We have

$$\left|\frac{1+\dfrac{zf''(z)}{f'(z)}+p}{1+\dfrac{zf''(z)}{f'(z)}-p(1-2\beta)}\right| \leqslant \frac{\sum_{n=0}^{\infty} a_{p+n}\dfrac{c+p+n+1}{c-p+1}(p+n)(2p+n)|z|^{2p+n}}{2p^2(1-\beta)-\sum_{n=0}^{\infty} a_{p+n}\dfrac{c+p+n+1}{c-p+1}(p+n)(n+2\beta p)|z|^{2p+n}}.$$

The last expression is bounded by 1 if

$$\sum_{n=0}^{\infty} a_{p+n}\frac{c+p+n+1}{c-p+1}[(p+n)(2p+n)+(p+n)(n+2\beta p)]|z|^{2p+n}\leqslant 2p^2(1-\beta).$$

That is $\sum_{n=0}^{\infty} a_{p+n}\dfrac{c+p+n+1}{c-p+1}[(p+n)(2p+2n+2\beta p)]|z|^{2p+n}\leqslant 2p^2(1-\beta)$,
which is equivalent to

$$\sum_{n=0}^{\infty} a_{p+n}\frac{(c+p+n+1)(p+n)(p+n+\beta p)}{(c-p+1)p^2(1-\beta)}|z|^{2p+n}\leqslant 1. \tag{6}$$

From theorem 2 we have

$$\sum_{n=0}^{\infty} a_{p+n}\frac{(p+n)(p+n+\alpha p)}{(1-\alpha)p^2}\leqslant 1.$$

Hence (6) is satisfied if

$$(c+p+n+1)(p+n+\beta p)/[(c-p+1)(1-\beta)]|z|^{2p+n}\leqslant (p+n+\alpha p)/(1-\alpha).$$

Solving for $|z|$ we obtain

$$|z|\leqslant\left[\frac{(c-p+1)(1-\beta)(p+n+\alpha p)}{(c+p+n+1)(1-\alpha)(p+n+\beta p)}\right]^{\frac{1}{2p+n}},\ n=0,1,\ 2\,\ldots.$$

The result follows.

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A STUDY FOR A UNIQUE SOLUTION FOR A VECTOR DIFFERENTIAL EQUATION WITH MAXIMUM

BY

M. M. SOBEIH and E. A. ALY

I. Introduction. The differential equations with maximums are widely appeared in automatic control and electric engineering. In 1971, A. R. Magomedov studied in [2] a first order scalar differential equation with maximum. Then in 1972, he continued in [2] his researches on the quantitative behaviour of the solution. Other different extensions and generalizations were carried out in several researches such as [3—5,7]. Our present work deals with the question of the existence and the uniqueness of the solution of the second order n-vector differential equation with maximum by using some famous types of approximations and also by using Schauder's and contraction mapping principles. Find the solution for the equation

$$x''(t)=F(t,x(t),x'(t),\max_{\tau\in[t_0,t]}x(\tau),\max_{\tau\in[t_0,t]}x'(\tau))\,;\ t\in J:=[t_0,T], \tag{1}$$

which satisfies the conditions

$$x^{(i)}(t_0)=x_0^{(i)}\ (i=0,1), \tag{2}$$

where x and F are n-vector functions in Banach space E. $\tau>0$ is a scalar and $\tau\in I:=[t_0,t]$. t_0, T are positive numbers. $x_0^{(i)}$ $(i=0,1)$ are constant n-*vectors*. Let $C^2[J;E]$ be the set of all twice differentiable continuous n-vector function $x(t)$; $t\in J$ whose values belong to E. And let G be the topological space in E for the vector function $x(t)$ i.e. $G=(C^2(J;E);E)$ and take in G the set

$$W:=\{x(t)\in G:\|x(t)-x_0\|\le a,\|x'(t)-x_0'\|\le b;\ a,b\ \text{consts.}\}.$$

Thus, the solution of (1)—(2) will be the vector function $x(t)$ which belongs to G and satisfies conditions (2).

II. Some basic concepts 1. Let $F(t,u_1,u_2,u_3,u_4)$ be defined, continuous in the region

$$R\equiv\{t\in J,\|x(t)-x_0\|\le a,\|x'(t)-x_0'\|\le b,\ a,b\ \text{consts.}\} \tag{3}$$

and satisfy the following conditions

(i) $\|F(t,u_1,u_2,u_3,u_4)\|\le M$; M positive number. (4)

(ii) $\|F(t,u_1,u_2,u_3,u_4)-F(t,\bar u_1,\bar u_2,\bar u_3,\bar u_4)\|\le\Phi(t,\|u_1-\bar u_1\|,\|u_2-\bar u_2\|,$
$\|u_3-\bar u_3\|,\|u_4-\bar u_4\|)$, (5)

where the function $\Phi(t, v_1, v_2, v_3, v_4)$ is continuous in R w.r.t. its all arguments and also is non-increasing w.r.t. v_i $(i = \overline{1, 4})$.

(iii)
$$\| F(t, u_1, u_2, u_3, u_4) - F(t, \bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4) \| \leq k \sum_{i=1}^{4} (\| u_i - \bar{u}_i \|), \tag{6}$$

where k is a positive constant.

2. Suppose that $\Phi(s, 0, 0, 0, 0) = 0$, so $\psi(t) = 0$ is the only solution of

$$\psi(t) \leq \int_{t_0}^{t} (t-s)\Phi(s, \psi(s), \psi'(s), \max_{\tau \in I} \psi(\tau), \max_{\tau \in I} \psi'(\tau))\, ds\,;\ t \in J, \tag{7}$$

and so also of $\psi'(t)$ where

$$\psi'(t) \leq \int_{t_0}^{t} \Phi(s, \psi(s), \psi'(s), \max_{\tau \in I} \psi(t), \max_{\tau \in I} \psi'(\tau))ds\,;\ t \in J. \tag{8}$$

3. If $S_m(t)$ $(m = n, n+1, \ldots)$ are continuous functions in J, then we will consider the uniform convergent series $\sum_{m=n}^{\infty} S_m(t)$ and $\sum_{m=n}^{\infty} S'_m(t)$ where

$$S_m(t) = \int_{t_0}^{t} (t - s)\, \Phi(s, S_{m-1}(s), S'_{m-1}(s), \max_{\tau \in I} S_{m-1}(\tau), \max_{\tau \in I} S'_{m-1}(\tau))ds;\ t \in J, \tag{9}$$

and

$$S'_m(t) = \int_{t_0}^{t} \Phi(s, S_{m-1}(s), S'_{m-1}(s), \max_{\tau \in I} S_{m-1}(\tau), \max_{\tau \in I} S'_{m-1}(\tau))\, ds\,;\ t \in J. \tag{10}$$

III. The existence of a unique solution using approximations. Problem (1)–(2) is equivalent to the integral equation

$$x(t) = x_0 + x'_0(t-t_0) + \int_{t_0}^{t} (t-s)F(s, x(s), x'(s), \max_{\tau \in I} x(\tau), \max_{\tau \in I} x'(\tau))\, ds;\ t \in J. \tag{11}$$

Thus, to answer the question on the existence of a unique solution for the integral equation (11), we need three types of approximations.

1. **Picard's approximations.** It is known that (11) can be put in the form

$$x_m(t) = x_0 + x'_0(t - t_0) + \int_{t_0}^{t} (t-s)\, F(s, x_{m-1}(s), x'_{m-1}(s), \max_{\tau \in I} x_{m-1}(\tau), \max_{\tau \in I} x'_{m-1}(\tau))\, ds\,;\ t \in J, \tag{12—1}$$

such that

$$x'_m(t) = x_0 + \int_{t_0}^{t} F(s, x_{m-1}(s), x'_{m-1}(s), \max_{\tau \in I} x_{m-1}(\tau), \max_{\tau \in I} x'_{m-1}(\tau))\, ds\,;\ t \in J, \tag{12—2}$$

where $m = 1; 2, \ldots$.

Here, we shall use the following notations:

$$\begin{aligned} \psi_m^{(i)}(t) &= \| x_{m+1}^{(i)}(t) - x_m^{(i)}(t) \|,\ \psi_{1,m}^{(i)}(t) = \| \max_{\tau \in I} x_{m+1}^{(i)}(\tau) - \max_{\tau \in I} x_m^{(i)}(\tau) \|, \\ \psi^i(t) &= \| x^{(i)}(t) - \bar{x}^{(i)}(t) \|,\ \psi_1^{(i)}(t) = \| \max_{\tau \in I} x^{(i)}(\tau) - \max_{\tau \in I} \bar{x}^{(i)}(\tau) \|. \end{aligned} \tag{13}$$

where $\bar{x}$ is some other solution for problem (1), (2) and $i = 0, 1$.

Theorem 1. *Suppose $F(t, u_1, u_2, u_3, u_4)$ is defined, continuous in R and satisfies conditions (4), (5). Also, let the inequalities*

$$\| x_0 \| + \| x'_0 \| r + 1/2\, Mr^2 \leq a,\quad \| x'_0 \| + Mr \leq b, \tag{14}$$

where $r = | T - t_0 |$, be satisfied. Then (12—1, 2) converge to the limits $x(t)$ and $x'(t)$ which represent the unique solution of problem (1), (2).

Proof. It is clear that $x_m(t)$, $x'_m(t)$; $m = 1, 2, 3, \ldots$ are bounded according to (14). Using (5), formulas (12—1, 2) lead to

$$\| x_{m+1}(t) - x_m(t) \| \leq \int_{t_0}^{t} (t-s)\Phi(s, \| x_m(s) - x_{m-1}(s) \|, \| x'_m(s) - x'_{m-1}(s) \|, \| \max_{\tau \in I} x_m(\tau) - \max_{\tau \in I} x_{m-1}(\tau) \|, \| \max_{\tau \in I} x'_m(\tau) - \max_{\tau \in I} x'_{m-1}(\tau) \|)\, ds, \tag{15}$$

$$\| x'_{m+1}(t) - x'_m(t) \| \leq \int_{t_0}^{t} F(s, \| x_m(s) - x_{m-1}(s) \|, \| x'_m(s) - x'_{m-1}(s) \|, \| \max_{\tau \in I} x_m(\tau) - \max_{\tau \in I} x_{m-1}(\tau) \|, \| \max_{\tau \in I} x'_m(\tau) - \max_{\tau \in I} x'_{m-1}(\tau) \|)ds.$$

Since

$$\| \max_{\tau \in I} x_m(\tau) - \max_{\tau \in I} x_{m-1}(\tau) \| \leq \max_{\tau \in I} \| x_m(\tau) - x_{m-1}(\tau) \|,$$

then, (13) gives

$$\psi_{1,m}^{(i)}(t) \leq \max_{\tau \in I} \psi_{m-1}^{(i)}(\tau)\ \ (i = 0, 1),$$

and (15) becomes

$$\psi_m(t) \leq \int_{t_0}^{t} (t-s)\Phi(s, \psi_{m-1}(s), \psi'_{m-1}(s), \max_{\tau \in I} \psi_{m-1}(\tau), \max_{\tau \in I} \psi'_{m-1}(\tau))\, ds,$$

such that

$$\psi'_m(t) \leq \int_{t_0}^{t} \Phi(s, \psi_{m-1}(s), \psi'_{m-1}(s), \max_{\tau \in I} \psi_{m-1}(\tau), \max_{\tau \in I} \psi'_{m-1}(\tau))\, ds,$$

which according to (9), (10) respectively yields

$$\psi_m(t) \leq S_m(t),\ \psi'_m(t) \leq S'_m(t).$$

So, for any $p = 1, 2, 3, \ldots$ we can have

$$\| x_{m+p}(t) - x_m(t) \| \leq S_{m+p-1}(t) + S_{m+p-2}(t) + \ldots + S_m(t) \leq \sum_{m=n}^{m+p-1} S_m(t) \leq \sum_{m=n}^{\infty} S_m(t),$$

uch that

$$\| x'_{m+p}(t) - x'_m(t) \| \leq S'_{m+p-1}(t) + S'_{m+p-2}(t) + \ldots + S'_m(t) \leq \sum_{m=n}^{m+p-1} S'_m(t) \leq \sum_{m=n}^{\infty} S'_m(t),$$

which converge uniformly on J.

Consequently, $\{x_m(t)\}$ and $\{x'_m(t)\}$ converge uniformly to $x(t)$ and $x'(t)$ on J, i.e problem (1), (2) has a solution $x(t)$. Now, if $\bar{x}(t)$ is another solution on J then,

$$\| x(t) - \bar{x}(t) \| \leq \int_{t_0}^{t} (t-s)\Phi(s, \| x(s) - \bar{x}(s) \|, \| x'(s) - \bar{x}'(s) \|,$$

$$\| \max_{\tau\in I} x(\tau) - \max_{\tau\in I} \bar{x}(\tau) \|, \ \| \max_{\tau\in I} x'(\tau) - \max_{\tau\in I} \bar{x}'(t) \|)\, ds.$$

Again, according to (13) we get

$$\psi(t) \le \int_{t_0}^{t} (t-s)\Phi(s, \psi(s), \psi'(s), \max_{\tau\in I}\psi(\tau), \max_{\tau\in I}\psi'(\tau))\, ds.$$

Since the inequality (7) has only zero solution therefore, the uniqueness of the solution $x(t)$ is just proved.

2. Tunel's approximations. Considering the Tunel's successive approximations, we construct the solution of equation (11) in the form.

$$x_n(t) = \begin{cases} x_0 + x_0'(t-t_0), & t_0 \le t \le T/n, \\ x_0 + x_0'(t-t_0) + \int_{t_0}^{t-\frac{T}{n}} (t-s)F(s, x_n(s), x_n'(s), \max_{\tau\in I} x_n(\tau), & \\ \qquad \max_{\tau\in I} x_n'(\tau))\, ds\,; & T/n < t \le T, \end{cases} \tag{16}$$

$$x_n'(t) = \begin{cases} x_0', & t_0 \le t \le T/n \\ x_0' + \int_{t_0}^{t-T/n} F(s, x_n(s), x_n'(s), \max_{\tau\in I} x_n(\tau), \max_{\tau\in I} x_n'(\tau))\, ds\,; & \frac{T}{u} < t \le T \end{cases}$$

where $n = 1, 2, 3, \ldots$.

Theorem 2. *Suppose $F(t, u_1, u_2, u_3, u_4)$ is defined, continuous in R and satisfies conditions (4), (5). Then (16) converges to the limits $x(t)$ and $x'(t)$ which represent the unique solution of problem (1), (2).*

Proof. If we suppose that $n \le m$ we can obtain from (16) that

$$\begin{aligned} x_m(t) = x_0 + x_0'(t-t_0) &+ \int_{t_0}^{t-T/m} (t-s)\, F(s, x_m(s), x_m'(s), \max_{\tau\in I} x_m(\tau), \\ &\max_{\tau\in I} x_m'(\tau))\, ds + \int_{t-T/n}^{t-T/m} (t-s)\, F(s, x_m(s), x_m'(s), \max_{\tau\in I} x_m(\tau), \\ &\max_{\tau\in I} x_m'(\tau))\, ds\,; \ \frac{T}{m} < t \le T, \end{aligned} \tag{17}$$

$$\begin{aligned} x_m'(t) = x_0' &+ \int_{t_0}^{t-T/0} F(s, x(s_m), x_m'(s), \max_{\tau\in I} x_m(\tau), \max_{\tau\in I} x_m'(\tau))\, ds + \\ &+ \int_{t-T/n}^{t-T/0} F(s, x_m(s), x_m'(s), \max_{\tau\in I} x_m'(\tau), \max_{\tau\in I} x_m(\tau))\, ds; \ \frac{T}{m} < t \le T. \end{aligned}$$

where $m = 1, 2, 3, \ldots$. So, formulas (16), (17) give

$$\begin{aligned} \| x_m(t) - x_n(t) \| \le &\int_{t-T/n}^{t-T/m} (t-s)F(s, x_m(s), x_m'(s), \max_{\tau\in I} x_m(\tau), \max_{\tau\in I} x_m'(\tau))\, ds + \\ &+ \int_{t_0}^{t-T/n} (t-s)\,\Phi(s, \|x_m(s) - x_n(s)\|, \| x_m'(s) - x_m'(s)) \|, \| \max_{\tau\in I} x_m(\tau) - \max_{\tau\in I} x_n(\tau) \|, \end{aligned}$$

$$\| \max_{\tau\in I} x_m'(\tau) - \max_{\tau\in I} x_n'(\tau) \|)\, ds; \ \frac{T}{n} < t \le T,$$

$$\begin{aligned} \| x_m'(t) - x_n'(t) \| \le &\int_{t-T/n}^{t-T/m} F(s, x_m(s), x_m'(s), \max_{\tau\in I} x_m(\tau), \max_{\tau\in I} x_m'(\tau))\, ds + \\ &+ \int_{t_0}^{t-T/n} \Phi(s, \| x_m(s) - x_n(s) \|, \| x_m'(s) - x_n'(s) \|, \| \max_{\tau\in I} x_m(\tau) - \max_{\tau\in I} x_n(\tau) \|, \\ &\| \max_{\tau\in I} x_m'(\tau) - \max_{\tau\in I} x_n'(\tau) \|)\, ds; \ \frac{T}{n} < t \le T, \end{aligned}$$

which yields

$$\| x_m(t) - x_n(t) \| \le \begin{cases} (T^2/(2n^2) - T^2/(2m^2))M\,; & T/m \le t \le T/n, \\ \left(\frac{T^2}{2n^2} - \frac{T^2}{2m^2}\right) M + \int_{t_0}^{t-T/n} (t-s)\,\Phi\,(s, \| x_m(s) - x_n(s) \|, & \\ \quad \| x_m'(s) - x_n'(x) \|, \| \max_{\tau\in I} x_m(\tau) - \max_{\tau\in I} x_n(\tau) \|, \| \max_{\tau\in I} x_m'(\tau) - & \\ \quad - \max_{\tau\in I} x_n'(\tau) \|)\, ds\,; & \frac{T}{n} < t \le T, \end{cases}$$

$$\| x_m'(t) - x_n'(t) \| \le \begin{cases} (T/n - T/m)\, M\,; & T/m \le t \le T/n, \\ \left(\frac{T}{n} - \frac{T}{m}\right) M + \int_{t_0}^{t-T/n} \Phi(s, \| x_m(s) - x_n(s) \|, \| x_m'(s) - x_n'(s) \|, & \\ \quad \| \max_{\tau\in I} x_m(\tau) - \max_{\tau\in I} x_n(\tau) \|, \ \| \max_{\tau\in I} x_m'(\tau) - \max_{\tau\in I} x_n'(\tau) \|)\, ds\,; & \frac{T}{n} < t \le T. \end{cases}$$

Using the notation $\psi_{n,m}^{(i)}(t) = \| x_m^{(i)}(t) - x_n^{(i)}(t) \|$, we obtain

$$\psi_{n,m}(t) \leqslant \begin{cases} (T^2/(2n^2) - T^2/(2m^2))\, M\,; & T/m \le t \le T/n. \\ \left(\frac{T^2}{2n^2} - \frac{T^2}{2m^2}\right) M + \int_{t_0}^{t-T/n} (t-s)\,\Phi\,(s, \psi_{n,m}(s), \psi_{n,m}'(s), \max_{\tau\in I} \psi_{n,m}(\tau), & \\ \quad \max_{\tau\in I} \psi_{n,m}(\tau))ds\,; & \frac{T}{n} < t \leqslant T, \end{cases}$$

such that

$$\psi_{n,m}'(t) \le \begin{cases} (T/n - T/m)\, M\,; & T/m \le t \le T/n, \\ \left(\frac{T}{n} - \frac{T}{m}\right) M + \int_{t_0}^{t-T/n} \Phi(s, \psi_{n,m}(s), \psi_{n,m}'(s), \max_{\tau\in I} \psi_{n,m}(\tau), & \\ \quad \max_{\tau\in I} \psi_{n,m}'(\tau))\, ds\,; & \frac{T}{n} < t \leqslant T. \end{cases}$$

Now, let $\mu_{n,m}(t)$ be the solution of

$$\mu_{n,m}(t)=\begin{cases}(T^2/(2n^2)-T^2/(2m^2))\,M, & T/m\le t\le T/n,\\ \left(\dfrac{T^2}{2n^2}-\dfrac{T^2}{2m^2}\right)M+\displaystyle\int_{t_0}^{t-T/n}(t-s)\,\Phi(s,\mu_{n,m}(s),\mu'_{n,m}(s),\max_{\tau\in I}\mu_{n,m}(\tau),\\ \qquad \max\mu'_{n,m}(\tau))\,ds\,; & \dfrac{T}{n}<t\leqslant T,\end{cases}$$

such that

$$\mu'_{nm}(t)=\begin{cases}(T/n-T/m)\,M, & T/m\le t\le T/n,\\ \left(\dfrac{T}{n}-\dfrac{T}{m}\right)M+\displaystyle\int_{t_0}^{t-T/n}\Phi(s,\mu_{n,m}(s),\mu'_{nm}(s),\max_{\tau\in I}\mu_{n,m}(\tau),\\ \qquad \max\mu'_{nm}(\tau))\,ds\,; & \dfrac{T}{n}<t<T.\end{cases}$$

Therefore

$$\psi_{n,m}(t)\le\mu_{n,m}(t),\ \text{ and }\ \psi'_{n,m}(t)\le\mu'_{n,m}(t).$$

But both $\lim\mu_{n,m}(t)$ and $\lim\mu'_{n,m}(t)$ tend to zero according to the basic concept 2 as n,m→∞. So,

$$\lim_{n,m\to\infty}\|x_m(t)-x_n(t)\|=0,\ \text{ and }\ \lim_{n,m\to\infty}\|x'_m(t)-x'_n(t)\|=0.$$

Thus, the sequences $\{x_n(t)\}$ and $\{x'_n(t)\}$ uniformly converge to the limits $x(t)$ and $x'(t)$ wich represent the solution $x(t)$ of problem (1), (2). The uniqueness of $x(t)$ can be proved similarly as in Theorem 1.

3. The lower-upper bounds approximations. Here, we construct for (11) the following two formulas

$$\rho(t)\le x_0+x'_0(t-t_0)+\int_{t_0}^{t}(t-s)\,F(s,\rho(s),\ \rho'(s),\ \max_{\tau\in I}\rho(\tau),\ \max_{\tau\in I}\rho'(\tau))\,ds,$$

and

$$\sigma(t)\ge x_0+x'_0(t-t_0)+\int_{t_0}^{t}(t-s)\,F(s,\sigma(s),\sigma'(s),\max_{\tau\in I}\sigma(\tau),\max_{\tau\in I}\sigma'(\tau))\,ds,$$

where $\rho(t)$, $\sigma(t)$ are the lower and the upper bounded functions for $x(t)$ such that

$$\rho(t_0)=\sigma(t_0)=x_0\ \text{ and }\ \rho'(t_0)=\sigma'(t_0)=x'_0.$$

Thus, the lower and the upper bounds approximations will be of the forms

$$\rho_m(t)\le x_0+x'_0(t-t_0)+\int_{t_0}^{t}(t-s)\,F(s,\rho_m(s),\rho'_m(s),\max_{\tau\in I}\rho_m(\tau),\max_{\tau\in I}\rho'_m(\tau))\,ds,\tag{18}$$

and

$$\sigma_m(t)\ge x_0+x'_0(t-t_0)+\int_{t_0}^{t}(t-s)\,F(s,\sigma_m(s),\sigma'_m(s),\max_{\tau\in I}\sigma_m(\tau),\max_{\tau\in I}\sigma'_m(\tau))\,ds,$$

such that

$$\rho'_m(t)\le x'_0+\int_{t_0}^{t}F(s,\rho_m(s),\rho'_m(s),\ \max_{\tau\in I}\rho_m(\tau),\ \max_{\tau\in I}\rho'_m(\tau))\,ds,$$

and

$$\sigma'_m(t)\ge x'_0+\int_{t_0}^{t}F(s,\sigma_m(s),\sigma'_m(s),\ \max_{\tau\in I}\sigma_m(\tau),\ \max_{\tau\in I}\sigma'_m(\tau))\,ds,$$

where $m=1,2,3,\dots$.

Applying the notations

$$x_m^{(i)}(t)=\|\rho_m^{(i)}(t)-\sigma_m^{(i)}(t)\|,$$

and

$$\bar{x}_m^{(i)}(t)=\|\max_{\tau\in I}\rho_m^{(i)}(\tau)-\max_{\tau\in I}\sigma_m^{(i)}(\tau)\|\quad(i=0,1),\tag{19}$$

we can prove the following.

Theorem 3. *Let $F(t,u_1,u_2,u_3,u_4)$ be defined, continuous in R and satisfy the conditions (4), (5). Then, the sequences $\{\rho_m(t)\}$, $\{\rho'_m(t)\}$, $\{\sigma_m(t)\}$ and $\{\sigma'_m(t)\}$ converge uniformly to the limit $x(t)$ and $x'(t)$ which give the unique solution of problem (1), (2).*

Proof. Refering to Picard's approximated formula (12), we can write

$$\rho_m(t)=x_0+x'_0(t-t_0)+\int_{t_0}^{t}(t-s)\,F(s,\rho_{m-1}(s),\rho'_{m-1}(s),\ \max_{\tau\in I}\rho_{m-1}(\tau),\max_{\tau\in I}\rho'_{m-1}(\tau))\,ds,$$

and (20)

$$\sigma_m(t)=x_0+x'_0(t-t_0)+\int_{t_0}^{t}(t-s)\,F(s,\sigma_{m-1}(s),\sigma'_{m-1}(s),\ \max_{\tau\in I}\sigma_{m-1}(\tau),\max_{\tau\in I}\sigma'_{m-1}(\tau))\,ds,$$

such that

$$\rho'_m(t)=x'_0+\int_{t_0}^{t}F(s,\rho_{m-1}(s),\rho'_{m-1}(s),\ \max_{\tau\in I}\rho_{m-1}(\tau),\max_{\tau\in I}\rho'_{m-1}(\tau))\,ds,$$

and

$$\sigma'_m(t)=x'_0+\int_{t_0}^{t}F(s,\sigma_{m-1}(s),\sigma'_{m-1}(s),\ \max_{\tau\in I}\sigma_{m-1}(\tau),\ \max_{\tau\in I}\sigma'_{m-1}(\tau))\,ds.$$

Thus, putting $m=1$ in (18) and $m=2$ in (20), we get

$$\rho_1(t)\le\rho_2(t)\le\sigma_2(t)\le\sigma_1(t)\,;\ \ \rho_1(t_0)=\sigma_1(t_0)=x_0,\ \ \rho_2(t_0)=\sigma_2(t_0)=x_0,$$

and so also

$$\rho'_1(t)\le\rho'_2(t)\le\sigma'_2(t)\le\sigma_1(t)\,;\ \ \rho'(t_0)=\sigma'_1(t_0)=x'_0,\ \ \rho'_2(t_0)=\sigma'_2(t_0)=x'_0.$$

Again, putting $m=2$ in (18) and $m=3$ in (20) we can obtain

$$\rho_2(t)\le\rho_3(t)\le\sigma_3(t)\le\sigma_2(t)\,;\ \rho_2(t_0)=\sigma_2(t_0)=x_0,\ \rho_3(t_0)=\sigma_3(t_0)=x_0.$$

and so also

$$\rho'_2(t)\le\rho'_3(t)\le\sigma'_3(t)\le\sigma'_2(t)\,;\ \rho'_2(t_0)=\sigma'_2(t_0)=x'_0,\ \rho'_3(t_0)=\sigma'_3(t_0)=x'_0,$$

Repeating the process, we can finally deduce that

$$\dots\le\rho_{m-1}(t)\le\rho_m(t)\le\sigma_m(t)\le\sigma_{m-1}(t)\le\dots$$

and also

$$\dots\le\rho'_{m-1}(t)\le\rho'_m(t)\le\sigma'_m(t)\le\sigma'_{m-1}(t)\le\dots.$$

for which $\{\rho_m(t)\}$ and $\{\rho'_m(t)\}$ seem to be monotonically increasing, bounded above by $\sigma_1(t)$ and $\sigma'_1(t)$ respectively. Also, $\{\sigma_m(t)\}$ and $\{\sigma'_m(t)\}$ are monotonically decreasing, bounded below by $\rho_1(t)$ and $\rho'_1(t)$ respectively. Hence $\rho(t)$, $\rho'(t)$ $\sigma(t)$ and $\sigma'(t)$ are uniformly bounded. Now, taking $\varepsilon>0$ sufficiently small and $\delta(\varepsilon)>0$, we can have

$$\| \rho_m(t_1)-\rho_m(t_2) \| < \varepsilon \text{ and } \| \rho'_m(t_1)-\rho'_m(t_2) \| < \varepsilon \text{ if } |t_1-t_2| < \delta$$

where $\delta=\varepsilon/M(T+|T-t_0|)$ and ε/M respectively. Consequently, $\{\rho_m(t)\}$, $\{\rho'_m(t)\}$ are equicontinuous on $[t_0, T]$ and so also $\{\sigma_m(t)\}$, $\{\sigma'_m(t)\}$. From above $\{\rho_m(t)\}$, $\{\sigma_m(t)\}$ and $\{\rho'_m(t)\}$, $\{\sigma'_m(t)\}$ will be uniformly convergent i.e. converge uniformly to $x(t)$ and $x'(t)$ respectively. Moreover, since

$$\| \rho_m(t)-\sigma_m(t) \| \leq \int_{t_0}^{t} (t-s)\Phi(s, \| \rho_{m-1}(s)-\sigma_{m-1}(s)\|, \|\rho'_{m-1}(s)-\sigma'_{m-1}(s)\|, \tag{21}$$
$$, \| \max_{\tau\in I} \rho_{m-1}(\tau)-\max_{\tau\in I} \sigma_{m-1}(\tau) \|, \| \max_{\tau\in I} \rho'_{m-1}(\tau)-\max_{\tau\in I} \sigma'_{m-1}(\tau) \|)\, ds,$$

and

$$\| \max_{\tau\in I} \rho^{(i)}_m(\tau)-\max_{\tau\in I} \sigma^{(i)}_m(\tau) \| \leq \max_{\tau\in I} \| \rho^{(i)}_m(\tau)-\sigma^{(i)}_m(\tau) \| \; (i=0, 1).$$

Then, using (19) we obtain $x^{(i)}_m(t)\leq \max x^{(i)}_m(t)$, and (21) becomes

$$x_m(t)\leq \int_{t_0}^{t} (t-s)\, \Phi\, (s, x_{m-1}(s), x'_{m-1}(s), \max_{\tau\in I} x_{m-1}(\tau), \max_{\tau\in I} x'_{m-1}(\tau))\, ds,$$

which has only the zero solution according to (7). Therefore

$$\lim_{m\to\infty} x_m(t)=\lim_{m\to\infty} \| \rho_m(t)-\sigma_m(t) \|, \text{ i.e. } x(t)=\rho(t)=\sigma(t).$$

IV. The existence of the solution by fixed point principles. Here, we introduce the theorem on the existence of a unique solution for problem (1), (2) by using Pirov's and Schauder's fixed point principles. We must take in account I and II besides the generalized norm [6] which is defined by

$$| x(t) |=\begin{pmatrix}\max_J \| x(t) \| \\ \max_J \| x(t) \|\end{pmatrix}. \tag{22}$$

Pirov's fixed point principle [6].

Theorem 4. *Suppose $F(t, u_1, u_2, u_3, u_4)$ is defined, continuous in R and satisfies (4), (6). Suppose also that the following inequalities*

$$\| x_0 \| + \| x'_0 \| r+\frac{M}{2} r^2\leq a, \; \| x'_0 \| + Mr\leq b, \tag{23}$$

and

$$rk(2+r)<1, \tag{24}$$

where $r=|T-t_0|$ are satisfied. Then, problem (1), (2) has a unique solution $x(t)$.

Proof. Since $G=(C^2(J; E); E)$, we take in G the set

$$W: =\{x(t)\in G : \| x(t)-x_0 \| \leq a, \| x'(t)-x'_0 \| \leq b; \; a, b \text{ consts.}\}. \tag{25}$$

Now, let (11) take the form $x(t)=A(x(t))$ where

$$A(x(t))=x_0+x'_0(t-t_0)+\int_{t_0}^{t} (t-s)\, F(s, x(s), x'(s), \max_{\tau\in I} x(\tau), \max_{\tau\in I} x'(\tau))\, ds. \tag{26}$$

It is clear by (23), that $(A(x(t))\in W$ since

$$\| A(x(t))-x_0 \| \leq \| x'_0 \| r+\frac{M}{2} r^2, \text{ and } \| A(x'(t))-x'_0 \| \leq Mr,$$

i.e. the operator A converts W into itself. If $x_1(t)$, $x_2(t)\in W$, we shall have

$$\| A(x_1(t))-A(x_2(t)) \| \leq \int_{t_0}^{t} (t-s)K(\| x_1(s)-x_2(s) \| + \| x'_1(s)-x'_2(s) \| +$$
$$+ \| \max_{\tau\in I} x_1(\tau)-\max_{\tau\in I} x_2(\tau) \| + \| \max_{\tau\in I} x'_1(\tau)-\max_{\tau\in I} x'_2(\tau) \|)\, ds,$$

which leads to

$$\| A(x_1(t))-A(x_2(t)) \| \leq r^2K \max (\| x_1(t)-x_2(t) \| + \|x'_1(t)-x'_2(t) \|), \tag{27}$$

ince

$$\| \max_{\tau\in I} x_1(\tau)-\max_{\tau\in I} x_2(\tau) \|\leq \max_{\tau\in I} \| x_1(\tau)-x_2(\tau) \|.$$

Also,

$$\| A(x'_1(t))-A(x'_2(t)) \| \leq 2rk \max (\| x_1(t)-x_2(t) \| + \| x'_1(t)-x'_2(t) \|), \tag{28}$$

according to definition (22) the inequalities (27), (28) can be written in the form

$$| Ax_1-Ax_2 | \leq \begin{pmatrix} r^2k & r^2k \\ 2rk & 2rk\end{pmatrix} | x_1-x_2 |,$$

where the included matrix seems to be the „a-matrix" [6] if and only if inequality (24) is satisfied. Consequently, from the contraction mapping principle [6], the operator A has a unique fixed point $x(t)\in G$, i.e. equation (11) or problem (1), (2) has a unique solution.

2. **Schauder's fixed point principle** [1].

Theorem 5. *Consider $F(t, u_1, u_2, u_3, u_4)$ which is defined, continuous in R and satisfies conditions (4), (6). If for $r=|T-t_0|$ the inequalities*

$$\| x_0\| + \| x'_0 \| r+\frac{M}{2} r^2 \leq a, \; \| x'_0 \| + Mr\leq b, \tag{29}$$

hold, then problem (1), (2) has at least one solution $x(t)$.

Proof. As we know from theorem 4, G is in E for all continuous differentiable n-vector functions $x(t)$, the set W and the operator A are defined as in (25) and (26) respectively. Now, it is known that the successive approximations for equation (11) take the form

$$x_n(t) = x_0 + x_0'(t - t_0) + \int_{t_0}^{t} (t-s)\, F(s, x_{n-1}(s),\ x'_{n-1}(s), \max_{\tau \in I} x_{n-1}(\tau),$$
$$\max_{\tau \in I} x'_{n-1}(\tau))\, ds\ ;\ n = 1, 2, 3, \dots .$$

such that

$$x_n'(t) = x_0' + \int_{t_0}^{t} F(x, x_{n-1}(s),\ x'_{n-1}(s), \max_{\tau \in I} x_{n-1}(\tau), \max_{\tau \in I} x'_{n-1}(\tau))\, ds\ ;\ n = 1, 2, 3, \dots .$$

If in W the sequences $\Sigma x_n(t)$ and $\Sigma x_n'(t)$ are uniformly convergent to the vector functions $x(t)$ and $x'(t)$ respectively then the vector function $f(t, x_n(t), x_n'(t), \max_{\tau \in I} x_n(\tau), \max_{\tau \in I} x_n'(\tau))$ will uniformly converge in J to $f(t, x(t), x'(t), \max_{\tau \in I} x(\tau), \max_{\tau \in I} x'(\tau))$ since $f(t, x(t), x'(t), \max_{\tau \in I} x(\tau), \max_{\tau \in I} x'(\tau))$ is continuous. Consequently $Ax_n \to Ax$ and $Ax_n' \to Ax'$, i.e. the operator A is continuous in W. It is clear, according to (29) that

$$\| A(x(t)) \| \le a, \ \| A(x'(t)) \| \le b, \tag{30}$$

which means that A is bounded and mapps W into itself. Also, taking $t_1, t_2 \in J$, $\varepsilon > 0$ sufficiently small and $\delta(\varepsilon) > 0$, we can have

$$\| A(x(t_1)) - A(x(t_2)) \|, \ \| A(x'(t_1)) - A(x'(t_2)) \| < \varepsilon \text{ if } | t_1 - t_2 | < \delta$$

where $\delta = \varepsilon/(M \mid T + r \mid)$ and ε/M respectively.

These results according to Arzela theorem [1] show that the continuous operator A converts the bounded set W into its compact part. So, using Schauder's principle [1] the operator A has at least one fixed point $x(t)$ which is the solution of problem (1), (2).

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EFFECTS OF DISPERSAL ON THE STABILITY OF TWO PREDATORS AND A COMMON PREY

BY

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Introduction. Stability of systems of interacting species has been investigated by various authors, for example Goh and Agenew [7]; Kazarinoff and Driessche [12]; El-Owaidy and Ammar [2, 3, 4, 5]. Effects of dispersal on the stability of the equilibrium state have also been investigated by various authors, for example: Gopalsamy [8]; Hadeler et al [9]; Levin [13]; Nallaswamy and Shukla [10, 11]; El-Owaidy and Ammar [6]. In particular, Nallaswamy and Shukla [10] investigated the effects of dispersal on the linear and nonlinear stability of the equilibrium state of a predator-prey system with functional response.

1. Basic systems. Consider a system modeling the interactions of two competing predator populations living exclusively on a common prey in a linear one dimensional finite habitat $(0 \le x \le L)$ with intraspecific interaction of prey and functional response. The system given by:

$$\begin{aligned} \frac{\partial N_1}{\partial t} &= N_1\left(a_1 - a_{11}N_1 - \frac{a_{12}N_2}{1+\alpha N_1} - \frac{a_{13}N_3}{1+\alpha N_1}\right) + \frac{\partial}{\partial x}\left(D_1 \frac{\partial N_1}{\partial x}\right), \\ \frac{\partial N_2}{\partial t} &= N_2\left(-a_2 + \frac{a_{21}N_1}{1+\alpha N_1} - a_{22}N_2\right) + \frac{\partial}{\partial x}\left(D_2 \frac{\partial N_2}{\partial x}\right), \\ \frac{\partial N_3}{\partial t} &= N_3\left(-a_3 + \frac{a_{31}N_1}{1+\alpha N_1} - a_{33}N_3\right) + \frac{\partial}{\partial x}\left(D_3 \frac{\partial N_3}{\partial x}\right), \end{aligned} \tag{1.1.}$$

where $N_1(x, t)$ is the population distribution of prey species; $N_2(x, t)$, $N_3(x, t)$ are the population distributions of predator species at time t; and D_1, D_2, D_3 are the variable coefficients of dispersal which are positive. The interaction coefficients a_1, a_2, a_3, a_{11}, a_{22}, a_{33}, a_{12}, a_{13}, a_{21}, a_{31}, and α are positive constants. The factor $(1+\alpha N_1)^{-1}$ represents the functional response in the model, and α determines the strength of this response. For a positive equilibrium (N_1^*, N_2^*, N_3^*) we have:

$$\begin{gathered} a_1 - a_{11}N_1^*)(1+\alpha N_1^*) = a_{12}N_2^* + a_{13}N_3^*, \quad (a_2 + a_{22}N_2^*)(1+\alpha N_1^*) = a_{21}N_1^*, \\ (a_3 + a_{33}N_3^*)(1+\alpha N_1^*) = a_{31}N_1^* \end{gathered} \tag{1.2.}$$