$$\Delta(p)=\begin{cases}\left.\begin{array}{ll}\mathrm{sp}\left\{\dfrac{\partial}{\partial x}(p)\right\}, & \text{when } x=0\\ T_pM, & \text{when } x\neq 0\end{array}\right\} & \text{if } a'(\lambda)\not\equiv 0\\ \mathrm{sp}\,\{X(p,\lambda_0)\}, & \text{if } a'(\lambda)\equiv 0 \ (\text{i.e.}\ a(\lambda)=\lambda_0)\end{cases}$$

and

$$\partial\Delta(p)=\begin{cases}\mathrm{sp}\left\{\dfrac{\partial}{\partial y}(p)\right\}, & \text{if } a'\not\equiv 0\\ 0, & \text{if } a'\equiv 0 \text{ or if } x=0.\end{cases}$$

We observe that the distribution Δ has the maximal integral manifold property if and only if $a'\equiv 0$ (then it is equal with the distribution spanned by only a vector field).

Indeed, if $a'\not\equiv 0$ and $x=0$, then it does not exist a maximal integral manifold which passes through p.

Now we shall verify that Δ has maximal integral manifold property if and only if $dX_{\lambda,t}(\partial\Delta(p))\subseteq\Delta(X_{\lambda,t}(p))$ (see Theorem 2.4). Let $p_1=\left(-1,\ y+\dfrac{a(\lambda)}{2}\right)$; the integral curve which passes through p_1 is

$$\begin{cases}x(t)=t-1,\\ y(t)=\dfrac{a(\lambda)t^2}{2}-a(\lambda)t+y+\dfrac{a(\lambda)}{2},\end{cases}\quad \text{and}$$

$$\partial\Delta(p_1)=\begin{cases}\mathrm{sp}\left\{\dfrac{\partial}{\partial y}(p)\right\}, & \text{if } a'\not\equiv 0\\ 0, & \text{if } a'\equiv 0\end{cases}$$

Let $t=1$ and $p_2=p_\lambda(1)=(0,y)$. It follows that

$$dX_{\lambda,1}(\partial\Delta(p_1))=\mathrm{sp}\left\{\begin{Vmatrix}1 & 0\\ a(\lambda) & 1\end{Vmatrix}\begin{Vmatrix}0\\ \alpha\end{Vmatrix};\ \text{where } \begin{Vmatrix}0\\ \alpha\end{Vmatrix}\in\partial\Delta(p_1)\right\}p_2=$$

$$=\begin{cases}\mathrm{sp}\left\{\dfrac{\partial}{\partial y}(p_2)\right\} & \text{if } \alpha\neq 0\\ 0 & \text{if } \alpha=0\end{cases}$$

We conclude that if $a'\not\equiv 0$, then $dX_{\lambda,1}(\partial\Delta(p_1)\not\subset\Delta(p_2)$ and if $a'=0$, then $dX_{\lambda,1}(\partial\Delta(p_1))=0\in\Delta(p_2)$. We can argue in a similar manner for any other point $p\in M$.

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Universitatea „Ştefan cel Mare"
Suceava, România

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SOME THEOREMS IN THE THEORY OF NONSIMPLE ELASTIC BODIES

BY

D. IEŞAN and R. QUINTANILLA

1. Introduction. Many papers have been devoted to the study of the theory of nonsimple elastic solids. For typical results in this direction we refer the reader to [1]—[5]. In this paper we establish some results in the dynamic theory. First, we derive a reciprocal theorem and prove that it leads to a uniqueness result of Brun type [6] and a minimum principle. We also obtain the classical reciprocity relation avoiding both the use of the Laplace transform and the incorporation of the initial conditions into the field equations. The uniqueness result is established with no definiteness assumption on the constitutive coefficients. Finally, we use results of the semigroup theory of linear operators to obtain an existence theorem. We also obtain a continuous dependence result.

2. Basic equations. We consider a body that at time $t=0$ occupies the region B of Euclidean three-dimensional space and is bounded by the piecewise smooth surface ∂B. The motion of the body is referred to a fixed system of rectangular Cartesian axes $0x_i$ $(i=1,2,3)$. We designate by $\mathbf{n}$ the outward normal of ∂B. Letters in boldface stand for tensors of an order $p\geq 1$, and if $\mathbf{v}$ has the order p, we write $v_{ij\ldots k}$ (p subscripts) for the components of $\mathbf{v}$ in the Cartesian coordinate frame. We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers (1, 2, 3), summation over repeated subscripts is implied and subscripts preced by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate. In all that follows, we use a superposed dot to denote partial differentiation with respect to the time.

We consider the linear theory of nonsimple elastic bodies. Let ϑ be an arbitrary material volume in the continuum bounded by a surface A at time t; we suppose that V is the corresponding volume in the initial undeformed state of the continuum, bounded by a surface A. We postulate (cf. [1]—[3]) an energy balance for an arbitrary material volume, in the form

$$\int_V \rho(v_i\dot{v}_i+\dot{U})\,dv=\int_V F_iv_i\,dv+\int_A (t_iv_i+\mu_{ji}v_{i,j})\,da, \qquad (2.1)$$

where u_i is the displacement vector, $v_i=\dot{u}_i$, ρ is the density in the reference configuration, U is the internal energy per unit mass, F_i is the body force per unit volume in the reference configuration, t_i is a stress vector, μ_{ij} is the dipolar surface force.

We suppose that the body has arrived at a given state at a time t through some prescribed motion. Following [7], we consider a second motion which differs from the given motion only by a constant superposed rigid body translational velocity, the body occupying the same position at time t and we assume that $\dot U$, F_i, t_i and μ_{ij} are unaltered by such superposed rigid body velocity. If we use equation (2.1) with v_i replaced by v_i+a_i, where a_i is an arbitrary constant, we obtain

$$\int_V \rho\dot v_i dv = \int_V F_i dv + \int_A t_i da. \tag{2.2}$$

From (2.2) we obtain

$$t_i = t_{ji}n_j \tag{2.3}$$

and

$$t_{ji,j} + F_i = \rho\ddot u_i. \tag{2.4}$$

Taking into account (2.3) and (2.4), the equation (2.1) reduces to

$$\int_V \rho\dot U dv = \int_V t_{ji}v_{i,j}dv + \int_A \mu_{ji}\, v_{i,j}da. \tag{2.5}$$

With an argument similar to that used in obtaining (2.3), we get

$$\mu_{ji}v_{i,j} = \mu_{rji}v_{i,j}n_r, \tag{2.6}$$

where μ_{rji} is the hyperstress tensor. With the help of (2.6), equation (2.5) reduces to

$$\rho\dot U = (t_{ji} + \mu_{rji,r})v_{i,j} + \mu_{rji}v_{i,jr}. \tag{2.7}$$

We introduce the notation

$$\tau_{ij} = t_{ij} + \mu_{rij,r}. \tag{2.8}$$

Let us now consider a motion of the body which differs from the given motion only by a superposed uniform rigid body angular velocity, the body occupying the same position at time t. With an argument similar to that used in obtaining (2.2), from (2.7) we obtain

$$\tau_{ij} = \tau_{ji}. \tag{2.9}$$

The relation (2.7) can be written in the form

$$\rho\dot U = \tau_{ij}\dot e_{ij} + \mu_{ijr}\dot\chi_{ijr}, \tag{2.10}$$

where

$$e_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}),\quad \chi_{ijr} = u_{r,ij}. \tag{2.11}$$

We restrict our attention to the linear theory of elastic bodies where the constitutive equations are

$$\tau_{ij} = A_{ijrs}e_{rs} + B_{ijpqr}\chi_{pqr},$$
$$\mu_{ijk} = B_{rsijk}e_{rs} + C_{ijkpqr}\chi_{pqr}. \tag{2.12}$$

The coefficients in (2.12) have the following symmetries

$$A_{ijrs} = A_{rsij} = A_{jirs},\quad B_{ijpqr} = B_{jipqr} = B_{ijqpr},$$
$$C_{ijkpqr} = C_{pqrijk} = C_{jikpqr}. \tag{2.13}$$

We assume that A, B and C are continuously differentiable on B.

On the basis of the constitutive equations, from (2.6) we obtain

$$\mu_{ji} = \mu_{rji}n_r. \tag{2.14}$$

It is known [7] that the skew symmetric part $\mu_{[ij]k}$ makes no contribution to the rate of work so that we take $\mu_{[ij]k}=0$.

With the help of (2.8), the equations (2.4) become

$$\tau_{ji,j} - \mu_{rji,rj} + F_i = \rho\ddot u_i. \tag{2.15}$$

We assume that the body occupies a bounded region B with the Lipschitz boundary ∂B which consists of a finite number of smooth surfaces. Let γ_α be intersections of two adjoined smooth surfaces and $C = \cup_\alpha\gamma_\alpha$. In the last integral of (2.1), $v_{i,j}$ is not independent of v_i on A ; only the normal component $Dv_i = v_{i,j}n_j$ is independent.

Following [1], [2], the total rate of work over the boundary of the body becomes

$$\int_{\partial B}(t_iv_i + \mu_{ji}v_{i,j})da = \int_{\partial B}(P_iv_i + R_iDv_i)da + \int_C Q_iv_ids, \tag{2.16}$$

where

$$P_i = (\tau_{ji} - \mu_{sji,s})n_j + (D_hn_h)\mu_{pji}n_pn_j - D_j(\mu_{pji}n_p),$$
$$R_i = \mu_{rsi}n_rn_s,$$
$$Q_i = \ll \mu_{pji}n_p\varepsilon_{jrq}s_rn_q \gg, \tag{2.17}$$

$\ll\gg$ denotes the difference of limits from both sides of C, s_i represent the components of the unit tangent vector to γ_α, ε_{ijp} is the alternating symbol and D_i are the components of the surface gradient.

Let us consider the subsets $\Sigma_s (s=1,2,3,4)$ of ∂B so that $\Sigma_1\cup\Sigma_2\cup C = \Sigma_3\cup\Sigma_4\cup C = \partial B$, $\Sigma_1\cap\Sigma_2 = \Sigma_3\cap\Sigma_4 = \emptyset$. We consider the following boundary conditions

$$u_i = \tilde u_i \text{ on } \Sigma_1\times I,\quad P_i = \tilde P_i \text{ on } \Sigma_2\times I,$$
$$Du_i = g_i \text{ on } \Sigma_3\times I,\quad \tilde R_i = \tilde R_i \text{ on } \Sigma_4\times I, \tag{2.18}$$
$$Q_i = \tilde Q_i \text{ on } C\times I,$$

where $\tilde u_i$, $\tilde P_i$, $\tilde R_i$, $\tilde Q_i$ are prescribed functions, and $I=[0,\infty)$.

The basic equations of the theory are: the equations of motion (2.15), the constitutive equations (2.12) and the geometrical equations (2.11). To the system of field equations we adjoin the boundary conditions (2.18) and the conditions

$$u_i(\mathbf{x},0) = a_i(x),\quad \dot u_i(\mathbf{x},0) = b_i(\mathbf{x}),\quad \mathbf{x}\in B, \tag{2.19}$$

where a_i and b_i are prescribed.

We assume that

(i) **a** and **b** are continuous on $\bar B$;

(ii) $\tilde{\mathbf{u}}$ is continuous on $\Sigma_1\times I$;

(iii) $\tilde{\mathbf{P}}$ is piecewise regular on $\Sigma_2\times I$ and continuous in time, $\tilde{\mathbf{g}}$ is piecewise regular on $\Sigma_3\times I$ and continuous in time $\tilde{\mathbf{R}}$ is piecewise regular on $\Sigma_4\times I$ and continuous in time, $\tilde Q$ is piecewise regular on $C\times I$ and continuous in time;

(iv) ρ is continuous on $\bar B$;

(v) $\mathbf{F}$ is continuous on $\bar B\times I$.

3. Reciprocity and uniqueness. Consider two external data system $L^{(\alpha)}=\{F_i^{(\alpha)},\ \tilde{u}_i^{(\alpha)},\ g_i^{(\alpha)},\ \tilde{P}_i^{(\alpha)},\ \tilde{R}_i^{(\alpha)},\ \tilde{Q}_i^{(\alpha)},\ a_i^{(\alpha)},\ b_i^{(\alpha)}\}\ (\alpha=1,\ 2)$. Let $A^{(\alpha)}=(u_i^{(\alpha)},\ e_{ij}^{(\alpha)},\ \chi_{ijr}^{(\alpha)},\ \tau_{ij}^{(\alpha)},\ \mu_{rij}^{(\alpha)})$ be a solution corresponing to $L^{(\alpha)}$.

Theorem 3.1. *Assume that* **A**, **B** *and* **C** satisfy the relations (2.13). *Let*

$$E_{\alpha\beta}(p,q)=\int_{\partial B}[P_i^{(\alpha)}(p)u_i^{(\beta)}(q)+R_i^{(\alpha)}(p)Du_i^{(\beta)}(q)]da+\int_C Q_i^{(\alpha)}(p)u_i^{(\beta)}(q)ds+$$
$$+\int_B F_i^{(\alpha)}(p)u_i^{(\beta)}(q)dv-\int_B \rho\ddot{u}_i^{(\alpha)}(p)u_i^{(\beta)}(q)dv, \tag{3.1}$$

for all $p,\ q\in I$. Then

$$E_{\alpha\beta}(p,\ q)=E_{\beta\alpha}(q,\ p), \tag{3.2}$$

for all $p,\ q\in I$.

Proof. Let

$$I_{\alpha\beta}(p,\ q)=\tau_{ij}^{(\alpha)}(p)u_{i,j}^{(\beta)}(q)+\mu_{ijk}^{(\alpha)}(p)\chi_{ijk}^{(\beta)}(q). \tag{3.3}$$

It follows from (2.12) and (3.3) that

$$I_{\alpha\beta}(p,q)=A_{ijrs}u_{r,s}^{(\alpha)}(p)u_{i,j}^{(\beta)}(q)+B_{ijrmn}(\chi_{rmn}^{(\alpha)}(p)u_{i,j}^{(\beta)}(q)+$$
$$+\chi_{rmn}^{(\beta)}(q)u_{i,j}^{(\alpha)}(p))+C_{ijkrmn}\chi_{ijk}^{(\beta)}(q)\chi_{rmn}^{(\alpha)}(p). \tag{3.4}$$

Thus we conclude from (2.13) and (3.4) that

$$I_{\alpha\beta}(p,\ q)=I_{\beta\alpha}(q,\ p). \tag{3.5}$$

On the other hand, in view of (2.11) and (3.3)

$$I_{\alpha\beta}(p,q)=[\tau_{ji}^{(\alpha)}(p)u_i^{(\beta)}(q)+\mu_{ijk}^{(\alpha)}(p)u_{k,i}^{(\beta)}(q)]_{,j}-\tau_{ij,j}^{(\alpha)}(p)u_i^{(\beta)}(q)-$$
$$-(\mu_{ijk,j}^{(\alpha)}(p)u_k^{(\beta)}(q))_{,i}+\mu_{ijk,ij}^{(\alpha)}(p)u_k^{(\beta)}(q).$$

If we integrate this relation over B, then we obtain, with the aid of the divergence theorem, that

$$\int_{\partial B} I_{\alpha\beta}(p,q)dv=\int_B [t_i^{(\alpha)}(p)u_i^{(\beta)}(q)+\mu_{ji}^{(\alpha)}(p)u_{i,j}^{(\beta)}(q)]da+$$
$$+\int_B[F_i^{(\alpha)}(p)u_i^{(\beta)}(q)-\rho\ddot{u}_i^{(\alpha)}(p)u_i^{(\beta)}(q)]dv. \tag{3.6}$$

From (2.16), (3.5) and (3.6) we obtain the desired result.

Let u and v be scalar fields on $B\times I$ that are continuous in time. We denote by $u*v$ the convolution of u and v,

$$[u*v](x,\ t)=\int_0^t u(x,\ t-\tau)v(x,\ \tau)d\tau.$$

We introduce the function

$$\gamma(t)=t,\quad t\in I. \tag{3.7}$$

An immediate consequence of Theorem 3.1 is the following result.

Corollary 3.1. *Assume that A, B and C satisfy the symmetry relations (2.13). Let $A^{(\alpha)}$ be a solution corresponding to the external data system $L^{(\alpha)}$ $(\alpha=1,\ 2)$. Then*

$$\int_{\partial B}\gamma*(P_i^{(1)}*u_i^{(2)}+R_i^{(1)}*Du_i^{(2)})da+\int_C\gamma*Q_i^{(1)}*u_i^{(2)}ds+\int_B f_i^{(1)}*u_i^{(2)}dv=$$
$$=\int_{\partial B}\gamma*(P_i^{(2)}*u_i^{(1)}+R_i^{(2)}*Du_i^{(1)})da+\int_C\gamma*Q_i^{(2)}*u_i^{(1)}ds+\int_B f_i^{(2)}*u_i^{(1)}dv, \tag{3.8}$$

where $f_i^{(\alpha)}=\gamma*F_i^{(\alpha)}+\rho(tb_i^{(\alpha)}+a_i^{(\alpha)})$.

Proof. If we take in (3.2), $p=\tau$ and $q=t-\tau$ and integrate with respect to τ from 0 to t, we arrive at

$$\int_{\partial B}(P_i^{(1)}*u_i^{(2)}+R_i^{(1)}*Du_i^{(2)})da+\int_C Q_i^{(1)}*u_i^{(2)}ds+\int_B F_i^{(1)}*u_i^{(2)}dv-$$
$$-\int_B\rho\ddot{u}_i^{(1)}*u_i^{(2)}dv=\int_{\partial B}(P_i^{(2)}*u_i^{(1)}+R_i^{(2)}*Dv_i^{(1)})da+\int_C Q_i^{(2)}*u_i^{(1)}ds+ \tag{3.9}$$
$$+\int_B F_i^{(2)}*u_i^{(1)}dv-\int_B\rho\ddot{u}_i^{(2)}*u_i^{(1)}dv.$$

Clearly,

$$\gamma*\ddot{u}_i^{(\alpha)}=u_i^{(\alpha)}-t\dot{u}_i^{(\alpha)}(x,\ 0)-u_i^{(\alpha)}(x,\ 0). \tag{3.10}$$

Taking the convolution of the relation (3.9) with γ, we conclude with the aid of (3.10) that (3.8) holds.

The proofs of the reciprocal theorems in linear dynamic theories of continuum mechanics are based on the use of the Laplace transform or on a characterization of the boundary-initial-value problem in which the initial conditions are incorporated into the field equations (see [8], [9]). The proof of Corollary 3.1 offers a new method to obtain reciprocity relations.

Corollary 3.2. *Assume that* **A**, **B** *and* **C** *satisfy the symmetry relations (2.13). Let*

$$G(p,\ q)=\int_{\partial B}[P_i(\mathbf{x},\ p)u_i(\mathbf{x},\ q)+R_i(\mathbf{x},\ p)Du_i(\mathbf{x},\ q)]da+$$
$$+\int_C Q_i(\mathbf{x},\ p)u_i(\mathbf{x},\ q)ds+\int_B F_i(\mathbf{x},\ p)u_i(\mathbf{x},\ q)dv, \tag{3.11}$$

for all $p,\ q\in I$. Then

$$\frac{d}{dt}\int_B\rho\mathbf{u}^2dv=\int_0^t[G(t-s,\ t+s)-G(t+s,\ t-s)]ds+\int_B\rho[\dot{u}_i(2t)u_i(0)+$$
$$+\dot{u}_i(0)u_i(2t))]dv. \tag{3.12}$$

Proof. By Theorem 3.1,

$$\int_0^t E_{11}(t+s,\ t-s)ds=\int_0^t E_{11}(t-s,\ t+s)ds. \tag{3.13}$$

Now we apply the relation (3.13) to the process $A=(u_i,\ e_{ij},\ \chi_{ijk},\ \tau_{ij},\ \mu_{ijk})$. It follows from (3.1) and (3.11) that

$$\int_0^t E_{11}(t+s,\ t-s)ds=\int_0^t G(t+s,\ t-s)ds-\int_0^t\int_B\rho\ddot{u}_i(t+s)u_i(t-s)dsdv. \tag{3.14}$$

Similarly,

$$\int_0^t E_{11}(t-s,\ t+s)ds=\int_0^t G(t-s,\ t+s)ds-\int_0^t\int_B\rho\ddot{u}_i(t-s)u_i(t+s)dsdv. \tag{3.15}$$

If we use the relations

$$\int_0^t\ddot{f}(t+s)h(t-s)ds=\dot{f}(2t)h(0)-\dot{f}(t)h(t)+\int_0^t\dot{h}(t-s)\dot{f}(t+s)ds,$$
$$\int_0^t\ddot{h}(t-s)f(t+s)ds=\dot{h}(t)f(t)-\dot{h}(0)f(2t)+\int_0^t\dot{h}(t-s)\dot{f}(t+s)ds. \tag{3.16}$$

then (3.13), (3.14) anj (3.15) imply (3.12).

An immediate consequence of Corollary 3.2 is the following uniqueness result of Brun type [6].

Corollary 3.3. *Assume that*

(*i*) ρ *is strictly positive*;

(*ii*) **A**, **B** and **C** *satisfy the symmetry relations (2.13).*

Then the boundary—initial—value problem of dynamic theory has at most one solution.

Proof. Suppose that there are two solutions. Then their difference $(u_i^0, e_{ij}^0, \chi_{ij}^0, \tau_{ijk}^0, \mu_{ijk}^0)$ corresponds to null data. Thus, we conclude from (3.12) that

$$\int_B \rho u_i^0 u_i^0 dv = 0,$$

so that $\mathbf{u}^0 = 0$ on $B \times I$.

4. A minimum principle. In this section we extend the result from [10] in order to establish a minimum principle in the dynamic theory of nonsimple elastic bodies. Our method to take into account the initial data is different from the one given in [10].

We say that ψ is bounded at infinity if $\lim_{t\to\infty} \psi(x, t)$ exists for each x in the domain of definition of ψ. Let X be the set of all functions ψ on $\bar{B} \times I$ with the following properties :

(*i*) ψ is of class $C^{4,2}$ on $B \times (0, \infty)$ and of class $C^{3,0}$ on $\bar{B} \times I$;

(ii) ψ, $\psi_{,t}$, $\dot{\psi}$, $\ddot{\psi}$, $\psi_{,ij}$, $\psi_{,ijk}$ and $\psi_{,mnpq}$ are bounded at infinity.

Following [10], we introduce the set Γ of admissible weight functions. We say that $g \in \Gamma$ if g is a function on I with the following properties

$$\int_0^\infty \int_0^\infty g^{(k)}(t+s)dtds \text{ exists for } k \geq 0, \tag{α}$$

$$g(t) = \int_0^\infty G(p)e^{-pt}dp, \quad t \in I, \tag{β}$$

where $g^{(k)} = \partial^k g/\partial t^k$, and G is continous and positive on I and has a finite limit at infinity. An example of weight function is $g(t) = (t+a)^{-n}$, $n > 2$, $a > 0$ with $G(t) = [t^{n-1}\exp(-at)]/(n-1)!$(see[10]).

Let Y be the set of all continous functions on $\bar{B} \times I$ that are bounded at infinity. Let $u, v \in Y$. We introduce the notation

$$[u, v]_g = \int_0^\infty \int_0^\infty \int_B g(t+s)u(x, t)v(x, s)dtdsdv, \tag{4.1}$$

where g is any fixed function of Γ. We note that

$$[u, v]_g = [v, u]_g. \tag{4.2}$$

Let A be a linear operator from X into Y. We denote by D_A the domain of definition of the operator A. We say that the operator A is g-symmetric if

$$[Au, v]_g = [u, Av]_g, \tag{4.3}$$

for any $u, v \in D_A$. A g-symmetric operator A is called g-positive if

$$[Au, u]_g \geq 0, \tag{4.4}$$

for any $u \in D_A$. We consider the equation

$$Au = F, \tag{4.5}$$

where $F \in Y$. Let Λ_g be the functional on D_A defined by

$$\Lambda_g\{u\} = [Au, u]_g - 2[u, F]_g$$

for every $u \in D_A$. We have the following well-known result

Lemma 4.1 *Let A be a g-positive operator and let Λ_g be the functional on D_A defined by (4.6). Further, let u be a solution of the equation (4.5) Then*

$$\Lambda_g\{u\} \leq \Lambda_g\{u^0\},$$

for every $u^0 \in D_A$.

In what follows we denote by $\hat{\psi}$ the Laplace transform with respect to time of the function ψ. By (β) and (4.1),

$$[u, v]_g = \int_0^\infty \int_B G(s)\hat{u}(\mathbf{x}, s)\hat{v}(\mathbf{x}, s)dsd\mathbf{x}, \tag{4.7}$$

for any $u, v \in X$.

We now consider the equations of the theory of nonsimple elastic bodies. To the field equations we adjoin the initial conditions (2.19) and the boundary conditions

$$u_i = 0 \text{ on } \Sigma_1 \times I, \quad Du_i = 0 \text{ on } \Sigma_3 \times I,$$

$$P_i = 0 \text{ on } \Sigma_2 \times I, \quad R_i = 0 \text{ on } \Sigma_4 \times I,$$

$$Q_i = 0 \text{ on } C \times I. \tag{4.8}$$

Following [8], [11], we have

Lemma 4.2. *The functions u_i, τ_{ij}, and μ_{ijr} satisfy the equations of motion (2.14) on $B \times (0, \infty)$ and the initial conditions (2.19), if and only if*

$$\gamma * (\tau_{ji,j} - \mu_{rji,rj}) + f_i = \rho u_i \text{ on } B \times I, \tag{4.9}$$

where

$$f_i = \gamma * F_i + \rho(tb_i + a_i).$$

The equations (4.9), (2.12) and (2.11) combine to yield the basic equations expressed in terms of the displacement field

$$\gamma * [A_{ijrs}u_{r,s} + B_{ijpqr}u_{r,pq} - (B_{mnrji}u_{m,n} + C_{rjipqs}u_{s,pq})_{,r}]_{,j} + f_i =$$
$$= \rho u_i \text{ on } B \times I. \tag{4.10}$$

Clearly, if **u** satisfies the equations (4.10) then the initial conditions (2.19) are satisfied.

We say that **u** is an admissible displacement field if $u_i \in X$. By a solution of the problem we mean an admissible displacement field that satisfies the equations (4.10) and the boundary conditions (4.8).

Let D be the set of all admissible displacement fields that satisfy the conditions (4.8). We introduce the operators L_i on D defined by

$$L_i\mathbf{u} = \rho u_i - \gamma * [A_{ijrs}u_{r,s} + B_{ijpq}u_{r,pq} - (B_{mnrji}u_{m,n} + C_{rjipqs}u_{s,pq})_{,r}]_{,j} \tag{4.11}$$

If we introduce the notations

$$\mathbf{L}\mathbf{u}=(L_1\mathbf{u},\ L_2\mathbf{u},\ L_3\mathbf{u}),\ \mathbf{f}=(f_1,\ f_2,\ f_3), \tag{4.12}$$

then the equation (4.10) can be written in the form of one vector equation

$$\mathbf{L}\mathbf{u}=\mathbf{f} \text{ on } B\times I. \tag{4.13}$$

Let $\mathbf{u}$ and $\mathbf{v}$ be vector fields on $\bar{B}\times I$ such that $u_i,\ v_i\in Y$. We introduce the notation

$$[\mathbf{u},\ \mathbf{v}]_g=\sum_{i=1}^{3}[u_i,\ v_i]_g. \tag{4.14}$$

The elastic potential is defined by

$$W=\frac{1}{2}A_{ijrs}e_{ij}e_{rs}+B_{ijpqr}e_{ij}\varkappa_{pqr}+\frac{1}{2}C_{ijkpqr}\varkappa_{ijk}\varkappa_{pqr}. \tag{4.15}$$

Theorem 4.1. *Assume that*

(i) *the density field* ρ *is strictly positive;*

(ii) $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$ *satisfy the symmetry relations (2.13);*

(iii) *the elastic potential* W *is a positive semi-definite quadratic form.*

Then the operator $\mathbf{L}$ *is* g*-positive.*

Proof. Consider two external data systems $L^{(\alpha)}$ with null boundary data. Let $\mathbf{u}$ and $\mathbf{v}$ be the displacement fields corresponding to $L^{(1)}$ and $L^{(2)}$, respectively. Then

$$\mathbf{L}\mathbf{u}=\mathbf{f}^{(1)},\quad \mathbf{L}\mathbf{v}=\mathbf{f}^{(2)}, \tag{4.16}$$

where $\mathbf{f}^{(\alpha)}=(f_1^{(\alpha)}\ f_2^{(\alpha)},\ f_3^{(\alpha)})$, $(\alpha=1,\ 2)$. It follows from (3.8) that

$$\int_B f_i^{(1)}*v_i dv=\int_B f_i^{(2)}*v_i dv.$$

If we take the Laplace transform of this relation we obtain with the help of (4.7), the relation

$$[\mathbf{L}\mathbf{u},\ \mathbf{v}]_g=[\mathbf{u},\ \mathbf{L}\mathbf{v}]_g \tag{4.17}$$

for any $\mathbf{u},\ \mathbf{v}\in D$.

From (3.4) and (3.6) we obtain

$$\begin{aligned}&\int_B[A_{ijrs}u_{r,s}(p)u_{i,j}(q)+B_{ijrmn}(u_{n,mr}(p)u_{i,j}(q)+u_{n,mr}(q)u_{i,j}(p))+\\&C_{ijkrmn}u_{k,ij}*(q)u_{n,rm}(p)]dv=\int_B F_i(p)u_i(q)dv-\int_B \rho\ddot{u}_i(p)u_i(q)dv.\end{aligned} \tag{4.18}$$

Now we take in (4.18), $p=\tau$ and $q=t-\tau$ and integrate with respect to τ from 0 to t. Then, we take the convolution of the resulting relation with γ. We obtain

$$\begin{aligned}&\int_B[A_{ijrs}u_{r,s}*u_{i,j}+2B_{ijrmn}u_{n,mr}*u_{i,j}+\\&+C_{ijkrmn}*u_{k,ij}*u_{n,rm}]dv=\int_B f_i*u_i dv-\int_B \rho u_i*u_i dv.\end{aligned} \tag{4.19}$$

If we take the Laplace transform of this relation we get

$$\begin{aligned}\int_B\hat{f}_i\hat{u}_i dv=\int_B[\rho\hat{u}_i\hat{u}_i+\hat{\gamma}(A_{ijrs}\hat{u}_{rs}\hat{u}_{i,j}+2B_{ijrmn}\hat{u}_{n,mr}\hat{u}_{i,j}+\\+C_{ijkrmn}\hat{u}_{k,ij}\hat{u}_{n,rm})]dv.\end{aligned} \tag{4.20}$$

Clearly, $\hat{\gamma}(s)=s^{-2}$. By (4.20), (4.7) and (4.1) we obtain

$$\begin{aligned}[\mathbf{L}\mathbf{u},\ \mathbf{u}]_g=&\int_0^\infty\int_B s^{-2}G(s)[A_{ijmn}\hat{u}_{i,j}(s)\hat{u}_{m,n}(s)+2B_{ijrmn}\hat{u}_{n,mr}(s)\hat{u}_{i,j}(s)+\\&+C_{ijkrmn}\hat{u}_{k,ij}(s)\hat{u}_{n,rm}(s)+\rho s^2\hat{u}_i(s)\hat{u}_i(s)]dsdv=\\=&\int_0^\infty\int_0^\infty\int_B g(t+s)[A_{ijmn}\bar{u}_{i,j}(t)\bar{u}_{m,n}(s)+2B_{ijrmn}\bar{u}_{i,j}(t)\bar{u}_{n,mr}(s)+\\&+C_{ijkrmn}\bar{u}_{k,ij}(t)\bar{u}_{n,rm}(s)+\rho u_i(t)u_i(s)]dtdsdv,\end{aligned} \tag{4.21}$$

where we have used the notation

$$\bar{h}(\mathbf{x},t)=\int_0^t h(x,s)ds.$$

By the hypotheses (i), (iii) and the relations (4.17) and (4.21) we obtain the desired result.

Let Φ_g be the functional on D defined by

$$\Phi_g\{\mathbf{u}\}=[\mathbf{L}\mathbf{u},\ \mathbf{u}]_g-2[\mathbf{u},\ \mathbf{f}]_g.$$

By (4.21) we obtain

$$\begin{aligned}\Phi_g\{\mathbf{u}\}=&\int_0^\infty\int_0^\infty\int_B g(t+s)[A_{ijmn}\bar{u}_{i,j}(t)\bar{u}_{m,n}(s)+2B_{ijrmn}\bar{u}_{i,j}(t)\bar{u}_{n,mr}(s)+\\&+C_{ijkr}\bar{u}_{k,ij}(t)\bar{u}_{n,rm}(s)+\rho u_i(t)u_i(s)-2f_i(t)u_i(s)]dtdsdv.\end{aligned} \tag{4.22}$$

In view of Lemma 4.1 and Theorem 4.1 we are led to the following result

Theorem 4.2. *Suppose that the density field is strictly positive and that* W *is a positive semi-definite quadratic form. Let* Φ_g *be the functional on* D *defined by (4.22). Further, let* $\mathbf{u}\in D$ *be a solution of the boundary-initial-value problem (2.11), (2.12), (2.15), (4.8) and (2.19). Then*

$$\Phi_g\{\mathbf{u}\}\leq\Phi_g\{\mathbf{u}^0\},$$

for every $\mathbf{u}^0\in D$.

5. Existence and continous dependence results. Throughout this section we assume that:

(i) the density ρ is strictly positive;

(ii) $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are of class C^2 on B and satisfy the symmetry relations (2.13);

(iii) the elastic potential is a positive definite quadratic form (see [4]), i.e. there exists a positive constant such that

$$A_{ijrs}\xi_{ij}\xi_{rs}+2B_{ijrmn}\xi_{ij}\eta_{rmn}+C_{ijkpqr}\eta_{ijk}\eta_{pqr}\leq c(\xi_{ij}\xi_{ij}+\eta_{ijk}\eta_{ijk}), \tag{5.1}$$

for all ξ_{ij} and η_{ijk} such that $\xi_{ij}=\xi_{ji}$ and $\eta_{ijk}=\eta_{jik}$.

In this section we use results of the semigroups theory of linear operators to obtain an existence theorem. To the system of field equations we adjoin the initial conditions (2.19) and the boundary conditions

$$u_i=0,\quad Du_i=0 \text{ on } \partial B\times R^+. \tag{5.2}$$

We assume that ∂B is smooth. Let

$$Z=\{w=(\mathbf{u},\ \mathbf{v});\ \mathbf{u}\in\mathbf{W}_0^{2,2}(B),\ \mathbf{v}\in\mathbf{L}^2(B)\}, \tag{5.3}$$

where $W_0^{2,2}(B)$ is the familiar Sobolev space (see e.g. [12]) and $\mathbf{W}_0^{2,2}(B) = [W_0^{2,2}(B)]^3$.

Let M_i be the operator on $\mathbf{W}_0^{2,2}(B)$ defined by

$$M_i u = \frac{1}{\rho}[A_{jirs}u_{r,s} + B_{jipqr}u_{r,pq} - (B_{mnrji}u_{m,n} + C_{rjipqm}u_{m,pq})_{,r}]_{,j}. \tag{5.4}$$

We introduce the notations

$$\mathbf{Mu} = (M_i\mathbf{u}),\ \mathbf{I} = \text{the identity operator.}$$

Consider now the matrix operator $\mathcal{A}$ on Z defined by

$$\mathcal{A} = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{M} & 0 \end{pmatrix}, \tag{5.5}$$

with the domain $D(\mathcal{A}) = (W_0^{2,2}(B) \cap W^4(B))^3 \times (W_0^{2,2}(B))^3$. Clearly $D(\mathcal{A})$ is dense in Z.

The boundary-initial-value problem (2.11), (2.12), (2.15), (2.19) and (5.2) can be transformed into the following abstract equation in the space Z

$$\frac{dw(t)}{dt} = \mathcal{A}w(t) + F(t),$$

$$w(0) = w_0, \tag{5.6}$$

where

$$F = \left(0,\ \frac{1}{\rho}\mathbf{F}\right),\quad w_0 = (\mathbf{a},\ \mathbf{b}). \tag{5.7}$$

Let $w = (\mathbf{u},\ \mathbf{v})$ and $\tilde{w} = (\tilde{\mathbf{u}},\ \tilde{\mathbf{v}})$. We introduce the inner product in $\mathbf{Z}$ defined by

$$(w, \tilde{w})_* = \int_B [\rho v_i \tilde{v}_i + A_{ijrs}u_{(r,s)}\tilde{u}_{(i,j)} + B_{ijpqr}(u_{(i,j)}\tilde{u}_{r,pq} + \tilde{u}_{(i,j)}u_{r,pq}) + C_{ijkpqr}u_{r,pq}\tilde{u}_{k,ij}]dv, \tag{5.8}$$

where $u_{(i,j)}$ denotes the symmetric part of $u_{i,j}$. By the hypothesis (5.1) and the first Korn inequality we conclude that the norm induced by the inner product (5.8) is equivalent to the usual norm in $\mathbf{Z}$.

Lemma 5.1. *The operator A satisfies the equality $(Aw, w)_* = 0$, for every $w \in D(A)$.*

Prof. By (5.5), (5.6) and (5.8),

$$(Aw, w)_* = \int_B \{v_i[\tau_{ji}(\mathbf{u}) - \mu_{rji,r}(\mathbf{u})]_{,j} + A_{ijrs}u_{(r,s)}v_{(i,j)} + B_{ijpqr}(u_{(i,j)}v_{r,pq} + v_{(i,j)}u_{r,pq}) + C_{ijkpqr}u_{r,pq}v_{k,ij}\}dv.$$

Using the divergence theorem, (2.3), (2.8), (2.14), (2.16) and the boundary conditions (5.2) we obtaine the desired result.

Lemma 5.2. *The operator $\mathcal{A}$ satisfies the range condition*

$$\text{Rang}(-\mathcal{A}) = \mathbf{Z}. \tag{5.9}$$

Proof. Let $\bar{w} = (\mathbf{f},\ \mathbf{g}) \in \mathbf{Z}$. We must prove that the equation

$$-\mathcal{A}w = \bar{w}, \tag{5.10}$$

has a solution $w = (\mathbf{u},\ \mathbf{v})$ in $D(\mathcal{A})$. From (5.10) we obtain

$$-\mathbf{Mu} = \mathbf{g},\quad -\mathbf{v} = \mathbf{f}.$$

To study the first equation, we consider the following bilinear form in $\mathbf{W}_0^{2,2}$

$$\mathcal{B}(\mathbf{u},\ \mathbf{u}^*) = -\langle \mathbf{u},\ \mathbf{Mu}^* \rangle_{L^2(B)}.$$

We note that in the static case, Theorem 3.1 implies that the operator $\mathbf{M}$ is symmetric. Moreover, we have

$$-\langle \mathbf{u}, \mathbf{Mu} \rangle_{L^2(B)} = \int_B [A_{ijrs}u_{(i,j)}u_{(r,s)} + 2B_{ijrmn}u_{(i,j)}u_{n,rm} + C_{ijpqr}u_{k,ij}u_{r,pq}]dv.$$

By using the hypothesis (5.1) we conclude that the bilinear form $\mathcal{B}$ is coercive. It follows from Lax-Milgram theorem [13, p. 85] the existence of $\mathbf{u} \in \mathbf{W}_0^{2,2}$ such that $-\mathbf{Mu} = \mathbf{g}$.

Theorem 5.1. *The operator $\mathcal{A}$ generates a contractive semigroup in Z.*

Proof. The proof follows from the Lemmas 5.1, 5.2 and the Lummer-Philips corollary to the Hille-Yosida theorem [14, p. 340].

The next theorem is an immediate consequence of Theorem 5.1.

Theorem 5.2. *Assume that $F_i \in C^1(\mathbf{R}^+; L_2(B)) \cap C^0(\mathbf{R}^+,\ W_0^{2,2}(\mathcal{B}))$ and $w_0 \in D(\mathcal{A})$.*

Then there exists a unique solution $w \in C^1(\mathbf{R}^+, Z)$ with values in $D(\mathcal{A})$ to the boundary initial-value problem (2.11), (2.12), (2.15), (2.19) and (5.2).

Remark 5.1. Since the sēmigroup defined by the operator $\mathcal{A}$ is contractive, we have the following estimate

$$\|w\|_Z \leq \left(\|w_0\|_Z + \int_0^t \|\mathbf{F}(s)\|_{L^2(B)}ds\right).$$

This inequality gives a result of continous dependence upon the initial data and body forces. Thus under the hypotheses (i)—(iii), the problem is well posed.

Remark 5.2. If $\mathbf{C}$ is continous on B and there exists a positive constant c_0 such that

$$C_{ijkpqr}\xi_i\xi_j\xi_p\xi_q\eta_k\eta_r \geq c_0\xi^4\eta^2. \tag{5.11}$$

for all $\xi,\ \eta \in \mathbf{R}^3$ then by using the Gärding inequality (see e.g. [15, p. 135]) we are led to a quasi-contractive semigroup.

Remark 5.3. By hypothesis (5.11) we may conclude the existence of a positive constant k such that:

$$\|\mathbf{u}\|_{W^{2,s}} \leq k(\|\mathbf{Mu}\|_{W^{2,s-4}} + \|\mathbf{u}\|_{L^2}),$$

whenever $s \leq 4$ and $\mathbf{u} \in W^{2,4}$. Then we see that

$$D(\mathbf{M}^r) \subset \mathbf{W}^{2,4r} \subset C^l$$

whenever $l < 4r - n/2$, because the Sobolev imbedding (see e.g. [12, p. 97]).

Using the basic properties of the symigroups theory we may conclude C^∞ - regularity of the solutions if the initial data (u_0, v_0) satisfies the conditions

$$u_0, v_0 \in \cap_{n \geq 1} D(M^n).$$

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Department of Mathematics University of Iaşi, 6600 Iaşi, Romania
Department Matematica Aplicada, II. Universidad Politecnica de Catalunya, Ap. 508, Terrassa, Spain

Assistant Professor Dr. $\boxed{\text{VASILE POPESCU}}$

It was completely unexpected that one of the most talented and intellectually endowed mathematicians of Iaşi, the assistant professor Vasile Popescu passed away on the 21-st of March 1990, leaving an impressive gap behind him for the Faculty of Mathematics of the "Al. I. Cuza" University of Iaşi.

Born on the 11-th of March 1946, in Armăşoaia, the comity of Vaslui, in a family of modest and hard-working householders, he, as well as his brothers and sisters, he received a sound instruction in the spirit of honesty and work ; he attended the primary school in his native village, and then went to Pungeşti where he attended school between 1957—1960. In 1960 he became, by competitive examination, a student of the Pedagogical High-school affiliated with the School Group in the district of Bîrlad which he attended for two years. As he distinguished himself there due to his remarkable intellectual abilities and a keen desire of through and deep-going knowledge, his teachers, urged him and finally convinced him to go for a theoretical highschool within the same school Group/ Complex. He successfully graduated highschool în 1964. During the highschool period, he took part in highschool level contests obtaining quite remarkable results, among which we may mention the third prize at the national contest in physics when he was a third year highschool student.

Going his own way and following his in most intellectual propensities, he was admitted, again by competitive examination, as a first year undergraduate of the Faculty of Mathematics & Physics of the "Al. I. Cuza" University of Iaşi which he graduated in 1969 with an honour's diploma. Due to his excellent results as an undergaduate, he was appointed, by ministerial decision, assistant lecturer with the Department of differential equations and numerical analysis where he worked until September 1971. Afterwards he did his military duty and he came back to work in 1972. As lecturer assistant, he held brilliant seminars on different subjects such as : ordinary differential equations, partially differential equations, integral equations, potential theory, special mathematics with the Physics & Chemistry faculties, numerical analysis, functional analysis, the optimal control theory and special chapters on differential equations. In 1977, he was promoted, by competitive examination, assistant professor at the same department. In this capacity he delivered courses of lectures and seminars in various subjects such as : Differential equations and integral equations, the Equations of mathematical physics, Bases of Informatics, Informatics Systems, Numerical Solving of partially differential equations, optimal control, Special methods of numerical analysis, Mathematics for Economists, Logic and mass theory and numerical theory.