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MEROMORPHIC P-VALENT CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS

BY

G. SOMANATHA

1. Introduction. Let $f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n}$ be regular and p-valent in the punctured disk $E = \{z : 0 < |z| < 1\}$. $f(z)$ is said to be p-valent meromorphically convex of order α if $\mathrm{Re}(1 + zf''(z)/f'(z)) < -p\alpha$ for $|z| < 1$. Let $\sigma_k(p, \alpha)$ denote the class of functions of the form $f(z) = \frac{1}{z^p} - \sum_{n=0}^{\infty} a_{p+n} z^{p+n}$, $(a_{p+n} \geq 0)$ which are p-valent meromorphically convex in E.

In this paper we obtain coefficient estimates, distortion and closure theorems for the class $\sigma_k(p, \alpha)$. We also obtain the class preserving integral operator of the form

$$F(z) = \frac{c-p+1}{z^{c+1}} \int_0^z t^c f(t)\,dt, \quad c > p-1 \tag{1}$$

for the functions f in the class $\sigma_k(p, \alpha)$. Conversely, when $F \in \sigma_k(p, \alpha)$ the disk in which $\mathrm{Re}(1 + zf''(z)/f'(z)) < -p\beta$ is obtained.

While considering the integrals we have been able to extend the result of S. K. Bajpai [1], for functions with negative coefficients.

In [3] Uralegaddi and Ganigi examined the meromorphic multivalent functions with positive coefficients, and in [4] they studied the meromorphic convex functions with negative coefficients.

We employ technique similar to those of Silverman [2].

2. Coefficient Inequalities.

Theorem 1. *Let* $f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n}$ *be regular and p-valent in E. If* $\sum_{n=0}^{\infty} (p+n)(p+n+\alpha p)|a_{p+n}| \leq (1-\alpha)p^2$ *then* $\mathrm{Re}\{1 + zf''(z)/f'(z)\} < -p\alpha$ *for* $|z| < 1$.

Proof. It suffices to show that

$$\left|\left(1 + \frac{zf''(z)}{f'(z)} + p\right)\Big/\left(1 + \frac{zf''(z)}{f'(z)} - p(1-2\alpha)\right)\right| < 1 \text{ for } |z| < 1.$$

We have

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p(1-2\alpha)}\right| = \left|\frac{\sum_{n=0}^{\infty}(p+n)(2p+n)a_{p+n}z^{2p+n}}{2p^2(1-\alpha)+\sum_{n=0}^{\infty}(p+n)(n+2p\alpha)a_{p+n}z^{2p+n}}\right| \leqslant$$

$$\leqslant \left(\sum_{n=0}^{\infty}(p+n)(2p+n)|a_{p+n}|\right)\Big/\left(2p^2(1-\alpha)-\sum_{n=0}^{\infty}(p+n)(n+2p\alpha)|a_{p+n}|\right).$$

The last expression is bounded by 1 if

$$\sum_{n=0}^{\infty}(p+n)(2p+n)|a_{p+n}| \leqslant 2p^2(1-\alpha)-\sum_{n=0}^{\infty}(p+n)(n+2\alpha p)|a_{p+n}|,$$

which is equivalent to

(2) $$\sum_{n=0}^{\infty}(p+n)(p+n+\alpha p)|a_{p+n}| \leqslant (1-\alpha)p^2.$$

But (2) is true by hypothesis. The result follows.

Theorem 2. *A function* $f(z)=\frac{1}{z^p}-\sum_{n=0}^{\infty}a_{p+n}z^{p+n}$, $(a_{p+n}\geqslant 0)$ *is in* $\sigma_k(p,\alpha)$ *if and only if*

$$\sum_{n=0}^{\infty}(p+n)(p+n+\alpha p)a_{p+n}\leqslant(1-\alpha)p^2.$$

Proof. In view of theorem 1, it suffices to show the only if part. Assume that

(3) $$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}=\mathrm{Re}\left(p^2-\sum_{n=0}^{\infty}(p+n)^2a_{p+n}z^{2p+n}\right)\Big/\left(-p-\sum_{n=0}^{\infty}(p+n)a_{p+n}z^{2p+n}\right)<$$

$$<-p\alpha \text{ for } |z|<1.$$

Choose values of z on the real axis so that $1+zf''(z)/f'(z)$ is real. Upon clearing the denominator in (3) and letting $z\to 1$ through real values we get

$$p^2-\sum_{n=0}^{\infty}(p+n)^2a_{p+n}\geqslant p^2\alpha+p\alpha\sum_{n=0}^{\infty}(p+n)a_{p+n}.$$ That is

$$\sum_{n=0}^{\infty}(p+n)(p+n+\alpha p)a_{p+n}\leqslant(1-\alpha)p^2.$$

Corollary 1. *If* $f\in\sigma_k(p,\alpha)$ *then* $a_{p+n}\leqslant(1-\alpha)p^2/[(p+n)(p+n+\alpha p)]$ *with equality for functions of the form*

$$f_n(z)=\frac{1}{z^p}-(1-\alpha)p^2/[(p+n)(p+n+\alpha p)]z^{p+n}.$$

3. Distortion theorem.

Theorem 3. *If* $f\in\sigma_k(p,\alpha)$ *then*

(4) $$\frac{1}{r^p}-\frac{1-\alpha}{1+\alpha}r^p\leqslant|f(z)|\leqslant\frac{1}{r^p}+\frac{1-\alpha}{1+\alpha}r^p\quad(|z|=r)$$

with equality for $f(z)=\frac{1}{z^p}-\frac{1-\alpha}{1+\alpha}z^p$ $(z=r,\ r\,e^{i\pi/2p})$ *and*

(5) $$\frac{p}{r^{p+1}}-\frac{1-\alpha}{1+\alpha}pr^{p-1}\leqslant|f'(z)|\leqslant\frac{p}{r^{p+1}}+\frac{1-\alpha}{1+\alpha}pr^{p-1}.$$

Proof. Let $f(z)=\frac{1}{z^p}-\sum_{n=0}^{\infty}a_{p+n}z^{p+n}$, $(a_{p+n}\geqslant 0)$. Then it follows from theorem 2 that, $\sum_{n=0}^{\infty}a_{p+n}\leqslant\frac{1-\alpha}{1+\alpha}$. Hence

$$|f(z)|\leqslant\frac{1}{r^p}+\sum_{n=0}^{\infty}a_{p+n}r^{p+n}\leqslant\frac{1}{r^p}+r^p\frac{1-\alpha}{1+\alpha}$$

and

$$|f(z)|\geqslant\frac{1}{r^p}-\sum_{n=0}^{\infty}a_{p+n}r^{p+n}\geqslant\frac{1}{r^p}-r^p\sum_{n=0}^{\infty}a_{p+n}\geqslant\frac{1}{r^p}-r^p\frac{1-\alpha}{1+\alpha}.$$

Thus (4) follows. Again from theorem 2 it follows that $\sum_{n=0}^{\infty}(p+n)a_{p+n}\leqslant\frac{p(1-\alpha)}{1+\alpha}$.

Since $f'(z)=\frac{-p}{z^{p+1}}-\sum_{n=0}^{\infty}(p+n)a_{p+n}z^{p+n-1}$, we have $|f'(z)|\leqslant\frac{p}{r^{p+1}}+\frac{1-\alpha}{1+\alpha}pr^{p-1}$.

Also $|f'(z)|\geqslant\frac{p}{r^{p+1}}-\frac{1-\alpha}{1+\alpha}pr^{p-1}$. Thus (5) follows.

4. Closure theorem. In this section we shall show that the class $\sigma_k(p,\alpha)$ is closed under convex linear combinations.

Theorem 4. *Let* $f_{-1}(z)=\frac{1}{z^p}$ *and*

$$f_n(z)=\frac{1}{z^p}-\frac{(1-\alpha)p^2}{(p+n)(p+n+\alpha p)}z^{p+n}\quad\text{for } n=0,1,2,\ldots$$

Then $f\in\sigma_k(p,\alpha)$ *if and only if* $f(z)$ *can be expressed in the form* $f(z)=\sum_{n=-1}^{\infty}\lambda_nf_n(z)$ *where* $\lambda_n\geqslant 0$ *and* $\sum_{n=-1}^{\infty}\lambda_n=1$.

Proof. Suppose

$$f(z)=\sum_{n=-1}^{m}\lambda_nf_n(z)=\lambda_{-1}f_{-1}(z)+\sum_{n=0}^{\infty}\lambda_nf_n(z)=$$

$$=\frac{\lambda_{-1}}{z^p}+\sum_{n=0}^{\infty}\lambda_n\left[\frac{1}{z^p}-\frac{(1-\alpha)p^2}{(p+n)(p+n+\alpha p)}z^{p+n}\right]=$$

$$=\frac{1-\sum\limits_{n=0}^{\infty}\lambda_n}{z^p}+\sum_{n=0}^{\infty}\frac{\lambda_n}{z^p}-\sum_{n=0}^{\infty}\frac{\lambda_n(1-\alpha)p^2}{(p+n)(p+n+\alpha p)}z^{p+n}=$$

$$=\frac{1}{z^p}-\sum_{n=0}^{\infty}\frac{\lambda_n(1-\alpha)p^2}{(p+n)(p+n+\alpha p)}z^{p+n}.$$

Then

$$\sum_{n=0}^{\infty}\frac{\lambda_n(1-\alpha)p^2}{(p+n)(p+n+\alpha p)}\left(\frac{(p+n)(p+n+\alpha p)}{(1-\alpha)p^2}\right)=\sum_{n=0}^{\infty}\lambda_n=1-\lambda_{-1}\leqslant 1.$$

By theorem 2, it follows that $f\in\sigma_k(p,\alpha)$. Conversely, let $f\in\sigma_k(p,\alpha)$. Since from corollary 1,

$a_{p+n}\leqslant(1-\alpha)p^2/[(p+n)(p+n+\alpha p)]$. We may set,

$$\lambda_n=\frac{(p+n)(p+n+\alpha p)}{(1-\alpha)p^2}a_{p+n},\ n=0,1,2,\ \dots\ \text{and}\ \lambda_{-1}=1-\sum_{n=0}^{\infty}\lambda_n.$$

Then $f(z)=\sum\limits_{n=-1}^{\infty}\lambda_n f_n(z)$.

5. Integral operators.

Theorem 5. *If* $f(z)=\frac{1}{z^p}-\sum\limits_{n=0}^{\infty}a_{p+n}z^{p+n}$, $(a_{p+n}\geqslant 0)$ *is in* $\sigma_k(p,\alpha)$ *then*

$$F(z)=\frac{c-p+1}{z^{c+1}}\int_0^z t^c f(t)dt=\frac{1}{z^p}-\sum_{n=0}^{\infty}\frac{c-p+1}{c+p+n+1}a_{p+n}z^{p+n},\quad c>p-1$$

also belongs to $\sigma_k\left(\frac{p+\alpha+\alpha c}{1+\alpha p+c}\right)$.

Proof. Suppose $f(z)\in\sigma_k(p,\alpha)$. Then $\sum\limits_{n=0}^{\infty}(p+n)(p+n+\alpha p)a_{p+n}\leqslant(1-\alpha)p^2$.

In view of theorem 2, we shall find the largest β for which

$$\sum_{n=0}^{\infty}\frac{(p+n)(p+n+\beta p)}{(1-\beta)p^2}\left(\frac{c-p+1}{c+p+n+1}\right)a_{p+n}\leqslant 1.$$

It suffices to find the range of values of β for which

$$(c-p+1)(p+n+\beta p)/[(1-\beta)(c+p+n+1)]\leqslant(p+n+\alpha p)/(1-\alpha)\quad\text{for each}$$

$$n=0,\ 1,\ 2,\ \dots$$

From the above inequality we obtain

$$\beta\leqslant[n^2+n(3p+\alpha c+\alpha)+2p(p+\alpha c+\alpha)]/[n^2+n(1+2p+\alpha p+c)+2p(1+\alpha p+c)].$$

That is

$$\beta\leqslant 1+[(1-\alpha)(p-c-1)(n+2p)]/[n^2+n(1+2p+\alpha p+c)+2p(1+\alpha p+c)],$$

$$c>p-1.$$

Since $[(p-c-1)(n+2p)]/[n^2+n(1+2p+\alpha p+c)+2p(1+\alpha p+c)]$ is an increasing function of n, $\beta\leqslant(p+\alpha c+\alpha)/(1+\alpha p+c)$. Thus the result follows.

The result is sharp for the function $f(z)=\frac{1}{z^p}-\frac{1-\alpha}{1+\alpha}z^p$.

Corollary 2. *If* $f(z)=\frac{1}{z^p}-\sum\limits_{n=0}^{\infty}a_{p+n}z^{p+n}$, $(a_{p+n}\geqslant 0)$ *is regular and p-valent in* $0<|z|<1$ *and satisfy* $\operatorname{Re}\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)<\frac{p}{c+1}$ *then* $F(z)=\frac{c-p+1}{z^{c+1}}\int_0^z t^c f(t)dt$, *for* $c>p-1$ *is p-valent convex.*

Putting $c=1$ and $p=1$ in the Corollary 2 we obtain

Corollary 3. *If* $f(z)=\frac{1}{z}-\sum\limits_{n=1}^{\infty}a_nz^n$, $a_n\geqslant 0$ *is regular and univalent and satisfies*

$\operatorname{Re}(1+zf''(z)/f'(z)<1/2$ *then* $F(z)=\frac{1}{z^2}\int_0^z tf(t)dt$ *belongs to* σ_k.

The above Corollary 2 is comparable to the following result of S.K. Bajpai [1] for wider class of functions.

If $f(z)=\frac{1}{z}+\sum\limits_{n=0}^{\infty}a_nz^n$ is convex then so is $F(z)=z^{-2}\int_0^z t\,f(t)dt$.

Theorem 6. *Let* $F(z)=\frac{1}{z^p}-\sum\limits_{n=0}^{\infty}a_{p+n}z^{p+n}$, $(a_{p+n}\geqslant 0)$ *be in* $\sigma_k(p,\alpha)$ *and*

$$f(z)=\frac{[zF'(z)+(c+1)F(z)]}{c-p+1}=\frac{1}{z^p}-\sum_{n=0}^{\infty}a_{p+n}\frac{c+p+n+1}{c-p+1}z^{p+n},\ c>p-1$$

then $\operatorname{Re}(1+zf''(z)/f'(z))<-p\beta$, $0\leqslant\beta<1$, *for* $0<|z|<r(\alpha,\beta,p)$, where

$$r(\alpha,\beta,p)=\operatorname*{Inf}_n\left[\frac{(c-p+1)(1-\beta)(p+n+\alpha p)}{(c+p+n+1)(1-\alpha)(p+n+\beta p)}\right]^{\frac{1}{2p+n}},\ n=0,1,2,\dots$$

The result is sharp for $F(z)=\frac{1}{z^p}-\frac{(1-\alpha)p^2}{(p+n)(p+n+\alpha p)}z^{p+n}$, *for some* $n=0,\ 1,\ 2,\ \dots$

Proof. It suffices to show that

$$\left|\left(1+\frac{zf''(z)}{f'(z)}+p\right)\Big/\left[1+\frac{zf''(z)}{f'(z)}-p(1-2\beta)\right]\right|<1\ \text{for}\ 0<|z|<r(\alpha,\beta,p).$$

We have

$$\left|\frac{1+\dfrac{zf''(z)}{f'(z)}+p}{1+\dfrac{zf''(z)}{f'(z)}-p(1-2\beta)}\right| \leqslant \frac{\sum\limits_{n=0}^{\infty} a_{p+n}\dfrac{c+p+n+1}{c-p+1}(p+n)(2p+n)|z|^{2p+n}}{2p^2(1-\beta)-\sum\limits_{n=0}^{\infty} a_{p+n}\dfrac{c+p+n+1}{c-p+1}(p+n)(n+2\beta p)|z|^{2p+n}}.$$

The last expression is bounded by 1 if

$$\sum_{n=0}^{\infty} a_{p+n}\frac{c+p+n+1}{c-p+1}[(p+n)(2p+n)+(p+n)(n+2\beta p)]|z|^{2p+n}\leqslant 2p^2(1-\beta).$$

That is $\sum\limits_{n=0}^{\infty} a_{p+n}\dfrac{c+p+n+1}{c-p+1}[(p+n)(2p+2n+2\beta p)]|z|^{2p+n}\leqslant 2p^2(1-\beta)$,

which is equivalent to

$$\sum_{n=0}^{\infty} a_{p+n}\frac{(c+p+n+1)(p+n)(p+n+\beta p)}{(c-p+1)p^2(1-\beta)}|z|^{2p+n}\leqslant 1. \tag{6}$$

From theorem 2 we have

$$\sum_{n=0}^{\infty} a_{p+n}\frac{(p+n)(p+n+\alpha p)}{(1-\alpha)p^2}\leqslant 1.$$

Hence (6) is satisfied if

$$(c+p+n+1)(p+n+\beta p)/[(c-p+1)(1-\beta)]|z|^{2p+n}\leqslant (p+n+\alpha p)/(1-\alpha).$$

Solving for $|z|$ we obtain

$$|z|\leqslant\left[\frac{(c-p+1)(1-\beta)(p+n+\alpha p)}{(c+p+n+1)(1-\alpha)(p+n+\beta p)}\right]^{\frac{1}{2p+n}},\ n=0,1,\ 2\ldots.$$

The result follows.

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Department of Mathematics.
G.B.R. College. Huvinahadagali—
583219 India

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A STUDY FOR A UNIQUE SOLUTION FOR A VECTOR DIFFERENTIAL EQUATION WITH MAXIMUM

BY

M. M. SOBEIH and E. A. ALY

I. Introduction. The differential equations with maximums are widely appeared in automatic control and electric engineering. In 1971, A. R. Magomedov studied in [2] a first order scalar differential equation with maximum. Then in 1972, he continued in [2] his researches on the quantitative behaviour of the solution. Other different extensions and generalizations were carried out in several researches such as [3—5,7]. Our present work deals with the question of the existence and the uniqueness of the solution of the second order n-vector differential equation with maximum by using some famous types of approximations and also by using Schauder's and contraction mapping principles. Find the solution for the equation

$$x''(t)=F(t,x(t),x'(t),\max_{\tau\in[t_0,t]}x(\tau),\max_{\tau\in[t_0,t]}x'(\tau))\,;\ t\in J:=[t_0,T], \tag{1}$$

which satisfies the conditions

$$x^{(i)}(t_0)=x_0^{(i)}(i=0,1), \tag{2}$$

where x and F are n-vector functions in Banach space E. $\tau>0$ is a scalar and $\tau\in I:=[t_0,t]$. t_0, T are positive numbers. $x_0^{(i)}$ $(i=0,1)$ are constant n-vectors. Let $C^2[J\,;E]$ be the set of all twice differentiable continuous n-vector function $x(t)$; $t\in J$ whose values belong to E. And let G be the topological space in E for the vector function $x(t)$ i.e. $G=(C^2(J\,;E)\,;E)$ and take in G the set

$$W:=\{x(t)\in G:\|x(t)-x_0\|\leq a,\|x'(t)-x_0'\|\leq b\,;\ a,b\ \text{consts.}\}.$$

Thus, the solution of (1)—(2) will be the vector function $x(t)$ which belongs to G and satisfies conditions (2).

II. Some basic concepts 1. Let $F(t,u_1,u_2,u_3,u_4)$ be defined, continuous in the region

$$R\equiv\{t\in J,\|x(t)-x_0\|\leq a,\|x'(t)-x_0'\|\leq b,\ a,b\ \text{consts.}\} \tag{3}$$

and satisfy the following conditions

(i) $\|F(t,u_1,u_2,u_3,u_4)\|\leq M\,;\ M$ positive number. (4)

(ii) $\|F(t,u_1,u_2,u_3,u_4)-F(t,\bar u_1,\bar u_2,\bar u_3,\bar u_4)\|\leq\Phi(t,\|u_1-\bar u_1\|,\|u_2-\bar u_2\|$
$\|u_3-\bar u_3\|,\|u_4-\bar u_4\|)$, (5)