

Proof. By a straightforward computation in local coordinates.

Proposition 4. *The tensor field Γ_K of type (1, 1) on T^*M defined by*

(15) $$\Gamma_K = -\mathcal{L}_K J$$

defines a connection on T^*M where $\mathcal{L}_K$ denotes the Lie derivative with respect to K.

Proof. From the property $J_\xi^2 = 0$ we get $\Gamma_K \cdot J_\xi = -J_\xi \cdot \Gamma_K$. Next, because of (14)

$$\Gamma_K V = -(\mathcal{L}_K J_\xi) V = -[K, J_\xi V] + J_\xi [K, V] = -V$$

for every vertical vector field V on T^*M. Since J_ξ is surjective onto the set of vertical vector fields over T^*M it follows $\Gamma_K J_\xi = -J_\xi$. Thus Γ_K satisfies the conditions (12) defining a connection on T^*M.

Remark. From the integrability condition fulfilled by J_ξ we get

$$\mathcal{L}_{J_\xi Z} J_\xi = J_\xi \cdot (\mathcal{L}_Z J_\xi) = -(\mathcal{L}_Z J_\xi) \cdot J_\xi\,; \quad Z \in \mathfrak{X}(T^*M).$$

The local coordinate expression of Γ_K is (8) where

$$\Gamma_{ij} = \frac{1}{2}\left\{K g_{ij} + g_{ik}\frac{\partial \xi^k}{\partial q^j} - g_{kj}\frac{\partial \varkappa_i}{\partial p_k}\right\}. \tag{16}$$

Consider the canonical 1-form θ on T^*M and let $\omega = d\theta$ be its exterior differential. The 2-form ω defines the canonical symplectic structure on T^*M. The local coordinate expressions are

$$\theta = p_i dq^i, \quad \omega = dp_i \wedge dq^i. \tag{17}$$

Remark that $\omega(V_1, V_2) = 0$ whenever V_1, V_2 are vertical i.e. VT^*M is Lagrangian for ω.

Recall that a vector field H on HT^*M is a Hamilton vector field if it is regular and $\mathcal{L}_H \omega = 0$. If H has the local coordinate expression

$$H = \xi^i \frac{\partial}{\partial q^i} + \varkappa_i \frac{\partial}{\partial p_i}$$

then the condition $\mathcal{L}_H \omega = 0$ is expressed locally by

$$\text{(i)}\ \frac{\partial \xi^i}{\partial p_j} = \frac{\partial \xi^j}{\partial p_i}, \quad \text{(ii)}\ \frac{\partial \varkappa_i}{\partial p_j} = -\frac{\partial \xi^j}{\partial q^i}, \quad \text{(iii)}\ \frac{\partial \varkappa_i}{\partial q^j} = \frac{\partial \varkappa_j}{\partial q^i}. \tag{18}$$

There exists (at least locally) a smooth function $h: T^*M \to \mathbf{R}$ (called the Hamiltonian) such that

$$i_H \omega = -dh.$$

In this paper we deal with syme generalizations of the Hamilton vector fields on T^*M.

Definition. The vector field K on T^*M defines a mechanical structure on T^*M if K is regular and the matrix $(g^{ij}) = (\partial \xi^j / \partial p_i)$ defined by its associated M-vector field is symmetric.

In the case of a vector field defining a mechanical structure on T^*M only the condition (18) (i) is fulfilled. There exists a smooth real function $h: T^*M \to \mathrm{R}$ such that $\xi^i = \partial h / \partial p_i$ but, generally, $\varkappa_i \neq -\partial h / \partial q^i$. We may write

$$\varkappa_i = -\frac{\partial h}{\partial q^i} + \eta_i.$$

Then

$$\eta = \eta_i dq^i \tag{19}$$

is a semibasic 1-form on T^*M, interpreted as the elementary work, and K is defined by the condition

$$i_K \omega = -dh + \eta. \tag{20}$$

We have

$$K = H + \eta^{\#} \tag{21}$$

where H is the Hamilton vector field on T^*M defined by the Hamiltonian h i. e. $i_H \omega = -dh$ and $\eta^{\#}$ is the vertical vector field on T^*M defined by $i_{\eta^{\#}} \omega = \eta$.

Definition. The vector field K on T^*M is a semi-Hamilton vector field if K is a regular and $\mathcal{L}_K \omega$ is a semibasic 2-form on T^*M i. e. $i_V(\mathcal{L}_K \omega) = 0$ whenever V is a vertical vector field on T^*M.

In the case of a semi-Hamilton vector field on T^*M only the conditions (18), (i), (ii) are fulfilled. The formulae (19), (20), (21) are valid with the mention that η is the pull-back by π of a 1-form on M.

The property of a vector field K on T^*M to define a mechanical structure may be expressed by

$$\omega(Z_1, J_\xi Z_2) + \omega(J_\xi Z_1, Z_2) = 0, \quad Z_1, Z_2 \in \mathcal{X}(T^*M) \tag{22}$$

where J is the integrable adapted almost tangent structure on T^*M defined by the associated M-vector field of K.

Proposition 5. *The vector field K on T^*M defines a mechanical structure if and only if*

$$(\mathcal{L}_K \omega)(V_1, V_2) = 0 \tag{23}$$

whenever V_1, V_2 are vertical.

Proof. Let $Z_1, Z_2 \in \mathcal{X}(T^*M)$, $V_1 = J_\xi Z_1$, $V_2 = J_\xi Z_2$. Then $(\mathcal{L}_K \omega)(J_\xi Z_1, J_\xi Z_2) = K(\omega(J_\xi Z_1, J_\xi Z_2)) + \omega(\Gamma_K Z_1, J_\xi Z_2) + \omega(J_\xi Z_1, \Gamma_K Z_2)$. Taking $\Gamma_K Z_1$, $\Gamma_K Z_2$ for Z_1, Z_2 in this relation and using (12) and the property $\Gamma_K^2 = I$ we get

$$(\mathcal{L}_K \omega)(J_\xi Z_1, J_\xi Z_2) = (\mathcal{L}_K \omega)(J_\xi \Gamma_K Z_1, J_\xi \Gamma_K Z_2) = \omega(Z_1, J_\xi Z_2) + \omega(J_\xi Z_1, Z_2)$$

Then the proposition 5 follows easily.

In the case of the vector field defining a mechanical structure on T^*M we may get another two connections. Let K be a vector field on T^*M defining a mechanical structure. Define the tensor field g of type (0, 2) on T^*M by

$$g(Z_1, Z_2) = -\omega(Z_1, J_\xi Z_2)\,; \quad Z_1, Z_2 \in \mathcal{X}(T^*M). \tag{24}$$

Then we have $g(Z_1, Z_2) = g(Z_2, Z_1)$ and $g(Z, V) = 0$ whenever V is vertical. The local coordinate expression is

$$g = g_{ij} dq^i \otimes dq^j.$$

Proposition 6. *If K defines a mechanical structure on T^*M then the the tensor field Γ_K' of type (1, 1) defined by*

$$\omega(Z_1, \Gamma_K' Z_2) = (\mathcal{L}_K g)(Z_1, Z_2)\,; \quad Z_1, Z_2 \in \mathcal{X}(T^*M) \tag{25}$$

determines a connection on T^*M.

Proof. We have

$$(\mathcal{L}_K g)(Z_1, Z_2) = -(\mathcal{L}_K \omega)(Z_1, J_\xi Z_2) + \omega(Z_1, \Gamma_K Z_2)\,; \ Z_1, Z_2 \in \mathcal{X}(T^*M)$$

Then

$$\omega(Z_1, \Gamma_K' J_\xi Z_2) = (\mathcal{L}_K g)(Z_1, J_\xi Z_2) = \omega(Z_1, \Gamma_K J_\xi Z_2) = -\omega(Z_1, J_\xi Z_2)$$

whence

$$\Gamma'_K J_\xi = -J_\xi .$$

Next

$$\omega(Z_1, J_\xi \Gamma'_K Z_2) = -\omega(J_\xi Z_1, \Gamma'_K Z_2) = -(\mathcal{L}_K g)(J_\xi Z_1, Z_2) = (\mathcal{L}_K \omega)(J_\xi Z_1, J_\xi Z_2) - $$
$$- \omega(J_\xi Z_1, \Gamma_K Z_2) = \omega(Z_1, J_\xi \Gamma_K Z_2) = \omega(Z_1, J_\xi Z_2)$$

whence $J_\xi \Gamma'_K = J_\xi$.

Thus Γ'_K defines a connection on T^*M. Its local coordinate expression is given by the coefficients

$$\Gamma'_{ij} = \frac{1}{2}\left\{ K g_{ij} + g_{ik}\frac{\partial \xi^k}{\partial q^j} + g_{kj}\frac{\partial \xi^k}{\partial q^i}\right\}. \tag{26}$$

Another connection Γ''_K on T^*M defined by a vector field K defining a mechanical structure is obtained in a similar way from $\tilde g = g_{ij}\partial/\partial p_i \otimes \partial/\partial p_j$. Its local coordinate expression is determined by the coefficients

$$\Gamma''_{ij} = \frac{1}{2}\left\{ K g_{ij} - g_{ik}\frac{\partial \varkappa_j}{\partial p_k} - g_{kj}\frac{\partial \varkappa_i}{\partial p_k}\right\}. \tag{27}$$

Remark that both Γ'_K, Γ''_K are symmetric.

Moreover, in the case of a semi-Hamilton vector field K we have

$$\Gamma_K = \Gamma'_K = \Gamma''_K .$$

The integrability condition for the almost product structure Γ_K is given by the vanishing of the Nijenhuis tensor fiel N_Γ. We get by a straightforward computation

$$N_\Gamma\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right) = 0, \quad N_\Gamma\left(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}\right) = 0, \quad N_\Gamma\left(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}\right) = 2\left[\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}\right] -$$
$$-2\Gamma_K\left[\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}\right] = -4R_{kij}\frac{\partial}{\partial p_k}$$

where

$$R_{kij} = \frac{\delta \Gamma_{kj}}{\delta q^i} - \frac{\delta \Gamma_{ki}}{\delta q^j}. \tag{28}$$

Thus

$$N_\Gamma = -4R_{kij}\frac{\partial}{\partial p_k}\otimes dq^i \otimes dq^j$$

and the distribution HT^*M is integrable if and only if the almost product structure Γ_K is integrable.

Remark that the components R_{kij} define an M-tensor field of type (0, 3) on T^*M. The following identity is fulfilled

$$\frac{\delta R_{hij}}{\delta q^k} + \frac{\delta R_{hjk}}{\delta q^i} + \frac{\delta R_{hki}}{\delta q^j} = -R_{lij}\frac{\partial \Gamma_{hk}}{\partial p_l} - R_{ljk}\frac{\partial \Gamma_{hi}}{\partial p_l} - R_{lki}\frac{\partial \Gamma_{hj}}{\partial p_l}. \tag{28'}$$

Similar results are obtained in the case of the connections Γ'_K, Γ''_K.

3. Semisprays on TM and regular vector fields on T*M A semispray (a second order differential equation vector field) on the tangent bundle TM is a vector field S on TM such that $JS = C$ where J is the natural almost tangent structure on TM and C is the Liouville vector field on TM (the local coordinate expression is $C = y^i \partial/\partial y^i$, see [6], [2], [4]). The local coordinate expression of the semispray S is

$$S(x, y) = y^i\frac{\partial}{\partial x^i} + \sigma^i(x, y)\frac{\partial}{\partial y^i}. \tag{29}$$

There is a connection Γ (thought of as an almost product structure) on TM derived from the semispray S by

$$\Gamma = -\mathcal{L}_S J. \tag{30}$$

The local coordinate expression of Γ is

$$\Gamma = \frac{\partial}{\partial x^i}\otimes dx^i - \frac{\partial}{\partial y^i}\otimes dy^i + \frac{\partial \sigma^i}{\partial y^j}\frac{\partial}{\partial y^i}\otimes dx^j \tag{31}$$

hence the coefficients of Γ are

$$\Gamma^i_j = -\frac{1}{2}\frac{\partial \sigma^i}{\partial y^j}. \tag{32}$$

Consider the splitting $TTM = VTM \oplus HTM$ defined by Γ. Then the local vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^j_i \frac{\partial}{\partial y^j}; \quad i = 1, \ldots, n \tag{33}$$

define a local frame in HTM.

The integrability condition for the almost product structure Γ expressed by the vanishing of its Nijenhuis tensor field is equivalent to the complete integrability of the differential system on TM defined by HTM. We get

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = -R^k_{ij}\frac{\partial}{\partial y^k}, \quad R^k_{ij} = \frac{\delta \Gamma^k_j}{\delta x^i} - \frac{\delta \Gamma^k_i}{\delta x^j} \tag{34}$$

and it follows that the components R^h_{ij} define an M-tensor field of type (1, 2) on TM.

Consider now a regular vector field K on T^*M. Then it is defined a local M-bundle diffeomorphism Φ from T^*M to TM given locally by (7) where ξ^i; $i = 1, \ldots, n$, define the M-vector field associated with K. Assume for convenience that Φ is a global diffeomorphism. Its inverse Φ^{-1} has the following local coordinate expression

$$q^i = x^i, \quad p_i = \zeta_i(x, y). \tag{35}$$

The components $\zeta_i(x, y)$ define an M-1-form on the tangent bundle TM. From the condition for Φ^{-1} to be the inverse of Φ we get

(36) $$\frac{\partial \zeta_i}{\partial y^j}\circ\Phi = g_{ij}, \quad \frac{\partial \zeta_i}{\partial x^j}\circ\Phi = -g_{ik}\frac{\partial \xi^k}{\partial q^j},$$

where $g^{ij}=\partial\xi^j/\partial p_i$ and g_{ij} are defined by (5). Then

(37) $$\Phi_*\frac{\partial}{\partial p_i} = (g^{ik}\circ\Phi^{-1})\frac{\partial}{\partial y^k}, \quad \Phi_*\frac{\partial}{\partial q^i}=\frac{\partial}{\partial x^i}+\left(\frac{\partial\xi^k}{\partial q^i}\circ\Phi^{-1}\right)\frac{\partial}{\partial y^k},$$
$$\Phi_*^{-1}\frac{\partial}{\partial y^i}=g_{ki}\frac{\partial}{\partial p_k}, \quad \Phi_*^{-1}\frac{\partial}{\partial x^i}=\frac{\partial}{\partial q^i}-g_{kh}\frac{\partial\xi^h}{\partial q^i}\frac{\partial}{\partial p_k}.$$

Theorem 7. *Let K be a regular vector field on T^*M and assume that its associated M-vector field defines a global M-bundle diffeomorphism $\Phi : T^*M\to TM$. Then the vector field Φ_*K is a semispray on TM whose derived connection Γ is the image by Φ of the connection Γ_K defined by (15).*

Proof. Using the local coordinate formulae (7), (13), (37) we get that Φ_*K is a semispray with the local coordinate expression

$$S=\Phi_*K=y^i\frac{\partial}{\partial x^i}+\sigma^i(x,y)\frac{\partial}{\partial y^i}.$$

where σ^i are given by

(38) $$\sigma^i\circ\Phi = K\xi^i = \xi^h\frac{\partial\xi^i}{\partial q^h}+\varkappa_h\frac{\partial\xi^i}{\partial p_h}.$$

Then, denoting by $\tilde\Phi$ the map induced by Φ at the level of tensor fields and using (3), (37) we have

$$\tilde\Phi J_\xi = (g_{ij}\circ\Phi^{-1})\Phi_*\frac{\partial}{\partial p_i}\otimes(\Phi^{-1})^* dq^j = \frac{\partial}{\partial y^i}\otimes dx^i = J.$$

Finally, using the properties of the Lie derivative

$$\Gamma = -\mathcal{L}_S J = -\mathcal{L}_{\Phi_*K}(\tilde\Phi J_\xi) = -\tilde\Phi(\mathcal{L}_K J_\xi) = \tilde\Phi\Gamma_K.$$

Thus Γ is the image of Γ_K by $\tilde\Phi$.

Remark. By using (36), (37) we get that the formula (38) may be written in the equivalent form

(38') $$\varkappa_i\circ\Phi^{-1} = S\zeta_i.$$

The theorem 7 gives us the following geometric result. The splitting $TT^*M=VT^*M\oplus HT^*M$ defined by Γ_K is mapped by Φ_* into the splitting $TTM=VTM\oplus HTM$ defined by Γ. We have the following local coordinate expressions

(39) $$\Phi_*\frac{\delta}{\delta q^i}=\frac{\delta}{\delta x^i}, \quad \Phi_*^{-1}\frac{\delta}{\delta x^i}=\frac{\delta}{\delta q^i},$$

(40) $$\Gamma^i_j\circ\Phi = -\frac{\delta\xi^i}{\delta q^j}, \quad \Gamma_{ij}\circ\Phi^{-1} = -\frac{\delta\zeta_i}{\delta x^j}.$$

Theorem 8. *Let $\xi^i(q,p)$; $i=1,\dots,n$ define a regular M-vector field on T^*M and let Γ^* be a connection on T^*M. Then there exists a regular vector field K on T^*M with its associated M-vector field ξ^i and such that $\Gamma^*=\Gamma_K$ if and only if the following integrability condition is fulfilled*

(41) $$\frac{\delta g_{kj}}{\delta q^i}-\frac{\delta g_{ki}}{\delta q^j} = g_{hi}\frac{\partial\Gamma_{kj}}{\partial p_h} - g_{hj}\frac{\partial\Gamma_{ki}}{\partial p_h}.$$

Proof. Denote by Φ the (local) M-bundle diffeomorphism defined by the M-vector field $\xi^i(q,p)$. If K is a vector field on T_*M, satisfying the condition of the theorem then Φ_*K is (locally) a semispray on TM and the connection Γ derived from this semispray does coincide with the image by $\tilde\Phi$ of the connection $\Gamma^*=\Gamma_K$. The coefficients Γ^i_j of the connection Γ satisfy the condition

(42) $$\frac{\partial\Gamma^k_i}{\partial y^j}=\frac{\partial\Gamma^k_j}{\partial y^i}.$$

Then, using (40), (37) we get

(42') $$\frac{\partial\xi^h}{\partial p_i}\frac{\partial}{\partial p_j}\left(\frac{\delta\xi^k}{\delta q^h}\right)=\frac{\partial\xi^h}{\partial p_j}\frac{\partial}{\partial p_i}\left(\frac{\delta\xi^k}{\delta q^h}\right)$$

and, after a strightforward computation we obtain (41). The converse may be proved remarking that the image of the connection Γ^* by $\tilde\Phi$ is a connection Γ on TM whose coefficients Γ satisfy (42). But this is just the condition for Γ to be derived from a semispray.

Remark. The theorem 8 may be proved too by studying the complete integrability of the partial differential system

(43) $$\frac{\partial\varkappa_i}{\partial p_k} = \left(\varkappa_h\frac{\partial g_{il}}{\partial p_h}+\xi^h\frac{\partial g_{il}}{\partial q^h}+g_{ih}\frac{\partial\xi^h}{\partial q^l}-2N_{il}\right)g^{kl}$$

obtained from (16), where the coordinates q^j are considered as parameters. Remark that, due to the linearity of the system (43) we may get a global solution using a partition of unity on M.

Consider now another problem related to the correspondence considered in theorem 7. Assume that S is a semispray on TM and let $\varkappa_1,\dots,\varkappa_n$ be n smooth functions on the domain $\pi^{-1}(U)$ of an induced local chart on TM. We ask for the local existence of a regular M-vector field ξ^i such that the image of S by the inverse Φ^{-1} of the associated (local) M-bundle diffeomorphism is just $K=\xi^i\partial/\partial q^i+\varkappa_i\,\partial/\partial p_i$.

Theorem 9. *Consider a semispray S on TM and let $\varkappa_i(q,p)$; $i=1,\dots,n$ be n smooth real valued functions on the domain $\pi^{-1}(U)$ of an induced local chart on T^*M. Assume that the ordinary differential system on $\tau^{-1}(U)$*

(44) $$\frac{dx^1}{y^1}=\dots=\frac{dx^n}{y^n}=\frac{dy^1}{\sigma^1}=\dots=\frac{dy^n}{\sigma^n}=\frac{d\zeta_1}{\varkappa_1(x,\zeta(x,y))}=\dots=\frac{\zeta_n}{\varkappa_n(x,\zeta(x,y))}$$

has n prime integrals $C_1,\dots,C_n$ such that

$$(45)\qquad \frac{D(C_1, \ldots, C_n)}{D(y^1, \ldots, y^n)} \neq 0, \qquad \frac{D(C_1, \ldots, C_n)}{D(\zeta_1, \ldots, \zeta_n)} \neq 0,$$

where the local coordinate expression of S is given by (25). Then there exists a local M-bundle diffeomorphism Φ^{-1} from TM to T^*M sending the semispray S in a regular vector field K on T^*M, with Φ the (local) M-bundle diffeomorphism corresponding to its associated M-vector field and the last n components $\varkappa_1, \ldots \varkappa_n$.

Proof. Consider the partial differential system

$$(46)\qquad y^h \frac{\partial \zeta_i}{\partial x^h} + \sigma^h(x, y) \frac{\partial \zeta_i}{\partial y^h} = \varkappa_i(x^k, \zeta_k(x, y))$$

obtained by using (38') from the condition $\Phi_*^{-1} S = K$ where Φ is given by (7) and K is given by (13). Then (see [1]) the characteristic system of (46) is just (44) and the condition (45) for the prime integrals $C_1, \ldots, C_n$ assures the obtaining, by using the Implicite Function Theorem, of the components $\zeta_1, \ldots, \zeta_n$ in formula (31) of the local M-bundle diffeomophism Φ^{-1}.

Remark. The first $2n-1$ equations in the system (44) define the integral paths of the semispray S.

Consider now the correspondence between the connections on T^*M, TM given in theorem 7 when these connections are not necessarily obtained from regular vector fields on T^*M and semisprays on TM. If $\Phi : T^*M \to TM$ is an M-bundle diffeomorphism and Γ^*, Γ are connections on T^*M, TM respectively then the correspondence is given by

$$(47)\qquad \tilde{\Phi}\Gamma^* = \Gamma, \qquad \tilde{\Phi}^{-1}\Gamma = \Gamma^*.$$

We shall say that Γ^*, Γ are Φ-related. The local coordinate formulae relating the coefficients $\Gamma_i^j(x, y)$, $\Gamma_{ij}(q, p)$ are (36).

Proposition 10. *If the connections Γ^*, Γ on T^*M, TM are Φ-related then*

$$(48)\qquad \text{(i)}\quad R_{hij}\frac{\partial \xi^k}{\partial p_h} = R^k_{ij}\circ\Phi, \qquad \text{(ii)}\quad R^h_{ij}\frac{\partial \zeta_k}{\partial y^h} = R_{kij}\circ\Phi^{-1}.$$

Proof. By a straightforward computation using (28), (34) or by using the relation $\tilde{\Phi}N_{\Gamma^*} = N_\Gamma$ between the Nijenhuis tensor fields of Γ^* and Γ.

Proposition 11. *If the connection Γ on TM is derived from a semispray then*

$$(49)\qquad \frac{\partial R^h_{ij}}{\partial y^k} + \frac{\partial R^h_{jk}}{\partial y^i} + \frac{\partial R^h_{ki}}{\partial y^j} = 0.$$

Proof. By a local coordinate cymputation using the formula

$$\frac{\partial}{\partial y^i}\left(\frac{\delta \Gamma^h_k}{\delta x^j}\right) = \frac{\delta}{\delta x^j}\left(\frac{\partial \Gamma^h_k}{\partial y^i}\right) - \frac{\partial \Gamma^l_j}{\partial y^i}\,\frac{\partial \Gamma^h_k}{\partial y^l}.$$

Remark. The connection $\Gamma^* = \tilde{\Phi}\Gamma$ on T^*M, corresponding by Φ to a connection Γ on TM, derived from a symispray must satisfy the identity

$$(49')\qquad g_{hi}\frac{\partial}{\partial p_h}\left(\frac{\partial \xi^m}{\partial p_l} R_{ljk}\right) + g_{hj}\frac{\partial}{\partial p_h}\left(\frac{\partial \xi^m}{\partial p_l} R_{lki}\right) + g_{hk}\frac{\partial}{\partial p_h}\left(\frac{\partial \xi^m}{\partial p_l} R_{lij}\right) = 0$$

corresponding to (49).

Remark. The identity (49') may be obtained too as the integrability condition of (41) thought of as a partial differential system satisfied by g_{ij}.

Let Γ^*, Γ be connections on T^*M, TM respectively and let us look for (at least locally) an M-bundle diffeomorphism $\Phi : T^*M \to TM$ such that the connections Γ^*, Γ are Φ-related. From (40) we get the following partial differential system

$$(50)\qquad \frac{\partial \xi^k}{\partial q^j} - \Gamma_{hj}\frac{\partial \xi^k}{\partial p_h} = -\Gamma^k_j(q, \xi^i(q, p))$$

satisfied by the components $\xi^i(q, p)$ of the M-vector field on T^*M, corresponding to Φ. The integrability condition for this system is (48) (i). Generally, (48) (i) defines an overdetermined partial differential system on T^*M with variables $q^1, \ldots, q^n$ as parameters. If we may find the general solution of (48) (i) then the existence problem for the solutions of (50) may be studied easily. A quite interesting situation arises in the case where (48) (i) is equivalent to a completely integrable system. Then the partial differential system obtained by considering together (50), (48) (i) is a completely integrable system and by using the Frobenius theorem we may get a local solution of our problem.

Remark. An equivalent problem may be considered if we look for the inverse M-bundle diffeomorphism $\Phi^{-1} : TM \to T^*M$ such that Γ, Γ^* are Φ^{-1}-related. Then the M-1-form on TM defined by $\zeta_k(x, y)$ corresponding to Φ^{-1} must satisfy the partial differential system

$$(50')\qquad \frac{\partial \zeta_k}{\partial x^j} - \Gamma^h_j\frac{\partial \zeta_k}{\partial y^h} = -\Gamma_{kj}(x, \zeta_l(x, y))$$

with the integrability condition (48) (ii).

If the connection Γ^* on T^*M is symmetric then the M-bundle diffeomorphism Φ^{-1} and the M-tensor field R^h_{ij} must satisfy the following identity

$$(51)\qquad R^h_{jk}\frac{\partial \zeta_i}{\partial y^h} + R^h_{kj}\frac{\partial \zeta_j}{\partial y^h} + R^h_{ij}\frac{\partial \zeta_k}{\partial y^h} = 0.$$

A quite interesting situation arises in the case where we look for the M-bundle diffeomeorphivm Φ i. e. for the M-vector field $\xi^i(q, p)$ on T^*M in the form

$$(52)\qquad \xi^i(q, p) = p_h g^{hi}(q)$$

thus $g^{hi}(x)$ are the components of a nondegenerate tensor field of type (2, 0) on M. If the tensor field $g_{ij}(x)$ on M is defined by (5) then

$$(52')\qquad \zeta_i(x, y) = g_{ih}(x)y^h.$$

If Γ is a connection on TM associated with a linear connection ∇ on M, defined by the connection coefficients $\Gamma^k_{ij}(x)$ i. e.

$$\Gamma^i_j(x, y) = \Gamma^i_{jh}(x)y^h$$

then Γ^* must be a connection on T^*M associated with another linear connection $\mathring{\nabla}$ on M defined by the connection coefficients $\mathring{\Gamma}^k_{ij}$ i. e.

$$\Gamma_{ij}(q,p) = -\mathring{\Gamma}^h_{ji}(q)p_h.$$

Moreover, the linear connections ∇, $\mathring{\nabla}$ must be conjugate (see [10]) with respect to the tensor field of type (0, 2) on M, defined by $g_{ij}(x)$ i. e.

$$(53)\qquad \frac{\partial g_{ki}}{\partial x^j} - g_{hi}\mathring{\Gamma}^h_{jk} - g_{kh}\Gamma^h_{ji} = 0.$$

The complete integrability condition for the partial differential system (53) is expressed by the following purely algebraic relation satisfied by g_{ij} and the curvature tensor fields R^l_{kij}, $\mathring{R}^l_{kij}$ of ∇, $\mathring{\nabla}$ respectively

$$(54)\qquad g_{kh}R^h_{lij} + g_{hl}\mathring{R}^h_{kij} = 0.$$

4. Regular vector fields and the inverse problem of Lagrangian dynamics. The inverse problem of Lagrangian dynamics is to give necessary and sufficient conditions for a system of second order differential equations to be the Euler Lagrange equations of some regular Lagrangian function (see [11], [2], [3]). One solution of this problem is known as the Helmholtz conditions. If S is a semispray on TM with the local coordinate expression (29) and Γ^i_j are the coefficients given by (32) of the connection Γ on TM derived from S then the Helmholtz conditions are concerned with the (local) existence of the functions g_{ij}; $i,j=1,\ldots,n$ on $\tau^{-1}(U)$ such that

$$(55)\qquad \begin{aligned} &\text{(i)} && \det(g_{ij})\neq 0,\ g_{ji}=g_{ij},\ \frac{\partial g_{ij}}{\partial y^k}=\frac{\partial g_{ik}}{\partial y^j}.\\ &\text{(ii)} && Sg_{ij}-g_{ik}\Gamma^k_j-g_{jk}\Gamma^k_i=0,\\ &\text{(iii)} && g_{ik}\left(S\Gamma^k_j-\frac{\partial\sigma^k}{\partial x^j}-\Gamma^k_h\Gamma^h_j\right)=g_{jk}\left(S\Gamma^k_i-\frac{\partial\sigma^k}{\partial x^i}-\Gamma^k_h\Gamma^h_i\right).\end{aligned}$$

Remark that the conditions (55) are not completely satisfactory since they involve the extraneous functions g_{ij} when one would hope for a condition on the semispray S alone. Remark that in [2] the Helmholtz conditions have been incorporated in the problem of existence of an almost symplectic structure on TM satisfying appropriate geometric conditions.

The functions g_{ij} in the Helmholtz conditions (55) may be thought of as $\partial\zeta_i/\partial y^j$ where ζ_i are the components of the M-bundle diffeomorphism Φ^{-1} given locally by (35). Then it is suggested to use an M-bundle diffeomorphism $\Phi^{-1}: TM\to T^*M$ to understand better the geometric content of the Helmholtz conditions.

Proposition 12. *Let S be a semispray on TM and let $\Phi^{-1}: TM\to T^*M$ be an M-bundle diffeomorphism given locally by (35). Then $K=\Phi_*^{-1}S$ is a vector field defining a mechanical structure on T^*M if and only if S and the functions $g_{ij}=\partial\zeta_i/\partial y^j$ satisfy the conditions (55) (i).*

Proof. Since Φ is a diffeomorphism we have $\det(g_{ij})\neq 0$. Using (36) and (18) (i) we see that K defines a mechanical structure on T^*M if and only if $\partial\zeta_i/\partial y^j=\partial\zeta_j/\partial y^i$.

Remark. We denoted $g_{ij}=g_{ij}\circ\Phi^{-1}=\partial\zeta_i/\partial y^j$ by abuse.

Remark. There always exist M-bundle diffeomorphisms $\Phi^{-1}: TM\to T^*M$ such that the corresponding functions g_{ij} satisfy the conditions (55) (i). These M-bundle diffeomorphisms may be obtained as the Legendre transformations associated with regular Lagrangian functions on TM.

Proposition 13. *Let S be a semispray on TM and let $\Phi^{-1}: TM\to T^*M$ be an M-bundle diffeomorphism given locally by (35). Then $K=\Phi_*^{-1}S$ is a semi-Hamilton vector field if and only if S and $g_{ij}=\partial\zeta_i/\partial y^j$ satisfy the conditions (55) (i), (ii) and*

$$(56)\qquad \frac{\delta\zeta_i}{\delta x^j}=\frac{\delta\zeta_j}{\delta x^i}$$

where $\delta/\delta x^i$ is defined by (33).

Proof. The conditions (18) (i) are fulfilled if and only if the functions g_{ij} satisfy (55) (i). If $K=\Phi_*^{-1}S$ then (38′) gives the relation between the components σ^i, of S and $\varkappa_i$ of K.

The condition (18) (ii) is written as follows

$$(57)\qquad \frac{\partial\zeta_j}{\partial x^i}-\frac{\partial\zeta_i}{\partial x^j}=Sg_{ij}-2g_{ih}\Gamma^h_j.$$

It follows by considering in (57) the symmetric and skew-symmetric parts that (57) is equivalent to (55) (ii) and (56).

Remark. We may show that (56) is related to (55) (ii). In fact, taking in (55) (ii) the partial derivative with respect to y^k, then interchanging j and k we get

$$\frac{\partial}{\partial y^i}\left(\frac{\delta\zeta_j}{\delta x^k}-\frac{\delta\zeta_k}{\delta x^j}\right)=0.$$

Thus (56) may be obtained from (55) (i) provided that suitable initial conditions are imposed to $\partial\zeta_j/\partial x^k(x,0)$ and $\partial\zeta_j/\partial y^k(x,0)$.

Finally, it follows that the image of a given semispray S on TM by an M-bundle diffeomorphism $\Phi^{-1}: TM\to T^*M$ is a Hamilton vector field on T^*M if and only if the Helmholtz conditions (55) and the condition (56) are fulfilled.

5. Horizontal regular vector fields on T*M. Let K be a regular vector field on T^*M with the corresponding connection Γ_K defined by (15) and $\Phi: T^*M\to TM$ the M-bundle diffeomorphism defined by the associated M-vector field ξ^i — (we assume for convenience that Φ is a global diffeomorphism).

Proposition 14. *The regular vector field K is horizontal with respect to the connection Γ_K if and only if its local coordinate components ξ^i, $\varkappa_i$ satisfy*

$$(58)\qquad \varkappa_i+\Gamma_{ij}\xi^j=0$$

with the coefficients Γ_{ij} given by

$$(59)\qquad \Gamma_{ij}=g_{ik}\left(\frac{\partial\xi^k}{\partial q^j}+\Gamma^k_{jl}(q)\xi^l\right)$$

where Γ^k_{jl} are the connection coefficients of a torsion free linear connection on M.

Proof. The condition for K to be horizontal with respect to Γ_K means that K is a combination of the local vector fields $\delta/\delta q^i$; $i=1, \ldots, n$, defining a local frame in HT^*M. Then it follows (58). In order to obtain informations about the connection Γ_K corresponding to K consider the semispray $S=\Phi_* K$ which, according to theorem 7, is horizontal with respect to Γ if and only if it is a spray (see [6], [2]) i. e.

$$\sigma^i = -\Gamma^i_{jk}(x) y^j y^k \tag{60}$$

where $\Gamma^i_{jk}(x)$ are the connection coefficients of a torsion free linear connection on M. Then, using (40) we get the expression (59) of the coefficients of the connection Γ_K.

Proposition 15. *Let K be a semi-Hamilton vector field on T^*M. Assume that K is horizontal with respect to Γ_K and denote by $\Gamma^k_{ij}(x)$ the connection coefficients of the torsion free linear connection on M, determining by (59) the connection Γ_K. Then*

$$\left(\frac{\partial \Gamma_{ih}}{\partial p_k} + \Gamma^k_{ih}\right) \xi^h = 0, \tag{61}$$

$$R_{hij}\xi^h = \alpha_{ij}(q) \tag{62}$$

where α_{ij} are independent of p. Moreover, the 2-form on M defined by the coefficients α_{ij} is closed.

Proof. If K is a semi-Hamilton vector field on T^*M the conditions (18) (i), (ii) are fulfilled. From (58) we get by using (18) (ii) and the property $g_{ji}=g_{ij}$ that

$$\frac{\delta \xi^k}{\delta q^i} = \frac{\partial \Gamma_{ih}}{\partial p_k} \xi^h. \tag{63}$$

Next use (59) to get (61). Using (63) and (28) we have

$$R_{hij} \frac{\partial \xi^h}{\partial p_k} = R^k_{hij}(q) \xi^h \tag{64}$$

where $R^k_{hij}(x)$ are the local coordinate components of the curvature tensor field of the connection ∇. Using (61) and (28) we have

$$\xi^h \left(\frac{\partial R_{hij}}{\partial p_k} + R^k_{hij}\right) = 0. \tag{65}$$

From (64), (65) it follows $\partial/\partial p_k(R_{hij}\xi^h)=0$ i. e. $R_{hij}\xi^j$ is independent of p. Finally, from (62) we get

$$\frac{\partial \alpha_{ij}}{\partial x^k} = \frac{\delta \xi^h}{\delta q^k} R_{hij} + \xi^h \frac{\delta R_{hij}}{\delta q^k}$$

from which we obtain, by using (63) and (28′)

$$\frac{\partial \alpha_{ij}}{\partial x^k} + \frac{\partial \alpha_{jk}}{\partial x^i} + \frac{\partial \alpha_{ki}}{\partial x^j} = 0.$$

Remark. From the relation (61) we get by differentiating with respect to p_j

$$\frac{\partial^2 \Gamma_{ih}}{\partial p_j \partial p_k} \xi^i \xi^h = 0. \tag{65}$$

6. Sections in the cotangent bundle tangent to regular vector fields. Let α be a 1-form on the manifold M, thought of as a section $\alpha: M \to T^*M$ in the cotangent bundle. Then $\alpha(M)$ is an n-dimensional smooth submanifold in T^*M and we should be interested in finding the conditions under which a given regular vector field K on T^*M is tangent to the submanifold $\alpha(M) \subset T^*M$ in the points of $\alpha(M)$. The tangent space to $\alpha(M)$ in a point $\alpha(x)$ is spanned by

$$\alpha_*\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial q^i} + \frac{\partial \alpha_j}{\partial x^i} \frac{\partial}{\partial p_j}; \quad i=1, \ldots, n$$

where $\alpha = \alpha_i dx^i$ is the local coordinate expression of α. The condition for $K(\alpha(x))$ to be tangent to $\alpha(M)$ in $\alpha(x)$ is obtained by expressing it as a combination of $\alpha_*(\partial/\partial x^i)$:

$$\frac{\partial \alpha_i}{\partial x^j} \xi^j - \varkappa_i = 0 \tag{66}$$

where the local coordinate expression of K is (13). Remark that the same condition is obtained from

$$\omega\left(K, \alpha_*\left(\frac{\partial}{\partial x^i}\right)\right) = 0.$$

Proposition 16. *Consider the ordinary differential system*

$$\frac{dx^1}{\xi^1(x, \alpha)} = \ldots = \frac{dx^n}{\xi^n(x, \alpha)} = \frac{d\alpha_1}{\varkappa_1(x, \alpha)} = \ldots = \frac{d\alpha_n}{\varkappa_n(x, \alpha)} \tag{67}$$

and let $C_1, \ldots, C_n$ be prime integrals such that

$$\frac{D(C_1, \ldots, C_n)}{D(\alpha_1, \ldots, \alpha_n)} \neq 0.$$

Then there exists a local section α in T^*M such that K is tangent to $\alpha(M)$ in the points of $\alpha(M)$.

Proof. Think of (66) as a partial differential system whose characteristic system is just (67). Then the local solution α of (66) is obtained from the prime integrals $C_1, \ldots, C_n$ by using the Implicite Function Theorem.

Remark. It should be important to get closed 1-forms α which are solutions of (66).

REFERENCES

1. Courant R., Hilbert D. — *Methods in Mathematical Physics*, vol II: Partial *Differential Equations*, Wiley Interscience, New York, London 1962, Russian Edition, Mir Moskwa, 1964, 830
2. Crampin M. — *On the differential geometry of the Euler Lagrange equations and the inverse problem of Lagrangian dynamics*, J. Physics, A. Math. Gen. 14 (1981), 2 567—2 575.

3. Crampin M., Prince G. E., Thompson G. — *A geometrical version of the Helmholtz conditions in time dependent dynamics*. J. Phys. A. Math. Gen. 17 (1984), 1437—1 447.
4. Crampin, M., Pirani, F. A. — *Applicable Differential Geometry*, Cambridge Univ. Press (1986), ii, 394
5. Dekret, A. — *Mechanical structures and connections*, Diff. Geom. Appl. Proceedings Conf. in Dubrovnik (1988), Faculty of Math. Univ. Belgrade, Inst. Math. Univ. Novi Sad (1989), 121—132.
6. Grifone, J. — *Structures presque tangentes et connexions*, Ann. Inst. Fourier Grenoble, 22 (1) (1972), 287—334, 22 (3), (1972), 291—338.
7. Klein J. — *Structures symplectiques ou J symplectiques homogénes sur l'espace tangent a une variété*. Symposia Math. 14 (1974), 181—192.
8. Klein, J. — *Almost symplectic structures in dynamics*. Diff. Geom. Appl. Proceedings Conf. in Brno (1986), D. Reidel Publ. Co. 1987, 79—90.
9. Mok, K. R., Patterson E. M., Wong Y. C. — *Structure of symmetric tensors of type (0,2) and tensors of type (1,1) on the tangent bundle*. Trans. Am. Math. Soc. 234 (1977), 253—278.
10. Norden, A. P. — *Prostranstva affinnoi sviaznosti*, Russian, Nauka, Moskwa 1976, 432.
11. Santilli, R. M. — *Foundations of Theoretical Mechanics I, The Inverse Problem in Newtonian Mechanics*, Texts and Monographs in Physics, Springer Verlag, 1978, xix, 266.

Received 24. I. 1990

Seminarul Matematic
Universitatea din Iași, România

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA"
Tomul XXXVII, s. I a, Matematică, 1991, f. 1.

ON A TYPE OF SEMI-SYMMETRIC METRIC CONNECTION ON A RIEMANNIAN MANIFOLD

BY

U. C. DE

Introduction. The present paper deals with a type of semi-symmetric metric connection D [1] on a Riemannian manifold (M^n, g) $(n>3)$ such that the curvature tensor R of D satisfies the relation

$$(\bar{D}_V \bar{D}_U R)(X, Y)Z = B(V)(\bar{D}_U R)(X, Y)Z + P(U, V)R(X, Y)Z, \tag{1}$$

where B is a i-form, P is a (0,2) tensor and $\bar{D}$ denotes the Levi-Civita connection. It is shown that if (1) is satisfied i.e., if the curvature tensor R of D is generalised 2-recurrent [2], then the manifold (M^n, g) is generalised conformally 2-recurrent [2].

1. Preliminaries. Let (M^n, g) be an n-dimensional $(n>3)$ Riemannian manifold with Levi-Civita connection $\bar{D}$. A linear connection D on (M^n, g) is said to be semi-symmetric if its torsion tensor T satisfies the relation

$$T(X, Y) = \omega(Y)X - \omega(X)Y \tag{1.1}$$

where ω is a 1-form.
It is known [1] that if D is a semi-symmetric metric connection, then

$$D_X Y = \bar{D}_X Y + \omega(Y)X - g(X, Y)\rho \tag{1.2}$$

where

$$g(X, \rho) = \omega(X) \tag{1.3}$$

for every vector field X. Further, it is also known [1] that if R and K denote the curvature tensor of D and $\bar{D}$ respectively, then

$$R(X, Y)Z = K(X, Y)Z - \alpha(Y, Z)X + \alpha(X, Z)Y - g(Y, Z)AX + g(X, Z)AY \tag{1.4}$$

where α is a tensor field of type (0,2) defined by

$$\alpha(X, Y) = (\bar{D}_X\omega)(Y) - \omega(X)\omega(Y) + \tfrac{1}{2}\omega(\rho)\, g(X, Y) \tag{1.5}$$

and A is a tensor field of type (1, 1) defined by

$$g(AX, Y) = \alpha(X, Y) \tag{1.6}$$

for any vector fields X, Y.
We shall use the above results in the next section.