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SMOOTH MAPPINGS BETWEEN FOLIATED MANIFOLDS WITH BOUNDARY

BY

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In the paper [5] J. Mather proves that a proper, infinitesimally stable mapping of one finite dimensional C^∞ manifold into another is structurally stable. The main theorems ([5], §3) only deals with manifolds without boundary but Erik von Essen in [1] extends Mather's theorem to a theorem which also deals with manifolds with boundary and corners.

In the special context of the foliated manifolds without boundary, L. A. Favaro, in [2], has presented the notion of stability in the tangential sense of smooth mappings between foliated manifolds. Moreover, he has presented the notion of infinitesimal stability in the tangential sense and he has shown that the latter is the sufficient condition of the former. In [4], S. Izumiya has shown that the converse of Favaro's theorem is true for foliation preserving mappings.

The purpose of this paper is to extend Favaro-Izumiya's theorems to a theorem which also deals with foliated manifolds with boundary. All manifolds, maps and foliations will be C^∞. Let M be a manifold with boundary ∂M and $j : \partial M \times [0, 1] \to U \subset M$ with $j(x, 0) = x$, $\forall x \in M$, be a collar neighborhood of the boundary ∂M in M. To the pair (M, j) an open manifold can be associated, namely, $M_j = M \sqcup_j (\partial M \times (-1, 1))$.

Definition 1. *M has a codimension q foliation, if there exists a collar j and a codimension q foliation $\mathcal{F}_M^j$ on M_j. The restriction $\mathcal{F}_M$ of $\mathcal{F}_M^j$ to M is called a codimension q foliation on M. A codimension q foliation on M is called* tangent *(to the boundary) if the leaf of $\mathcal{F}_M^j$ through each point $(x, 0)$, $x \in \partial M$, is contained in $\partial M \times \{0\}$, and is called* transverse *(to the boundary), if the leaf of $\mathcal{F}_M^j$ through each point $(x, 0)$, $x \in \partial M$, is a immersed submanifold transversal to $\partial M \times \{0\}$.*

Let S be the stratification of M defined by the int M and the boundary ∂M, that is the set of components of $\{x \in M, \text{index}\,(x) = j\}$, $j = 0, 1$. For $x \in M$, we let str(x) denote the stratum which contains x, and let $\hat{T}M = \bigcup_{X \in S} TX$, $\hat{T}M_x = \hat{T}M \cap T_xM$. Obviously, $\hat{T}M$ is the set of tangent vectors $v \in TM$, to which there exists a C^∞ curve $c : (-\varepsilon, \varepsilon) \to M$ whose tangent vector in 0 is v.

If $T\mathcal{F}_M$ is the subbundle of TM with fiber $T_x\mathcal{F}_M$, the tangent space to the leaf of $\mathcal{F}_M$ at x, then, by the previous definition, we have $T\mathcal{F}_M \subset \hat{T}M$ for tangent foliations.

Now, let M and M' be C^∞ — manifolds of dimensions m and m' respectively and $\mathcal{F}_M$, $\mathcal{F}_{M'}$ be C^∞ — foliations on M and M' of codimensions q and q' respectively.

The double structure (foliated and stratified) of the manifolds M and M' permits to consider the following two subsets in $C^\infty(M, M')$, which are associated to a mapping $f \in C^\infty(M, M')$:

$$C^\infty_S(M, M'; f) = \{g \in C^\infty(M, M'),\ \mathrm{str} f(x) = \mathrm{str}\, g(x),\ \forall x \in M\} \text{ and}$$

$$C^\infty_{\mathcal{F}}(M, M'; f) = \{g \in C^\infty(M, M'),\ g(x) \text{ and } f(x) \text{ belong to the same leaf of } \mathcal{F}_{M'} \text{ for each } x \in M\}.$$

Obviously, if the foliations $\mathcal{F}_M$ and $\mathcal{F}_{M'}$ are the tangent foliations, we have $C^\infty_{\mathcal{F}}(M, M'; f) \subset C^\infty_S(M, M'; f)$.

We denote by $C^\infty_{S,\mathcal{F}}(M, M'; f) = C^\infty_S(M, M'; f) \cap C^\infty_{\mathcal{F}}(M, M'; f)$ and we define a concept of stability for f in the set $C^\infty_{S,\mathcal{F}}(M, M'; f)$.

Definition 2. *We call the mapping $f \in C^\infty(M, M')$ structurally stratum stable in the tangential sense if there exists a neighborhood $U_f \subset C^\infty_{S,\mathcal{F}}(M, M'; f)$ of f in the Whitney C^∞-topology, such that for each $g \in U_f$ there exists C^∞-diffeomorphisms $h: M \to M$, taking each leaf of $\mathcal{F}_M$ onto itself, and $h': M' \to M'$, taking each leaf of $\mathcal{F}_{M'}$ onto itself, such that $f = h' o g o h$.*

To produce an infinitesimal concept of stability we consider $T\mathcal{F}_M$ (resp. $T\mathcal{F}_{M'}$), the subbundle of TM (resp. of TM') with fiber $T_x\mathcal{F}_M (T_{x'}\mathcal{F}_{M'})$, the tangent space to the leaf of $\mathcal{F}_M$ (resp. of $\mathcal{F}_{M'}$) at x (resp. at x').

We define the following sets:

$$\hat{T}\mathcal{F}_M = \hat{T}M \cap T\mathcal{F}_M, \quad \hat{T}_x\mathcal{F}_M = T_x\mathcal{F}_M \cap \hat{T}M \text{ and}$$

$$C^\infty(\hat{T}\mathcal{F}_M) = \{u \in C^\infty(T\mathcal{F}_M),\ u(M) \subset \hat{T}\mathcal{F}_M\}.$$

If $f \in C^\infty(M, M')$, we shall write:

$$C^\infty(f^*\hat{T}M') = \{w \in C^\infty(f^*TM'),\ w(M) \subset \hat{T}M'\},$$

interpreting $w \in C^\infty(f^*TM')$ as a vector field along f. Similarly, we shall put:

$$C^\infty(f^*\hat{T}\mathcal{F}_{M'}) = \{w \in C^\infty(f^*T\mathcal{F}_{M'}),\ w(M) \subset \hat{T}\mathcal{F}_{M'}\}.$$

Obviously, $\hat{T}M$, $\hat{T}M'$, $\hat{T}\mathcal{F}_M$ and $\hat{T}\mathcal{F}_{M'}$ are not vector bundles, not even manifolds. However, $C^\infty(\hat{T}M)$ is a submodule of $C^\infty(\hat{T}M)$, $C^\infty(\hat{T}\mathcal{F}_M)$ is a submodule of $C^\infty(\hat{T}M)$, $C^\infty(f^*\hat{T}M')$ is a submodule of $C^\infty(f^*TM')$ and $C^\infty(f^*\hat{T}\mathcal{F}_{M'})$ is a submodule of $C^\infty(f^*\hat{T}M')$.

The mapping $tf: C^\infty(TM) \to C^\infty(f^*TM')$ and $\omega f: C^\infty(TM') \to C^\infty(f^*TM')$, $(tf)(u) = (Tf) o u$, $(\omega f)(v) = v o f$, define, by restriction, the mappings:

$$\hat{t}f: C^\infty(\hat{T}M) \to C^\infty(f^*\hat{T}M'), \quad \hat{\omega}f: C^\infty(\hat{T}M') \to C^\infty(f^*\hat{T}M'),$$

where $\hat{t}f$ is a $C^\infty(M)$ — module homomorphism and $\hat{\omega}f$ is a module homomorphism over the ring homomorphism $f^*: C^\infty(M') \to C^\infty(M)$.

Definition 3. *The mapping $f \in C^\infty(M, M')$ is infinitesimally stratum stable in the tangential sense if, given $w \in C^\infty(f^*\hat{T}\mathcal{F}_{M'})$, there are $u \in C^\infty(\hat{T}\mathcal{F}_M)$ and $v \in C^\infty(\hat{T}\mathcal{F}_{M'})$ such that*

$$w = (\hat{t}f)(u) + (\hat{\omega}f)(v).$$

Remark. If we denote by $C^\infty_{\mathcal{F}}(M, M')$ the set of foliation preserving mappings between the manifolds M and M', for each $f \in C^\infty_{\mathcal{F}}(M, M')$ we have $C^\infty_{\mathcal{F}}(M, M'; f) \subset C^\infty_{\mathcal{F}}(M, M')$ and the notion of infinitesimal stratum stability in the tangential sense is equivalent with the following condition:

$$(\hat{t}f + \hat{\omega}f)(C^\infty(\hat{T}\mathcal{F}_M) + C^\infty(\hat{T}\mathcal{F}_{M'})) = C^\infty(f^*\hat{T}\mathcal{F}_{M'}).$$

To show that the infinitesimal stability is locally a condition of finite order, we consider a local version of the concept of infinitesimal stratum stability. We shall use the symbol $[h]_p$ to indicate the germ of h at p.

Definition 4. *The mapping $f \in C^\infty(M, M')$ is called locally infinitesimally stratum stable in the tangential sense at $x \in M$ if the germ $[f]_x$ is infinitesimally stratum stable, i. e. for every germ of a vector field $w \in C^\infty(f^*\hat{T}\mathcal{F}_{M'})$ there exist germs $[u]_x$, $[v]_{f(x)}$ of vector fields $u \in C^\infty(\hat{T}\mathcal{F}_M)$, $v \in C^\infty(\hat{T}\mathcal{F}_{M'})$ so that*

$$[w]_x = [(Tf) o u]_x + [f^*v]_x \tag{1}$$

Our first result is the following:

Theorem 1. *Let $(M, \mathcal{F}_M)$ be a m-dimensional compact manifold foliated by the q-codimensional foliation $\mathcal{F}_M$, $(M', \mathcal{F}_{M'})$ be a m'-dimensional manifold foliated by the q'-codimensional tangent foliation $\mathcal{F}_{M'}$ and $f \in C^\infty_{\mathcal{F}}(M, M')$. Then, the following conditions are equivalent:*

a) f is infinitesimally stratum stable;

b) f is locally infinitesimally stratum stable at x, for every $x \in M$;

c) for every $x' \in M'$ and each finite subset $S \subset f^{-1}(x')$ with no more than $(m' - q' + 1)$ — points it follows that:

$$J^{m'-q'}(f^*T\mathcal{F}_{M'})_S = (Tf)(J^{m'-q'}(\hat{T}\mathcal{F}_M)_S) + f^*(J^{m'-q'}(T\mathcal{F}_{M'})_{x'}),$$

where, the linear spaces involved in the previous relation are the spaces of jets of the sections which belong to the modules from the paranthESES.

Proof. It is clear that a)$\Rightarrow$b). So, the proof goes: b)$\Rightarrow$c), c)$\Rightarrow$a).

Firstly, let $x \in \mathrm{int}\, M$ be a fixed point so that $x' = f(x) \in \mathrm{int}\, M'$, (U, φ), $\varphi(U) = U_1 \times U_2 \subset R^{m-q} \times R^q$, $\varphi(x) = (x_1, x_2)$ be a foliated chart on $(M, \mathcal{F}_M)$, based at x, and (V, ψ), $\psi(V) = V_1 \times V_2 \subset R^{m'-q'} \times R^{q'}$, $\psi(x') = (x'_1, x'_2)$ be a foliated chart on $(M', \mathcal{F}_{M'})$, based at $f(x) = x'$, so that $f(U) \subset V$. We have: $(\psi o f o \varphi^{-1})(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$, $\forall x = (x_1, x_2) \in U_1 \times U_2$.

We will compute the equation (1) in these coordinates.

It is obvious that, locally, $C^\infty(T\mathcal{F}_M)$ and $C^\infty(f^* T\mathcal{F}_{M'})$ are finite generated modules over $C^\infty(M)$ and $C^\infty(T\mathcal{F}_{M'})$ is a finite generated module over $C^\infty(M')$. But, over the neighborhoods U, V, $C^\infty(\hat{T}\mathcal{F}_M)=C^\infty(T\mathcal{F}_M)$, $C^\infty(\hat{T}\mathcal{F}_{M'})=$ $=C^\infty(T\mathcal{F}_{M'})$ and $C^\infty(f^*\hat{T}\mathcal{F}_{M'})=C^\infty(f^* T\mathcal{F}_{M'})$. So, if $\left(\frac{\partial}{\partial x^i}\right)_{i=\overline{1,(m-q)}}$, $\left(\frac{\partial}{\partial x^{i'}}\right)_{i'=\overline{1,(m'-q')}}$, generate locally the modules $C^\infty(T\mathcal{F}_M)$ and $C^\infty(T\mathcal{F}_{M'})$, respectively, the equation (1) becomes:

$$[w]_x=[Tf_1 \circ u]_{(x)}+[v\circ f]_{(x)}. \tag{2}$$

Since $[f]_x$ is infinitesimally stratum stable in the tangential sense iff: $\left(e_{i'}=\mathrm{pr}\left(f^*\frac{\partial}{\partial x^{i'}}\right)\right)_{i'=\overline{1,(m'-q')}}$ generate the $C^\infty_{x'}(M')$-module $M^x_f=C^\infty(f^* T\mathcal{F}_{M'})_x/A$ (via $f^*_{x,x'}$), where $A=\{[Tf_1\circ u]_x,\ u\in C^\infty(T\mathcal{F}_M)_x\}$ and $\mathrm{pr}: C^\infty(f^* T\mathcal{F}_{M'})_x\to M^x_f$, we apply the considerations from [3], Ch. IV, Th. 3.10, to see that $[f]_x$ is infinitesimally stratum stable in the tangential sense iff the module $M^x_f/\mathcal{M}^{m'-q'+1}_x(M)\cdot M^x_f$ is generated over $C^\infty_{x'}(M')$ by the projections of $(e_{i'})_{i'=\overline{1,(m'-q')}}$. Recall that $\mathcal{M}^{m'-q'+1}_x(M)$ consists of germs of functions at x whose Taylor series start with terms of order $m'-q'+1$ or greater. But, the last statement is easily seen to be equivalent to solving equation (2) to order $m'-q'$.

Now, let $x\in \mathrm{int}\, M$ and $x'=f(x)\in\partial M'$. Let $j':\partial M'\times[0,1)\to U'\subseteq M'$ be a collar of the boundary $\partial M'$ and $\mathcal{F}^{j'}_{M'}$ be the foliation on the open manifold $M'_{j'}=M'\sqcup_{j'}(\partial M'\times(-1,1))$ the restriction of whose is $\mathcal{F}_{M'}$. Let (U,φ) be a foliated chart in x, $\varphi(U)=U_1\times U_2\subseteq R^{m-q}\times R^q$ and (V,ψ) be a foliated chart in $(x',0)\in M'_{j'}$, for $\mathcal{F}^{j'}_{M'}$, $\psi(V)=V_1\times V_2\subseteq R^{m'-q'}\times R^{q'}$ with $V\subset \mathrm{imm}_2(\partial M'\times(-1,1))$, $\mathrm{imm}_2=\pi\circ \mathrm{in}_2$, $\mathrm{in}_2:\ \partial M'\times(-1,1)\to M'\sqcup(\partial M'\times(-1,1))$, $\pi: M'\sqcup(\partial M'\times(-1,1))\to M'_{j'}$.

Let $V'=j'(\mathrm{imm}_2^{-1}(V))$ which is an open subset of M'. We define: $\psi'=$ $=\psi|V'$. If $\mathcal{F}_{M'}$ is a tangent foliation of M', we obtain that ψ' is a homeomorphism of V' on an open subset $V'_1\times V'_2\subseteq R^{m'-q'}\times H^{q'}_\lambda$, $\lambda\in L(R^{q'},R)$, $\lambda\neq 0$.

So, the pair (V',ψ') defines a chart in $x'\in M'$ which defines, by restriction, a foliated chart in $\partial M'$ for the restriction of $\mathcal{F}_{M'}$ to $\partial M'$. We have that $C^\infty(\hat{T}\mathcal{F}_{M'})_{x'}=C^\infty(T\mathcal{F}_{M'})_{x'}$, $C^\infty(\hat{T}\mathcal{F}F_M)_x=C^\infty_x(T\mathcal{F}_M)$, $C^\infty(f^*\hat{T}\mathcal{F}_{M'})_x=C^\infty(f^* T\mathcal{F}_M)_x$ and the conclusion is the same as in the previous case.

Similar, if $x\in\partial M$, we obtain that $[f]_x$ is infinitesimally stratum stable in the tangential sense iff the equation (1) can be solved to order $m'-q'$ in the case $f(x)\in \mathrm{int}\, M'$ and to order m' in the case $f(x)\in\partial M'$.

Now, we will prove the implication c)⇒a).

Let $\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)=\{x\in M,\ \dim(T_xf(\hat{T}_x\mathcal{F}_M)\cap T_{f(x)}\mathcal{F}_{M'})<m'-q'\}$. If $f\in C^\infty(M,M')$ satisfies a), then, for each $x'\in M'$, $\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)\cap f^{-1}(x')$ has no more than $m'-q'$ points.

In fact, we will show first that for each $x'\in M'$, $\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)\cap f^{-1}(x')$ has no more than $m'-q'$ points. We shall argue by contradiction. Suppose $S=\{x_1,\dots,x_{m'-q'+1}\}$ of distinct points, $S\subset\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)\cap f^{-1}(x')$. Let $\hat{H}_i=$ $=(T_{x_i}f)(\hat{T}_{x_i}\mathcal{F}_M)\cap T_{x'}\mathcal{F}_{M'}$. We apply lemma 1.7, Ch. V [3], to show that $\{\hat{H}_i\}_{i=\overline{1,m'-q'+1}}$ are in general position as subspaces of $T_{x'}\mathcal{F}_{M'}$. Now let $\{z_i\}_{i=\overline{1,m'-q'+1}}\subset T_{x'}\mathcal{F}_{M'}$ and choose a vector field $w\in C^\infty(f^* T\mathcal{F}_{M'})$ so that $w(x_i)=z_i$. By c) we must choose vector fields $u\in C^\infty(\hat{T}\mathcal{F}_M)$, $v\in C^\infty(T\mathcal{F}_{M'})$ so that $w=Tf\circ u+v\circ f$ on a neighborhood of S. Let $h_i=(T_{x_i}f)(u(x_i))$ and $z=v(x')$. We have $w(x_i)=z_i=h_i+z$, $\forall_i=\overline{1,(m'-q'+1)}$. Thus, $m'-q'\geqslant$ $\geqslant\mathrm{codim}(\hat{H}_1\cap\dots\cap\hat{H}_{m'-q'+1})=\sum_{i=0}^{m'-q'+1}\mathrm{codim}\,\hat{H}_i\geqslant m'-q'+1$ (the last inequality holds since if $x_i\in\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)$, $\mathrm{codim}\,\hat{H}_i\geqslant 1$). This contradiction shows that $\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)\cap f^{-1}(x')$ has no more than $m'-q'$ points.

Now, we suppose that for some $x'\in M'$ there are the distinct points $x_1,\dots,x_{m'-q'+1}$ in $\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)$ such that $f(x_i)=x'$, $\forall i=\overline{1,(m'-q'+1)}$. For each $i\in\{1,\dots,m'-q'+1\}$, let $(x_{ij})_{j\geqslant 1}$ be the sequence of points of $\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)$ converging to x_i and $H_i=\{z\in T_{x'}\mathcal{F}_{M'},\ z\in\lim_{j\to\infty}(d_{x_{ij}}f)(u_{ij}),\ u_{ij}\in(T_{x_{ij}}f)^{-1}(T_{f(x_{ij})}\mathcal{F}_{M'})\cap\hat{T}_{x_{ij}}\mathcal{F}_M\}$. It is obvious that H_i is a vector subspace of $T_{x'}\mathcal{F}_{M'}$ and, since $\dim(\hat{T}_{x_{ij}}f(T_{x_{ij}}\mathcal{F}_M)\cap T_{f(x_{ij})}\mathcal{F}_{M'})<m'-q'$, there results that $\dim H_i<m'-q'$. Then $\mathrm{codim}\,H_i\geqslant 1$ in $T_{x'}\mathcal{F}_{M'}$ and $\sum_{i=1}^{m'-q'+1}\mathrm{codim}\,H_i\geqslant m'-q'+1$. Therefore $\{H_i\}_{i=\overline{1,(m'-q'+1)}}$ are not in general position in $T_{x'}\mathcal{F}_{M'}$. This means that there exists $\bar{w}_i,\dots,\bar{w}_{m'-q'+1}\in T_{x'}\mathcal{F}_{M'}$ such that the equations $\bar{w}_i=h_i+z$ has no solution for $h_i\in H_i$ and $z\in T_{x'}\mathcal{F}_{M'}$. But this is in contradiction with the fact that f is simultaneously infinitesimally stratum stable in the tangential sense for all subset S with no more than $m'-q'+1$ points, $S\subseteq f^{-1}(x')$, $x'\in M'$ (this last fact is implied by c)).

Let now $w\in C^\infty(f^* T\mathcal{F}_{M'})$. To prove that f is infinitesimally stable we need to show that there exist vector fields $u\in C^\infty(\hat{T}\mathcal{F}_M)$ and $v\in C^\infty(T\mathcal{F}_{M'})$ so that $w=(\hat{t}f)(u)+(\hat{\omega}f)(v)$.

We first show that this equation can be solved on a neighborhood of $\hat{\Sigma}(\mathcal{F}_{M'},\mathcal{F}_{M'},f)$. For this aim we prove that there exist open sets $(U_i)_{i=\overline{1,p}}$

in M, $(V_i)_{i=\overline{1,p}}$ in M', $(W_i)_{i=\overline{1,p}}$ in M', vector fields $(u_i)_{i=\overline{1,p}}$ on U_i, $(v_i)_{i=\overline{1,p}}$ on V_i satisfying:

1) $f(U_i)\subset V_i$,

2) $\overline{W}_i\subset V_i$,

3) $f(\overline{\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)})\subset \bigcup_{i=1}^{p} W_i$,

4) $f^{-1}(\overline{W}_i)\cap \overline{\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)}\subset U_i$,

5) $w=(\hat{t}f)(u_i)+(\hat{\omega}f)(v_i)$ on U_i.

One needs only construct U, V, W, u and v for a given $x'\in f(\overline{\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)})$ satisfying 1), 2), 4), 5). Indeed, $\overline{\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)}$, $f(\overline{\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)})$ being the compact sets, the necessary p will exists. By the condition c), which obviously is equivalent with the fact that f is simultaneously locally infinitesimally stratum stable in the tangential sense at $p_1,\dots,p_k\in S$, we may choose open neighborhood U of $\Sigma_{x'}=\overline{\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)}\cap f^{-1}(x')$ in M and V of x' in M' and vector fields u on U, v on V, so that 1) and 5) are satisfied. Next we choose W satisfying 4). We claim that there exists an open neighborhood W in M' so that $f^{-1}(W)\cap\overline{\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)}\subset U$. If not, there results that, for every open neighborhood W of x', $f^{-1}(W)\cap\Sigma_{x'}\not\subset U$. So, for every W, there exists a point $x_W\in f^{-1}(W)\cap\Sigma_{x'}\notin U$. Let $\{W_\alpha\}_{\alpha\in N}$ be a sequence of open neighborhoods of x' and $x_\alpha=x_{W_\alpha}$ the corresponding points; we have $x_\alpha\notin\overline{U}$, $x_\alpha\in\Sigma_{x'}$ and $\lim_{\alpha\to\infty} f(x_\alpha)=x'$. Since M is compact, we may assume that $\lim_{\alpha\to\infty} x_\alpha=x$. By continuity, we have $f(x)=x'$ and $x\in\overline{\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)}$. So, $x\in\Sigma_{x'}$. This fact is a contradiction since x is in $M\setminus U$ and $(M\setminus U)\cap\Sigma_{x'}=\emptyset$. By strinking W, we may assume that $f^{-1}(\overline{W})\cap\overline{\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)}\subset U$ and $\overline{W}\subset U$.

Next, we choose a partition of unity $\{\rho_i\}_{i=\overline{1,p}}$ on $W=\bigcup_{i=1}^{p} W_i$ with supp $\rho_i\subset W_i$ and an open neighborhood Z of $\overline{\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)}$ such that $f^{-1}(\overline{W}_i)\cap Z\subset U_i$, $\forall i=\overline{1,p}$. (This last fact is possible because, otherwise, there exists a sequence $(x_\alpha)_{\alpha\in N}$, $\lim_{\alpha\to\infty} x_\alpha=x\in\overline{\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)}$ with $x_\alpha\in f^{-1}(\overline{W}_i)\cap(M-U_i)$. Since both $f^{-1}(\overline{W}_i)$ and $M-U_i$ are closed, there results that $x\in f^{-1}(\overline{W}_i)$ and $x\notin U_i$. Contradiction).

Choose also a bump function $\rho\in C^\infty(M,R)$, with supp $\rho\subset Z$ and $\rho\equiv 1$ on a neighborhood of $\overline{\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)}$. Let $u=\sum_{i=1}^{p}\rho(\rho_i\circ f)\cdot u_i$, $v=\sum_{i=1}^{p}\rho_i v_i$. This u is well-defined on M because supp $\rho\cdot(\rho_i\circ f)\subset f^{-1}(\overline{W}_i)\cap Z\subset U_i$ and $u\in C^\infty(T\mathcal{F}_M)$.

The vector fields u and v satisfy the condition $w=(\hat{t}f)(u)+(\hat{\omega}f)(v)$ on a neighborhood of $\overline{\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)}$.

If $m-q<m'-q'$ then $\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)=M$ and the proof is finished. If $m-q\geq m'-q'$, we consider the open subset of M: $M_1=M-\overline{\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)}$ and let $f_1=f/M_1$, $w_1=w/M_1$. We prove that there exists a vector field u_1 on M_1 so that $(\hat{t}f_1)(u_1)=w_1$. For any $x\in M_1$, $T_xf\colon \hat{T}_x F_M\to(T_xf)(\hat{T}_x(\mathcal{F}_M))\cap$ $\cap\, T_{f(x)}\mathcal{F}_{M'}=T_{f(x)}\mathcal{F}_{M'}$ is onto. So, moreover, $T_xf\colon T_x\mathcal{F}_M\to T_{f(x)}\mathcal{F}_M$ is onto.

So, Ker $T_xf/T_x\mathcal{F}_M$ has constant dimension and forms a subbundle H_1 of $T\mathcal{F}_M$. Let H_2 be a subbundle complementary to H_1 in $T\mathcal{F}_M$. Clearly, $T_xf\colon (H_2)_x\to T_{f(x)}\mathcal{F}_{M'}$ is an isomorphism and thus induces an isomorphism on sections. Moreover, by hypotheses, the restriction of this isomorphism to the submodule $C^\infty(\hat{T}\mathcal{F}_M)$ defines an isomorphism on the submodule $C^\infty(f^*T\mathcal{F}_{M'})'$.

Thus there exists a vector field u_1 on M_1 so that $(\hat{t}f_1)(u_1)=w_1$. Choose a smooth function $\rho\colon M\to R$ which is zero on $\overline{\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)}$ and 1 off a neighborhood U of $\overline{\hat{\Sigma}(\mathcal{F}_M,\mathcal{F}_{M'},f)}$. Then ρu_1 is globally defined on M and $(\hat{t}f)(\rho u_1)=$ $=w$ since, by the first part of the proof, we may assume that supp $w\subset M-U$.

Our next objective is to establish a relation between the infinitesimal stratum stability of a mapping f and the stratum stability of f (both in the tangential sense).

The natural objects for which we attack this objective are the foliation preserving mappings.

Theorem 2. *Let $(M,\mathcal{F}_M)$ be a compact foliated manifold of dimension m, $(M',\mathcal{F}_{M'})$ be a foliated q'-codimensional manifold of dimension m' with $\mathcal{F}_{M'}$ a tangent foliation and $f\in C^\infty_{\mathcal{F}}(M,M')$ be a smooth foliation preserving mapping. Then, if f is infinitesimally stratum stable in tangential sense, f is stratum stable in the tangential sense.*

Proof. In view of the Remark which follows the Definition 3 and by using the algebraic machinery from [5], the assumption is equivalent with the following: the mixed homomorphism

$$\hat{\mu}=(\hat{\omega}f,\hat{t}f,C^\infty(T\mathcal{F}_{M'}),C^\infty(\hat{T}\mathcal{F}_M),C^\infty(f^*T\mathcal{F}_{M'}))$$

over $f^*\colon C^\infty(M')\to C^\infty(M)$, $f^*(h)=h\circ f$, is a surjective one. Throughout the remainder of this paper, we will use the following notations. We let I denote the unit interval $[0,1]$. For any manifold X, we let π_X denote the projection of $X\times I$ on X and π_I the projection on I.

If h maps $X\times I$ into Y and $t\in I$, we let h_t denote the mapping of X into Y defined by $h_t(x)=h(x,t)$. We let $\partial h/\partial t$ denote the mapping of $X\times I$ into TY which sends (x,t) into the tangent vector at $s=0$ to the curve $s\to$

$\to h(x, t+s)$. We let $T'h : \pi_X^* TX : TX \times I \to TY$ be the unique mapping such that $(T'h)_t = Th_t$, $\forall t \in I$. We define $l'h : C^\infty(\pi_X^* TX) \to C^\infty(h^* TY)$, $\omega' h : C^\infty(\pi_Y^* TY) \to C^\infty(h^* TY)$ by $(l'h)(\xi) = (T'h) \circ \xi$, $(\omega' h)(\eta) = \eta \circ (h, \pi_I)$, $(h, \pi_I) : X \times I \to Y \times I$, $(h, \pi_I)(x, t) = (h(x, t), \forall (x), t) \in X \times I$.

So, we may associate to the mixed homomorphism $\hat{\mu}$, the mixed homomorphism :

$$\hat{\mu}' = (\alpha, \beta, C^\infty(\pi_M^* T\mathcal{F}_{M'}), C^\infty(\pi_M^* \hat{T}\mathcal{F}_M), C^\infty((f \circ \pi_M)^* T\mathcal{F}_{M'})) \text{ over}$$

$$(f \times \mathrm{id}_I)^* : C^\infty(M' \times I) \to C^\infty(M \times I), \text{ with } \alpha = \hat{\omega}'(f \times \pi_M) | C^\infty(\pi_M^* T\mathcal{F}_{M'}).$$

$$\beta = \hat{l}'(f \circ \pi_M) | C^\infty(\pi_M^* \hat{T}\mathcal{F}_M).$$

From [5], we have an homomorphism of $C^\infty(T\mathcal{F}_{M'}) \times C^\infty(M' \times I)$ with $C^\infty(\pi_{M'}, T\mathcal{F}_{M'})$. By restriction, we get a bijection of $C^\infty(T\mathcal{F}_{M'}) \otimes C^\infty(M' \times I)$ with $C^\infty(\pi_M^* T\mathcal{F}_{M'})$ which is also an isomorphism. So, if we consider the natural transformation $\iota : C^\infty \to C'^\infty$ defined by $\iota_N = \pi_N^* : C^\infty(N) \to C^\infty(N \times I)$ for any manifold N, where the associated mixed homomorphism

$$\iota\hat{\mu} = ((\hat{\omega} f) \otimes (f \times id_I)^*, (\hat{l}f) \otimes id_I, C^\infty(T\mathcal{F}_{M'}) \otimes C^\infty(M' \times I),$$
$$C^\infty(\hat{T}\mathcal{F}_M) \otimes C^\infty(M \times I), C^\infty(f^* T\mathcal{F}_{M'}) \otimes C^\infty(M \times I))$$

is isomorphic to the mixed homomorphism $\hat{\mu}'$.

To make the parameter space, we consider the topological space :

$$\hat{X} = \{g \in C^\infty(M \times I, M'),\ g_0 = f,\ g(z \times I) \subset \mathrm{str}(f(z)),\ g(z \times I) \text{ belongs of same leaf as } f(z),\ \forall z \in M\},$$

with base point $x_0 = f \circ \pi_M$ and let $f \in C^{\hat{X}}(M \times I, M')$ denote the germ of the inclusion of $\hat{X}$ into $C^\infty(M \times I, M')$. We let $C^{\hat{X}}(\pi_M^*, T\mathcal{F}_{M'})$ denote the following set :

$C^{\hat{X}}(\pi_{M'}^* T\mathcal{F}_{M'}) = \{u$, u is the germ at x_0 of a mapping $\hat{X} \to C^\infty(M' \times I, T\mathcal{F}_{M'})$, $\tilde{\pi}_{M'}(x) = \hat{\pi}_{M'} \circ \tilde{u}(x)$, for all x near x_0, $\tilde{\pi}_{M'}$ and $\tilde{u}$ being the suitable representatives of the germs $\pi_{M'}$ and u respectively and $\hat{\pi}_{M'}$ the projection $T\mathcal{F}_M \to M'\}$. (Here $\pi_{M'} \in C^\infty(M' \times I, M')$ is identified with the constant mapping of $\hat{X}$ in $C^\infty(M' \times I, M')$).

Similarly, we define $C^{\hat{X}}(\pi_M^* \hat{T}\mathcal{F}_M)$ and let :

$C^{\hat{X}}(\hat{\hat{f}}\ \hat{T}\mathcal{F}_{M'}) = \{w$, w is the germ at x_0 of a mapping $\hat{X} \to C^\infty(M \times I, \hat{T}\mathcal{F}_{M'})$ so that $\tilde{\hat{\hat{f}}}(x) = \hat{\pi}_{M'} \circ \tilde{w}(x)$, for all x near x_0, $\tilde{\hat{\hat{f}}}$ and $\tilde{w}$ being the suitable representatives of the germs $\hat{\hat{f}}$ and $w\}$.

We define $l\hat{\hat{f}} : C^{\hat{X}}(\pi_M^* \hat{T}\mathcal{F}_M) \to C^{\hat{X}}(f^* T\mathcal{F}_{M'})$,

$$\omega\hat{\hat{f}} : C^{\hat{X}}(\pi_{M'}^* T\mathcal{F}_{M'}) \to C^{\hat{X}}(f^* T\mathcal{F}_{M'}),$$

as follows.

For a given $\zeta \in C^{\hat{X}}(\pi_M^* \hat{T}\mathcal{F}_M)$ let $(l\hat{\hat{f}})(\zeta)$ be the germ at x_0 of the mapping $g \to (l'g)(\tilde{\xi}(g))$, $\forall g \in \hat{X}$, where $\tilde{\zeta}$ is a representative of ζ.

For any $\eta \in C^{\hat{X}}(\pi_{M'}^* T\mathcal{F}_{M'})$, let $(\omega\hat{\hat{f}})(\eta)$ be the germ at x_0 of the mapping $g \to (\omega' g)(\tilde{\eta}(g))$, $\tilde{\eta}$ being a representative of η.

From the Proposition 1, §2 [5], it follows that $(\omega\hat{\hat{f}})(\eta)$, $(l\hat{\hat{f}})(\zeta)$ are the germs of the continuous mappings. Also, using the definition of $\hat{X}$, it follows directly that $(l\hat{\hat{f}})(\zeta)$, $(\omega\hat{\hat{f}})(\eta) \in C^{\hat{X}}(\hat{\hat{f}}^* T\mathcal{F}_{M'})$, $\forall \zeta \in C^{\hat{X}}(\pi_M^* \hat{T}\mathcal{F}_M), \eta \in C^{\hat{X}}(\pi_{M'}^*, T\mathcal{F}_{M'})$. Now, let $<\hat{\hat{f}}, \pi_I>$ be denoted the germ at x_0 of the mapping of $\hat{X}$ into $C^\infty(M \times I, M' \times I)$ which takes g to (g, π_I), let

$C^{\hat{X}}(<\hat{\hat{f}}, \pi_I>) : C^{\hat{X}}(M' \times I) \to C^{\hat{X}}(M \times I)$ the ring homomorphism defined by : for all $g' \in C^{\hat{X}}(M' \times I)$, $C^{\hat{X}}(<\hat{\hat{f}}, \pi_I>)(g')$ be the germ at x_0 of the mapping $x \to \tilde{g}'(x) \circ <\hat{\hat{f}}, \pi_I>(x)$, where with „$\sim$“ we designe the representative of the corresponding germs. Here, as in [5], for a manifold N, $C^{\hat{X}}(N)$ denote the ring of germs at x_0 of continuous mappings of $\hat{X}$ into $C^\infty(N)$.

We have a mixed homomorphism :

$$\hat{\nu} = (\omega\hat{\hat{f}}, l\hat{\hat{f}}, C^{\hat{X}}(\pi_{M'}^*, T\mathcal{F}_{M'}), C^{\hat{X}}(\pi_M^* \hat{T}\mathcal{F}_M), C^{\hat{X}}(\hat{\hat{f}}^* T\mathcal{F}_{M'}))$$

over $C^{\hat{X}}(<\hat{\hat{f}}, \pi_I>)$ and an isomorphism of ev $\hat{\nu}$ with $\hat{\mu}'$. Now, $\hat{\mu}$ surjective implies that $\hat{\mu}'$ is a surjective mixed homomorphism so that, via the isomorphism of $\hat{\mu}'$ with $\hat{\mu}$, we obtain that $\hat{\mu}'$ is a surjective mixed homomorphism over $(f \times \mathrm{id}_I)^*$. So, ev $\hat{\nu}$ is a surjective mixed homomorphism. Hence, it follows from Prop. 1, §6 [5] and from the fact that the modules involved are finitely generated that Ker ev ν is surjective. (Here ev : $C^{\hat{X}} \to C^\infty$ is the natural transformation given by the evaluation at x_0).

Let $\hat{\hat{\Delta}} : \hat{X} \to C^\infty(M \times I, T\mathcal{F}_{M'})$, $\hat{\hat{\Delta}}(g) = \dfrac{\partial g}{\partial t}$, so that $\hat{\hat{\Delta}}(g) \in C^\infty(g^* T\mathcal{F}_{M'})$ and it depends continuously on g.

Hence the germ $\hat{\Delta}$ of $\hat{\hat{\Delta}}$ at x_0 is in $C^{\hat{X}}(\hat{\hat{f}}^* T\mathcal{F}_{M'})$ and, clearly, $\hat{\hat{\Delta}}(x_0) = 0$. Hence $\hat{\Delta} \in \mathrm{Ker}\ \mathrm{ev}_{M \times I} \cdot C^{\hat{X}}(\hat{\hat{f}}^* T\mathcal{F}_{M'})$ where by $\mathrm{ev}_{M \times I}$ we denote the mapping $\mathrm{ev}_{M \times I} : C^{\hat{X}}(M \times I) \to C^\infty(M \times I)$ given by the evaluation at x_0. So, there exists $\tilde{\zeta} \in \mathrm{Ker}\ \mathrm{ev}_{M \times I} C^{\hat{X}}(\pi_M^* \hat{T}\mathcal{F}_M)$, $\hat{\eta} \in \mathrm{Ker}\ \mathrm{ev}_{M \times I} C^{\hat{X}}(\pi_{M'}^* T\mathcal{F}_{M'})$ such that $\hat{\Delta} = (l\hat{\hat{f}})(\zeta) + (\omega\hat{\hat{f}})(\eta)$.

Let $\hat{\tilde{\zeta}} : V \to C^\infty(\pi_M^* \hat{T}\mathcal{F}_M)$, $\hat{\tilde{\eta}} : V \to C^\infty(\pi_{M'}^*, \hat{T}\mathcal{F}_{M'})$ be the continuous representative of $\hat{\zeta}$ and $\hat{\eta}$ where V is a suitable neighborhood of x_0 in X.

We shall use the following result :

Lemma 1. *Let $(M', \mathcal{F}_{M'})$ be a foliated manifold with boundary so that $\mathcal{F}_{M'}$ is a tangent foliation. Then M' has a family of geodesics (U, ν) with the following property : when $(x_0, x_1) \in U$ and x_0, x_1 belong to the same leaf then $\nu((x_0, x_1) \times I)$ belong to the same leaf.*

Proof. Since this lemma is an extension of the Lemma 3 from §2 of [5], it is easily seen that, if the charts (V_i, φ_i) used in the proof of this lemma are the foliated charts of M', the family of geodesics constructed as in the proof of this lemma fulfils the claim of the lemma 1.

Now, let (U, ν) be a family of geodesics of the foliated manifold $(M', \mathcal{F}_{M'})$ as in the previous lemma. This family permits to construct a neighborhood $\mathcal{U}$ of f in $C^\infty_{\mathcal{F}, S}(M, M', f) = C^\infty_{\mathcal{F}}(M, M', f)$ such that $(f, g)(M) \subset U$, $\forall g \in \mathcal{U}$. We define $\hat{\nu}_* : \mathcal{U} \to C^\infty(M \times I, M')$ by $(\hat{\nu}_*(g))(z, t) = \nu(f(z), g(z), t)$. Note that $\hat{\nu}_*(g)(z \times I) \subset \mathrm{Str}\, f(z)$. Moreover, $\hat{\nu}_*(g)(z \times I) \subset$ the leaf of $\mathcal{F}_{M'}$ through $f(z)$, $\forall g \in \mathcal{U}$ and $z \in M$.

It is easy to see that $\hat{\nu}_*$ is a continuous mapping and that $\hat{\nu}_*(f) = (f \circ \pi_M) = x_0$. So we may choose $\mathcal{U}$ so that $\hat{\nu}_*(\mathcal{U}) \subset V$. Then $\tilde{\zeta}(\hat{\nu}_*(g))$ and $\tilde{\eta}(\hat{\nu}_*(g))$ are defined for all $g \in \mathcal{U}$.

To finish the proof we will need the following lemma.

Lemma 2. *Let $(M, \mathcal{F}_M)$ be a foliated manifold with boundary. Then there exists a neighborhood $\hat{O}_M$ of O in $C^\infty(\pi_M^* \hat{T}\mathcal{F}_M)$ such that for each $\zeta \in \hat{O}_M$ there exists $\hat{\Theta}(\zeta) \in C^\infty(M \times I, M)$ such that for each $t \in I$ $\hat{\Theta}(\zeta)_t$ is a diffeomorphism of M taking each leaf of $\mathcal{F}_M$ onto itself, $\hat{\Theta}(\zeta)_0 = id_M$ and* $\dfrac{\partial \Theta(\zeta)_t}{\partial t} \circ \hat{\Theta}(\zeta)_t^{-1} = \zeta_t$.

Proof. Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be a family of charts of M such that $\{U_\alpha\}_{\alpha \in I}$ is a locally finite cover. Choose $K_\alpha \subset L_\alpha \subset U_\alpha$ such that K_α and L_α are compact $\{K_\alpha\}_{\alpha \in I}$ cover M and L_α is a neighborhood of K_α, $\forall \alpha \in I$. Let $\hat{U}_\alpha$ be an open set in $\mathbb{R}^m$ such that $\varphi_\alpha(U_\alpha) = \hat{U}_\alpha \cap R^m_+$ and a compact set $\hat{L}_\alpha \subset \hat{U}_\alpha$ such that $\varphi_\alpha(K_\alpha) \subset \operatorname{int} \hat{L}_\alpha$ and $\varphi_\alpha(L_\alpha) = \hat{L}_\alpha \cap R^m_+$. For each $\alpha \in I$, let ε_α denote the distance from $\varphi_\alpha(K_\alpha)$ to $\hat{U}_\alpha - \hat{L}_\alpha$. Then $\varepsilon_\alpha > 0$ for each $\alpha \in I$. We define Θ_M to be the set of all $\zeta \in C^\infty(\pi_M^* T\mathcal{F}_M)$ such that for each $\alpha \in I$, the principal part ζ_α of the vector field $(\varphi_\alpha \times id_I)_*(\zeta | U_\alpha) \in C^\infty(T(\varphi_\alpha(U_\alpha) \times I) = (\varphi_\alpha(U_\alpha) \times I) \times R^{m+1})$, satisfies the condition : $\|\zeta_\alpha(x, t)\| < \varepsilon_\alpha$ for all $x \in \varphi_\alpha(L_\alpha)$ and all $t \in I$. We write $\hat{O}_M = O_M \cap C^\infty(\pi_M^* \hat{T}\mathcal{F}_M)$. For each $x \in K_\alpha$ and each $\zeta \in O_M$, there exists a unique curve $\nu_\alpha(\varphi_\alpha(x), .) : [0, \varepsilon) \to \hat{L}_\alpha$ such that $\nu_\alpha(\varphi_\alpha(x), 0) = \varphi_\alpha(x)$ and $\dfrac{\partial \nu_\alpha(\varphi_\alpha(x), t)}{\partial t} = \zeta_\alpha(\nu_\alpha(\varphi_\alpha(x), t), t)$.

If $\zeta \in \hat{O}_M$ we can obviously choose $\varepsilon = 1$ and define

$$\hat{\Theta}(\zeta)(z, t) = \varphi_\alpha^{-1}(\nu_\alpha(\varphi_\alpha(z), t)), \quad \forall (z, t) \in K_\alpha \times I.$$

That this definition has the properties states in the lemma there results in the same way as in Lemma 2, §7 [5], by using the fact that the flow generated by a vector field from $C^\infty(\hat{T}\mathcal{F}_M)$ is a diffeomorphism of M taking each leaf of F_M onto itself.

Now, we complete the proof of Theorem 2.

Since $\hat{\tilde{\zeta}}$ and $\hat{\tilde{\eta}}$ are continuous and $\hat{\tilde{\zeta}}(x_0) = 0$, $\hat{\tilde{\eta}}(x_0) = 0$, we may choose $\mathcal{U}$ so that $-\hat{\tilde{\zeta}}(\hat{\nu}_*(\mathcal{U})) \subseteq \hat{O}_M$ and $\hat{\tilde{\eta}}(\hat{\nu}_* U)) \subseteq \hat{O}_{M'}$. Then $\hat{\Theta}(-\hat{\tilde{\zeta}}(\hat{\nu}_*(g)))$ and $\hat{\Theta}(\hat{\tilde{\eta}}(\hat{\nu}_*(g)))$ are defined for all $g \in \mathcal{U}$. Let $H_1(g) = \hat{\Theta}(-\hat{\tilde{\zeta}}(\hat{\nu}_*(g)))_1$, $H_2(g) = \hat{\Theta}(\hat{\tilde{\eta}}(\hat{\nu}_*(g)))_1$.

From the previous lemma, it follows that $H_1(g) \in \mathrm{Diff}^\infty(M)$ and takes each leaf of $\mathcal{F}_M$ onto itself, $H_2(g) \in \mathrm{Diff}^\infty(M')$ and takes each leaf of $\mathcal{F}_{M'}$ onto itself, for all $g \in \mathcal{U}$. Since $\hat{\tilde{\zeta}}(\hat{\nu}_*(f)) = \hat{\tilde{\zeta}}(x_0) = 0$ and $\dfrac{\partial \hat{\Theta}(\zeta)_t}{\partial t} \circ \hat{\Theta}(\zeta)^{-1} = \zeta_t$ is equivalent with $\dfrac{\partial \hat{\Theta}(\zeta)}{\partial t}(x, t) = \zeta(\hat{\Theta}(\zeta)(x, t), t)$, $\forall x \in M$, $\forall t \in I$, we obtain that $\dfrac{\partial \hat{\Theta}(0)}{\partial t}(x, t) = 0$ $\forall x \in M$, $\forall t \in I$. So, we have $\hat{\Theta}(-\hat{\tilde{\zeta}}(\hat{\nu}_*(f)))_1 = H_1(f) = \hat{\Theta}(-\hat{\tilde{\zeta}}(\hat{\nu}(f)))_0 = id_M$. Similarly, we have $H_2(f) = id_{M'}$.

For any $g \in \mathcal{U}$ we have $y = \hat{\nu}(g) \in V$ and the mappings $h = \hat{\Theta}(-\hat{\tilde{\zeta}}(\hat{\nu}_*(g)))$, $h' = \hat{\Theta}(\hat{\tilde{\eta}}(\hat{\nu}_*(g)))$ which satisfy the hypothesis of Lemma 1, §7 [5], so that we obtain :

$$y_t(m) = h'_t \circ f \circ h_t^{-1}(m), \quad \forall m \in M, \ \forall t \in I.$$

Substituting $t = 1$, in the previous formula and since $\hat{\nu}_*(g)_1 = g$, we obtain : $g = H_2(g) \circ f \circ H_1(g)$ for all $g \in \mathcal{U}$. Clearly, this result implies Theorem 2.

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CENTROAFINNE GEOMETRY OF COMPLEX HYPERSURFACES

BY

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A complex centroaffine space C_n can be viewed as a real centroaffine space $\tilde{C}_{2n}$ endowed with an automorphism J which satisfies the condition $J^2 = -I$. Then, the group of the automorphisms of C_n appears as the subgroup of the automorphisms of $\tilde{C}_{2n}$ which commute with J. In addition a complex submanifold S_m in C_n appears as a real submanifold $\tilde{S}_{2m}$ in $\tilde{C}_{2n}$ whose tangent spaces at all points are invariant by J. Sometimes $\tilde{S}_{2m}$ is called an *invariant* submanifold in $\tilde{C}_{2n}$.

In particular, a complex hypersurface in C_n, can be viewed as an invariant submanifold of codimension 2 in $\tilde{C}_{2n}$. Moreover, under some assumptions the plane spanned by the vector of position r and the vector $J(r)$ can be taken as normal space at every point of the hypersurface. Thus, the geometry of these hypersurfaces can be developed similarly to the geometry of real hypersurfaces in real centroaffine spaces [3—6]. However one seems that it has a more rich and more varied content.

In this work, our purpose is to sketch the theory of complex hypersurfaces in the complex centroaffine spaces. Further details will be given in a forthcoming paper. We shall use, with some exceptions, the terms and the notations from [5].

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Let $\tilde{C}_{2n+2}$ be a real centroaffine space endowed with the complex structure J and let $\tilde{S}_{2n}$ be a complex hypersurface given locally by the equation

$$r = r(u^i), \quad (i, j, k = 1, 2, \ldots, 2n). \tag{1}$$

We assume that the analytic vector function r satisfies the condition

$$(r, \tilde{r}, r_1, \ldots, r_{2n}) \neq 0, \tag{2}$$

where $\tilde{r} = J(r)$ and $r_i = \partial r / \partial u^i$.

Since $\tilde{S}_{2n}$ is an invariant submanifold, we have also

$$\tilde{r}_j = J(r_j) = F_j^i r_i \tag{3}$$