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CENTROAFINNE GEOMETRY OF COMPLEX HYPERSURFACES

BY

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A complex centroaffine space C_n can be viewed as a real centroaffine space $\tilde{C}_{2n}$ endowed with an automorphism J which satisfies the condition $J^2 = -I$. Then, the group of the automorphisms of C_n appears as the subgroup of the automorphisms of $\tilde{C}_{2n}$ which commute with J. In addition a complex submanifold S_m in C_n appears as a real submanifold $\tilde{S}_{2m}$ in $\tilde{C}_{2n}$ whose tangent spaces at all points are invariant by J. Sometimes $\tilde{S}_{2m}$ is called an *invariant* submanifold in $\tilde{C}_{2n}$.

In particular, a complex hypersurface in C_n, can be viewed as an invariant submanifold of codimension 2 in $\tilde{C}_{2n}$. Moreover, under some assumptions the plane spanned by the vector of position r and the vector $J(r)$ can be taken as normal space at every point of the hypersurface. Thus, the geometry of these hypersurfaces can be developed similarly to the geometry of real hypersurfaces in real centroaffine spaces [3—6]. However one seems that it has a more rich and more varied content.

In this work, our purpose is to sketch the theory of complex hypersurfaces in the complex centroaffine spaces. Further details will be given in a forthcoming paper. We shall use, with some exceptions, the terms and the notations from [5].

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Let $\tilde{C}_{2n+2}$ be a real centroaffine space endowed with the complex structure J and let $\tilde{S}_{2n}$ be a complex hypersurface given locally by the equation

$$r = r(u^i), \quad (i, j, k = 1, 2, \dots, 2n). \tag{1}$$

We assume that the analytic vector function r satisfies the condition

$$(r, \bar{r}, r_1, \dots, r_{2n}) \neq 0, \tag{2}$$

where $\bar{r} = J(r)$ and $r_i = \partial r / \partial u^i$.

Since $\tilde{S}_{2n}$ is an invariant submanifold, we have also

$$\bar{r}_j = J(r_j) = F^i_j r_i \tag{3}$$

from which, by applying again J one obtains

$$F^i_j F^j_k = -\delta^i_k, \tag{4}$$

i.e. F is the tensor field which defines the complex structure induced by J on $\tilde{S}_{2n}$. Denoting by ρ (resp. $\tilde{\rho}$) the covector of the tangent hyperplan which contains the tangent subspace to $\tilde{S}_{2n}$ and the vector $\tilde{r}$ (resp. r), at every point of $\tilde{S}_{2n}$ we have

$$\rho r=1,\ \rho\tilde{r}=0,\ \rho r_i=0,\ \tilde{\rho} r=0,\ \tilde{\rho}\tilde{r}=1,\ \tilde{\rho} r_i=0, \text{ from which it follows} \tag{5}$$

$$\rho=\frac{[\tilde{r},r_1,\dots,r_{2n}]}{(r,\tilde{r},r_1,\dots,r_{2n})}, \quad \tilde{\rho}=-\frac{[r,r_1,\dots,r_{2n}]}{(r,\tilde{r},r_1,\dots,r_{2n})}. \tag{6}$$

Also from (3) and (5) one obtains

$$\tilde{\rho}=-J^*(\rho), \quad \rho\tilde{r}_i=0, \quad \tilde{\rho}\tilde{r}_i=0. \tag{7}$$

Now, from (5) it results

$$\rho_j r_i+\rho\partial_j r_i=0, \quad \partial_j\rho_i r+\rho_i r_j=0, \tag{8}$$

and the other ones which are obtained from these by replacing r by $\tilde{r}$ or ρ by $\tilde{\rho}$.

We assume in the sequel that $\tilde{S}_{2n}$ is regular i.e. satisfies the condition

$$(\rho,\ \tilde{\rho},\ \rho_1,\ \dots,\ \rho_{2n})\neq 0. \tag{9}$$

By considering the centroaffine normalisation i.e. taking r and $\tilde{r}=J(r)$ as normals, while ρ and $\tilde{\rho}=-J^*(\rho)$ as conormals, we can write the fundamental equations

$$\begin{aligned} \partial_j r_i &= {}'\Gamma^k_{ji} r_k + {}'G_{ji} r + {}'H_{ji}\tilde{r}, \\ \partial_j \rho_i &= {}''\Gamma^k_{ji}\rho_k + {}''G_{ji}\rho + {}''H_{ji}\tilde{\rho}. \end{aligned} \tag{10}$$

It comes out that $'\Gamma$ and $''\Gamma$ are symmetric linear connections called the connections of the *first* and the *second kind*, respectively, while $'G$, $'H$, $''G$, $''H$ are symmetric tensor fields called *Eulerian* tensor fields, induced by the centroaffine normalisation of $\tilde{S}_{2n}$. From (5), (7), (8) and (10) one obtains

$$\begin{aligned} {}'G_{ji} &= \rho\partial_j r_i = -\rho_j r_i = \partial_i\rho_j r = {}''G_{ij}, \\ {}'H_{ji} &= \tilde{\rho}\partial_j r_i = -\tilde{\rho}_j r_i = \partial_i\tilde{\rho}_j r = -{}''H_{ij}, \end{aligned} \tag{11}$$

i.e. $'G={}''G$ and $'H=-{}''H$.

Deleting the prime on G and H we can write the fundamental equations as follows.

$$'\nabla_j r_i = G_{ji} r + H_{ji}\tilde{r}, \quad ''\nabla_j\rho_i = G_{ji}\rho - H_{ji}\tilde{\rho}. \tag{12}$$

Differentiating covariantly with respect to $'\Gamma$ the both sides of the equality (3) and taking into account (12) we obtain

$$'\nabla_k F^i_j=0, \quad G_{kj}=H_{ki}F^i_j, \quad H_{kj}=-G_{ki}F^i_j. \tag{13}$$

The first relation tells us that $'\Gamma$ is a symmetric F-connection. The last two relations, multiplied with F give us

$$G_{ki}F^k_h F^i_j = -G_{hj}, \quad H_{ki}F^k_h F^i_j = -H_{hj} \tag{14}$$

i.e. the quadratic forms G and H are skew-invariant with respect to F.

Form (5) and (14) we obtain

$$(\rho,\ \tilde{\rho},\ \rho_1,\ \dots,\ \rho_{2n})(r,\ \tilde{r},\ r_1,\ \dots,\ r_{2n}) = \det(G_{ij}). \tag{15}$$

So, $\tilde{S}_{2n}$ is regular if and only if the tensor field G is nondegenerate.

Putting

$$\tilde{\rho}_j = -J^*(\rho_j) = A^i_j\rho_i + B_j\rho$$

one obtains by (4) (5) and (11)

$$\tilde{\rho}_j = -F^i_j\rho_i \tag{16}$$

from which, by covariant differentiation with respect to $''\Gamma$ one obtains again the last two equalities from (13) as well as

$$''\nabla_k F^i_j = 0. \tag{17}$$

This tells us that $''\Gamma$ is also an F-connection.

The equalities $G_{ji}=-\rho_j r_i$ and $H_{ji}=-\tilde{\rho}_j r_i$ lead to

$$\begin{aligned} \partial_k G_{ji} - {}'\Gamma^h_{kj} G_{hi} - {}''\Gamma^h_{ki} G_{jh} &= 0, \\ \partial_k H_{ji} - {}'\Gamma^h_{kj} H_{hi} - {}''\Gamma^h_{ki} H_{jh} &= 0. \end{aligned} \tag{18}$$

Therefore we have obtained

Proposition 1. *The pair of connections* ($'\Gamma$, $''\Gamma$) *is conjugate* [5] *with respect to each of the tensor fields G and H.*

The integrability conditions of the fundamental equations (12) give us :

$$\begin{aligned} {}'R^h_{kji} &= G_{ki}\delta^h_j - G_{ji}\delta^h_k + H_{ki}F^h_j - H_{ji}F^h_k, \\ {}'\nabla_k G_{ji} &= {}'\nabla_j G_{ki}, \quad {}'\nabla_k H_{ji} = {}'\nabla_j H_{ki}, \end{aligned} \tag{19}$$

$$\begin{aligned} {}''R^h_{kji} &= G_{ki}\delta^h_j - G_{ji}\delta^h_k + H_{ki}F^h_j - H_{ji}F^h_k, \\ {}''\nabla_k G_{ji} &= {}''\nabla_j G_{ki}, \quad {}''\nabla_k H_{ji} = {}''\nabla_j H_{ki} \end{aligned} \tag{20}$$

Therefore $'R={}''R$ and the tensor fields $'\nabla G$, $'\nabla H$, $''\nabla G$, $''\nabla H$ are completely symmetric.

Taking $k=h$ in (19_1) and (20_1) one obtains for the Ricci tensors of $'\Gamma$ and $''\Gamma$ the expression

$$'R_{ji} = {}''R_{ji} = -2(n-1)G_{ji}. \tag{21}$$

These equalities and (13) lead to

$$G_{ji} = -\frac{1}{2(n-1)}\,{}'R_{ji}, \quad H_{ji} = \frac{1}{2(n-1)}\,{}'R_{ji}F^h_i. \tag{22}$$

which say that G and H are determined by F and the connection $'\Gamma$ or $''\Gamma$. As G is symmetric, it follows that the Ricci tensors $'R_{ji}$ and $''R_{ji}$ are symmetric, hence the both connections $'\Gamma$ and $''\Gamma$ are equiaffine.

By using (22) and (19_{1-2}) one obtains

$$'R^h_{kji}=\frac{1}{2(n-1)}('R_{ji}\delta^h_k-'R_{ki}\delta^h_j+'R_{km}F^m_iF^h_j-'R_{jm}F^m_iF^h_k), \tag{23}$$
$$'\nabla_k{}'R_{ji}={}'\nabla_j{}'R_{ki}$$

For any symmetric F-connection on a complex manifold of real dimension $2n$, the tensor field P of H-projective curvature is defined by

$$P^h_{kji}=R^h_{kji}+\delta^h_kP_{ji}-\delta^h_jP_{ki}-(P_{kj}-P_{jk})\delta^h_i+F^h_kQ_{ji}-F^h_jQ_{ki}-(Q_{kj}-Q_{jk})F^h_i. \tag{24}$$

where

$$P_{ji}=-\frac{1}{2(n+1)}\left[R_{ji}+\frac{1}{n-1}O^{hk}_{ji}(R_{hk}+R_{kh})\right], \tag{25}$$
$$O^{hk}_{ji}=\frac{1}{2}(\delta^h_j\delta^k_i-F^h_jF^k_i),\quad Q_{ji}=-P_{jh}F^h_i.$$

When the connection is equiaffine, we have

$$P_{ji}=-\frac{1}{2(n-1)}R_{ji}. \tag{26}$$

Hence

$$P^h_{kji}=R^h_{kji}-\frac{1}{2(n-1)}(\delta^h_kR_{ji}-\delta^h_jR_{ki}+R_{km}F^m_iF^h_j-R_{jm}F^m_iF^h_k). \tag{27}$$

The F-connection Γ is H-projectively flat $(n\geq 2)$ if and only if the tensor field P vanishes [2].

From (19), (22) and (23) it results

Proposition 2. *The connections of first and second kind, induced by the centroaffine normalisation on a complex hypersurface, are H-projectively flat, equiaffine and symmetric F-connections.*

The fundamental equations (12), the integrability conditions (19) or (20) and the formulae (22) imply the following

Proposition 3. *An analytic manifold $\tilde{S}_{2n}$, endowed with a complex structure F and an F-connection $'\Gamma$ (resp. $''\Gamma$) which is symmetric, equiaffine and H-projectively flat, can be immersed as a complex hypersurface in a complex centroaffine space $(\tilde{C}_{2n+2}, J)$ such that F becomes the complex structure induced by J and $'\Gamma$ (resp. $''\Gamma$) be the induced linear connection of first kind (resp. second kind). The immersion is determined up to a complex centroaffine automorphism.*

Introducing the mean connection Γ and the deformation tensor h, for the pair of connections $('\Gamma, ''\Gamma)$ as follows

$$\Gamma^k_{ji}=\frac{1}{2}('\Gamma^k_{ji}+''\Gamma^k_{ji}),\quad h^k_{ji}=\frac{1}{2}('\Gamma^k_{ji}-''\Gamma^k_{ji}) \tag{28}$$

we can write

$$'\Gamma^k_{ji}=\Gamma^k_{ji}+h^k_{ji},\qquad ''\Gamma^k_{ji}=\Gamma^k_{ji}-h^k_{ji}. \tag{29}$$

From (13) and (17) it follows

$$\nabla_kF^i_j=0,\qquad \nabla_kG_{ji}=0,\quad \nabla_kH_{ji}=0. \tag{30}$$

Thus we have

Proposition 4. *The pairs of tensors (F, G) and (F, H) determine on a complex hypersurface, complex metric structures* [1].

By means of the mean connection and of the deformation tensor, the fundamental equations become

$$\nabla_jr_i=h^k_{ji}r_k+G_{ji}r+H_{ji}\tilde{r},\qquad \nabla_j\rho_i=-h^k_{ji}\rho_k+G_{ji}\rho-H_{ji}\tilde{\rho} \tag{31}$$

The integrability conditions of these give us :

$$R^s_{kji}=\nabla_jh^s_{ki}-\nabla_kh^s_{ji}+h^m_{ki}h^s_{jm}-h^m_{ji}h^s_{km}+G_{ki}\delta^s_j-G_{ji}\delta^s_k+H_{ki}F^s_j-H_{ji}F^s_k, \tag{32}$$

$$h^m_{ji}G_{km}-h^m_{ki}G_{jm}=0,\quad h^m_{ji}H_{km}-h^m_{ki}H_{jm}=0, \tag{33}$$

$$R^s_{kji}=\nabla_kh^s_{ji}-\nabla_jh^s_{ki}+h^m_{ki}h^s_{jm}-h^m_{ji}h^s_{km}+G_{ki}\delta^s_j-G_{ji}\delta^s_k+H_{ki}F^s_j-H_{ji}F^s_k. \tag{34}$$

Putting

$$h_{ijk}=h^m_{ij}G_{km} \tag{35}$$

from (33_1) one obtains

$$h_{ijk}=h_{ikj}, \tag{36}$$

which says that the tensor field h is completely symmetric.

Then, from (33_2) it results

$$h_{ijm}F^m_k=h_{ikm}F^m_j. \tag{37}$$

By substracting and adding the equations (32) and (34) one obtains, respectively,

$$\nabla_kh^s_{ji}=\nabla_jh^s_{ki}, \tag{38}$$

$$R^s_{kji}=h^m_{ki}h^s_{jm}-h^m_{ji}h^s_{km}+G_{ki}\delta^s_j-G_{ji}\delta^s_k+G_{jh}F^h_iF^s_k-G_{kh}F^h_iF^s_j. \tag{39}$$

Putting

$$t_i=\frac{1}{2n}h^s_{is} \tag{40}$$

we obtains from (38)

$$\nabla_kt_j=\nabla_jt_k, \tag{41}$$

which says that the 1-form

$$t=t_jdu^j \tag{42}$$

is closed.

Then, considering

$$\omega = (r,\ \bar{r},\ r_1,\ \ldots,\ r_{2n}),\quad G = \det(G_{ij}) \tag{43}$$

we obtain

$$'\Gamma^h_{ih} = \frac{\partial \ln |\omega|}{\partial u^i},\qquad \Gamma^h_{ih} = \frac{1}{2}\frac{\partial \ln |G|}{\partial u^i}, \tag{44}$$

and from this

$$t_i = \frac{1}{2n}\frac{\partial}{\partial u^i}\ln\frac{|\omega|}{\sqrt{|G|}}, \tag{45}$$

i.e. the 1-form t is just exacte.

From the fundamental equations (31) and their integrability conditions, it follows

Proposition 5. *An analytic manifold $\tilde{S}_{2n}$, endowed with a complex structure F, a tensor field G of type $(0, 2)$ symmetric and nondegenerate and a tensor field h of type $(1, 2)$ symmetric, which satisfy the equations (14_1), (30_{1-2}), (33_1), (35), (38), and (39), can be immersed as a complex hypersurface in the complex centroaffine space $(\tilde{C}_{2n+2}, J)$ such that F, G, h be, respectively, the complex structure and the tensor fields of type $(0, 2)$ and $(1, 2)$ induced be the centroaffine normalization. The immersion is determined up to a complex centroaffine automorphism.*

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LINEAR ALGEBRA OF CURVATURE TENSORS ON HERMITIAN MANIFOLDS

BY

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Introduction. Let V be an n-dimensional real vector space with a positive definite inner product g. In [21] S i n g e r and T h o r p e gave a decomposition of the Riemannian curvature tensor over V into $O(n)$– irreducible components.

In the case when V is a $2n$–dimensional real vector space with a complex structure, T r i c e r r i and V a n h e c k e studied this decomposition, with respect to the action of $\mathfrak{U}(n)$ [25].

Further on, supposing V endowed with different structures too, several authors considered similar problems for spaces of curvature tensors having special properties. This way, many interesting geometrical applications have been found (see, for example [1], [7], [9], [16], [17], [20], [24], [26]).

In particular, the Weyl conformal tensor ([20], [21]) and the Bochner tensor ([16], [22]) have been obtained using the decomposition theory. Moreover, in [11], V a n h e c k e and J a n s s e n s defined a Bochner curvature tensor for a class of almost contact metric structures, by means of the decomposition of the space of „C–curvature" tensors under the action of $\mathfrak{U}(n)\times 1$.

Recently, in the Riemannian case, N. B o k a n obtained the complete decomposition with respect to $SO(n)$, of the vector space $\mathfrak{R}(V)$ given by the tensors of type $(0,4)$ over V, which have the same symmetries as the curvature tensor of an arbitrary linear symmetric connection, defined on a Riemannian manifold ([5], [6]). The purpose of this paper, again in the Hetmitian case is to study the decomposition for the action of $\mathfrak{U}(n)$ on the subspace $\mathfrak{K}(V)$ of $\mathfrak{R}(V)$ which consists of the tensors satisfying the Kähler identity. These curvature tensors are strictly related to the holomorphically projective transformations (and hence to the holomorphically projective tensor) on a Hermitian manifold [27]. Furthermore, the decomposition of $\mathfrak{K}(V)$ is also essential to construct in a different way the C–projective curvature tensor, on normal almost-contact manifolds (see [15]).

In the last section, we shall explain briefly these geometrical applications which will be developed separately in two forthcoming papers ([14], [19]).

The present paper is mostly devoted to the study of the algebraic aspects related to the decomposition of $\mathfrak{K}(V)$.

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