Then, considering

$$\omega=(r,\ \bar{r},\ r_1,\ \ldots,\ r_{2n}),\quad G=\det(G_{ij}) \tag{43}$$

we obtain

$$'\Gamma^h_{ih}=\frac{\partial \ln|\omega|}{\partial u^i},\qquad \Gamma^h_{ih}=\frac{1}{2}\frac{\partial \ln|G|}{\partial u^i}, \tag{44}$$

and from this

$$t_i=\frac{1}{2n}\frac{\partial}{\partial u^i}\ln\frac{|\omega|}{\sqrt{|G|}}, \tag{45}$$

i.e. the 1-form t is just exacte.

From the fundamental equations (31) and their integrability conditions, it follows

Proposition 5. *An analytic manifold $\tilde{S}_{2n}$, endowed with a complex structure F, a tensor field G of type (0, 2) symmetric and nondegenerate and a tensor field h of type (1, 2) symmetric, which satisfy the equations (14_1), (30_{1-2}), (33_1), (35), (38), and (39), can be immersed as a complex hypersurface in the complex centroaffine space $(\tilde{C}_{2n+2}, J)$ such that F, G, h be, respectively, the complex structure and the tensor fields of type (0, 2) and (1, 2) induced be the centroaffine normalization. The immersion is determined up to a complex centroaffine automorphism.*

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LINEAR ALGEBRA OF CURVATURE TENSORS ON HERMITIAN MANIFOLDS

BY

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Introduction. Let V be an n-dimensional real vector space with a positive definite inner product g. In [21] S i n g e r and T h o r p e gave a decomposition of the Riemannian curvature tensor over V into $O(n)$ – irreducible components.

In the case when V is a $2n$ – dimensional real vector space with a complex structure, T r i c e r r i and V a n h e c k e studied this decomposition, with respect to the action of $\mathfrak{U}(n)$ [25].

Further on, supposing V endowed with different structures too, several authors considered similar problems for spaces of curvature tensors having special properties. This way, many interesting geometrical applications have been found (see, for example [1], [7], [9], [16], [17], [20], [24], [26]).

In particular, the Weyl conformal tensor ([20], [21]) and the Bochner tensor ([16], [22]) have been obtained using the decomposition theory. Moreover, in [11], V a n h e c k e and J a n s s e n s defined a Bochner curvature tensor for a class of almost contact metric structures, by means of the decomposition of the space of „C–curvature" tensors under the action of $\mathfrak{U}(n)\times 1$.

Recently, in the Riemannian case, N. B o k a n obtained the complete decomposition with respect to $SO(n)$, of the vector space $\mathfrak{R}(V)$ given by the tensors of type (0,4) over V, which have the same symmetries as the curvature tensor of an arbitrary linear symmetric connection, defined on a Riemannian manifold ([5], [6].). The purpose of this paper, again in the Hetmitian case is to study the decomposition for the action of $\mathfrak{U}(n)$ on the subspace $\mathcal{H}(V)$ of $\mathfrak{R}(V)$ which consists of the tensors satisfying the Kähler identity. These curvature tensors are strictly related to the holomorphically projective transformations (and hence to the holomorphically projective tensor) on a Hermitian manifold [27]. Furthermore, the decomposition of $\mathcal{H}(V)$ is also essential to construct in a different way the C–projective curvature tensor, on normal almost-contact manifolds (see [15]).

In the last section, we shall explain briefly these geometrical applications which will be developed separately in two forthcoming papers ([14], [19]).

The present paper is mostly devoted to the study of the algebraic aspects related to the decomposition of $\mathcal{H}(V)$.

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1. Preliminaries. Let V be a $2n$-dimensional real vector space with a complex structure J and a Hermitian product g; then, for all $x, y\in V$, we have

$$J^2=-I_V,\quad g(Jx, Jy)=g(x, y)$$

where I_V is the identity transformation of V. A tensor of type (1, 3) over V is a bilinear mapping

$$R: V\times V\to \mathrm{Hom}(V, V),\quad (x, y)\to R(x, y).$$

R is called a „curvature tensor" if it has the following properties:

$$R(x, y)=-R(y, x) \tag{1.1}$$

$$\underset{xyz}{\sigma} R(x, y)z=0, \tag{1.2}$$

where σ denotes the cyclic sum (this is the first Bianchi identity).

Let $\mathfrak{R}(V)$ be the vector space of the curvature tensors over V. $\mathfrak{R}(V)$ is a subspace of the tensor space of type (1, 3) over V and has a natural inner product induced from that of V:

$$\langle R, \tilde R\rangle=\sum_{i,j,k=1}^{2n} g(R(e_i, e_j)e_k, \tilde R(e_i, e_j)e_k),$$

where $R, \tilde R\in\mathfrak{R}(V)$ and $\{e_i\}$, $i=1, 2, \dots, 2n$, is an arbitrary orthonormal basis of V. Moreover, the standard representation α of the unitary group $\mathfrak{U}(n)$ in V induces a representation $\tilde\alpha$ of $\mathfrak{U}(n)$ in $\mathfrak{R}(V)$ by

$$\tilde\alpha: \mathfrak{U}(n)\to Gl(\mathfrak{R}(V)),\quad a\to\tilde\alpha(a),$$

where

$$\tilde\alpha(a)(R)(x, y, z, w)=R(\alpha(a^{-1})x, \alpha(a^{-1})y, \alpha(a^{-1})z, \alpha(a^{-1})w)$$

for all $x, y, z, w\in V$, $R\in\mathfrak{R}(V)$, $a\in\mathfrak{U}(n)$, and $R(x, y, z, w)=g(R(x, y)z, w))$

We shall consider now the subspace $\mathfrak{K}(V)$ of $R(V)$ given by the curvature tensors which satisfy the Kähler identity: then, if we denote by K the elements of $\mathfrak{K}(V)$, we have

$$J_0K(x, y)=K(x, y)\circ J, \tag{1.3}$$

or equivalentiy

$$K(x, y, z, w)=K(x, y, Jz, Jw)$$

where $K(x, y, z, w)=g(K(x, y)z, w)$.

Since, for any $a\in\mathfrak{U}(n)$, $g(a^{-1}x, a^{-1}y)=g(x, y)$ and $J\circ a^{-1}=a^{-1}\circ J$ hold, we have the following

Proposition 1.1. *$\mathfrak{K}(V)$ is an invariant subspace of $\mathfrak{R}(V)$ for $\tilde\alpha$.*

Our purpose is to study the decomposition of $\mathfrak{K}(V)$ into mutually orthogonal factors, invariant and irreducible with respect to the action of $\mathfrak{U}(n)$.

At first we observe that, being $\langle\tilde\alpha(a)R, \tilde\alpha(a)\tilde R\rangle=\langle R, \tilde R\rangle$, for all $a\in\mathfrak{R}(n)$ and $R, \tilde R\in\mathfrak{R}(V)$, the orthogonal complement of an invariant subspace of

$\mathfrak{R}(V)$ is also invariant and the standard representation of $\mathfrak{U}(n)$ in $\mathfrak{R}(V)$ is completely reducible.

Moreover the properties of $K\in\mathfrak{K}(V)$ imply that there exist only two possible independent traces associated with K. One of these, is the Ricci tensor of type (0, 2)

$$\rho(K)(x, y)=\sum_{i=1}^{2n} K(e_i, x, y, e_i)\quad x, y\in V, \tag{1.4}$$

and the second, one $\gamma(K)$, is defined by

$$\gamma(K)(x, y)=\sum_{i=1}^{2n} K(e_i, x, e_i, y)\quad x, y\in V \tag{1.5}$$

where $\{e_i\}$ is an arbitrary orthonormal basis of V.

ρ and γ are $\mathfrak{U}(n)$ – concomitants [8]. Generally, they are neither symmetric nor antisymmetric, but, after some computations, we can prove that we have for γ:

$$\gamma(K)(x, y)=\gamma(K)(Jx, Jy). \tag{1.6}$$

Finally, to do our decomposition, we shall need the scalar curvature $\tau(K)$ of K, i.e.

$$\tau(K)=\sum_{i=1}^{2n}\rho(K)(e_i, e_i),$$

and also the function $\tilde\tau$ defined by the equation

$$\tilde\tau(K)=\sum_{i=1}^{2n}\rho(K)(Je_i, e_i).$$

2. Invariant curvature tensors. Let $K\in\mathfrak{K}(V)$ be a curvature tensor invariant under the action of $\mathfrak{U}(n)$: that means that $\tilde\alpha(a)K=K$ for every $a\in\mathfrak{U}(n)$. Then K determines a 1-dimensional invariant and irreducible subspace of $\mathfrak{K}(V)$.

It is known that, for $n\geq 2$, the invariant tensors of type (0, 4) on V form a vector space with as a basis the 12 tensors $F_{\alpha_1\alpha_2}$ defined as follows ([10], [25])

$$F_{\alpha_1\alpha_2}(x, y, z, w)=\begin{cases}\Omega_{\alpha_1}(x, y)\Omega_{\alpha_2}(z, w)\\ \Omega_{\alpha_1}(x, z)\Omega_{\alpha_2}(y, w)\\ \Omega_{\alpha_1}(x, w)\Omega_{\alpha_2}(y, z)\end{cases} \tag{2.1}$$

where $\alpha_1, \alpha_2=0, 1$ and $\Omega_0(x, y)=g(x, y)$, $\Omega_1(x, y)=g(x, Jy)$. Imposing that these tensors belong to $\mathfrak{K}(V)$, we obtain:

Theorem 2.1. *The set of the invariant curvature tensors of $\mathfrak{K}(V)$ forms 2-dimensional subspace of $\mathfrak{K}(V)$ which has as basis the tensors π_1 and π_2 defined by the equations*

$$\pi_1(x, y)z=g(x, z)y-g(y, z)x+g(Jx, z)Jy-g(Jy, z)Jx+2g(Jx, y)Jz. \tag{2.2}$$

$$\pi_2(x, y)z=g(Jx, z)y-g(Jy, z)x+2g(Jx, y)z-g(x, z)Jy+g(y, z)Jx \tag{2.3}$$

Moreover, the spaces $W_5=\mathfrak{L}(\pi_1)$ and $W_6=\mathfrak{L}(\pi_2)$ generated by π_1 and π_2 respectively are mutually orthogonal.

Proof: The proof follows from the above results and from

(2.4) $$\langle K,\pi_1\rangle = -8\tau(K), \quad \langle K,\pi_2\rangle = 8\bar{\tau}(K), \quad K\in\mathcal{K}(V)$$

since $\tau(\pi_2)=0$ and $\bar{\tau}(\pi_1)=0$.

So, we obtain also

Proposition 2.2. *The orthogonal complement of* $W_5\oplus W_6$ *in* $\mathcal{K}(V)$ *consists of the tensors* $K\in\mathcal{K}(V)$ *such that* $\tau(K)=\bar{\tau}(K)=0$. *We shall denote that space with* W_0.

Remark 2.3. A curvature tensor $K\in\mathcal{K}(V)$ has constant holomorphic sectional curvature $4c$ if $K=c\Pi_1$, $c\in\mathcal{R}$, and then if $K\in W_5$.

Remark 2.4. For $n=1$, it is weasy to see that dim $\mathcal{K}(V)=2$ and the orthogonal complement of $W_5\oplus W_6$ is equal to $\{0\}$. Therefore, from this point on, we shall always consider $n\geq 2$.

3. The complete decomposition of $\mathcal{K}(V)$. In the present section we shall define all the components of the decomposition of $\mathcal{K}(V)$ and we shall give the most important theorems of our work which will be proved in the next sections.

Definition 3.1. *Let* Φ *be the operator defined by*

(3.1) $$\Phi(K)(x, y, z, w)=K(Jx, Jy, z, w) \quad K\in\mathcal{K}(V).$$

We easily obtain the

Proposition 3.1. *Let* K *be a curvature tensor of* $\mathcal{K}(V)$. *Then* $\Phi(K)\in\mathcal{K}(V)$ *and* $\Phi^2(K)=K$. *Moreover,* Φ *is a* $\mathfrak{U}(n)$*—concomitant and it is also an isometry with respect to the product of* $\mathcal{K}(V)$.

Definition 3.2. *Let* $\mathcal{K}^+$ and $\mathcal{K}^-$ *be the vector subspaces of* $\mathcal{K}(V)$ *given by*

(3.2) $$\mathcal{K}^+=\{K\in\mathcal{K}(V)|\Phi(K)=K\}, \quad \mathcal{K}^-=\{K\in\mathcal{K}(V)|\Phi(K)=-K\}.$$

We can state the following

Theorem 3.2. $\mathcal{K}^+$ *and* $\mathcal{K}^-$ *are* $\mathfrak{U}(n)$*—invariant, orthogonal subspaces of* $\mathcal{K}(V)$ *and* $\mathcal{K}(V)=\mathcal{K}^+\oplus\mathcal{K}^-$.

Proof: $\mathcal{K}^+$ and $\mathcal{K}^-$ are the orthogonal invariant eigenspaces corresponding to the eigenvalues $+1$ and -1 of Φ respectively.

Hence, every $K\in\mathcal{K}(V)$ can be decomposed uniquely and orthogonally as follows:

(3.3) $$K=\frac{1}{2}(K+\Phi(K))+\frac{1}{2}(K-\Phi(K)),$$

where $1/2(K+\Phi(K))\in\mathcal{K}^+$ and $1/2(K-\Phi(K))\in\mathcal{K}^-$. To define all the components of the decomposition we need to consider some other subspaces of $\mathcal{K}(V)$.

Definition 3.3.

$\mathcal{K}_H=\{K\in\mathcal{K}(V)|\rho(K)=0\}$,

$\mathcal{K}_0=\{K\in\mathcal{K}(V)|\rho(K)=\gamma(K)=0\}$.

$\mathcal{K}^s_\rho=\{K\in W_0|\rho(K)\neq 0$ *and* $\rho(K)(x, y)=\rho(K)(y, x)\}$,

$\mathcal{K}^a_\rho=\{K\in W_0|\rho(K)\neq 0$ *and* $\rho(K)(x, y)=-\rho(K)(y, x)\}$,

$\mathcal{K}^s_\gamma=\{K\in\mathcal{K}(V)|\gamma(K)(x, y)=\gamma(K)(y, x)\}$,

$\mathcal{K}^a_\gamma=\{K\in\mathcal{K}(V)|\gamma(K)(x, y)=-\gamma(K)(y, x)\}$.

Definition 3.4. *We put*

$W_9=\{K\in\mathcal{K}^+\cap\mathcal{K}_0|K(x, y, z, w)=-K(x, y, w, z)\}$,

$W_{10}=\{K\in\mathcal{K}^+\cap\mathcal{K}_0|K(x, y, z, w)=K(x, y, w, z)\}$,

W_{11}=*orthogonal complement of* $W_9\oplus W_{10}$ *in* $\mathcal{K}^+\cap\mathcal{K}_0$,

$W_1=\mathcal{K}^+\cap\mathcal{K}^s_\rho$,

$W_3=\mathcal{K}^+\cap\mathcal{K}^a_\rho$,

$W_5=\mathfrak{L}(\pi_1)$,

$W_6=\mathfrak{L}(\pi_2)$,

W_7=*orthogonal complement of* $\mathcal{K}_0$ *in* $\mathcal{K}_H\cap\mathcal{K}^s_\gamma$,

W_8= *orthogonal complement of* $\mathcal{K}_0$ *in* $\mathcal{K}_H\cap\mathcal{K}^a_\gamma$,

$W_{12}=\mathcal{K}^-\cap\mathcal{K}_H$

$W_2=\mathcal{K}^-\cap\mathcal{K}^s_\rho$=*orthogonal complement of* W_1 *in* $\mathcal{K}^s_\rho$,

$W_4=\mathcal{K}^-\cap\mathcal{K}^a_\rho$=*orthogonal complement of* W_3 *in* $\mathcal{K}^a_\rho$.

Thus we obtain

Theorem 3.3. *If* $dim V=2n$, $n\geq 3$, *then*

(3.4) $$\mathcal{K}(V)=W_1\oplus\ldots\oplus W_{12};$$

if $n=2$, $W_{11}=W_{12}=\{0\}$ *and*

(3.5) $$\mathcal{K}(V)=W_1\oplus\ldots\oplus W_{10};$$

These subspaces are mutually orthogonal and invariant under the action of $U(n)$.

Recalling that an invariant subspace is irreducible if it does not contain a non trivial invariant subspace, we have also

Theorem 3.4. *The decomposition of* $\mathcal{K}(V)$ *given above is irreducible under the action* $\mathfrak{U}(n)$.

Remark 3.5. The decomposition (3.4) is not unique (for another decomposition see [19].

4. Algebraic characterization of the subspaces W_i, $i=1, 2, 3, 4$. Let V^* denote the dual space of V; for any $S\in\overset{2}{\otimes}V^*=V^*\otimes V^*$, we can consider the following two mappings from $\overset{2}{\otimes}V^*$ in $\overset{4}{\otimes}V^*$:

(4.1) $$\lambda(S)(x, y, z, w)=S(x, z)g(y, w)-S(y, z)g(x, w)+ \\ +S(x, y)g(z, w)-S(y, x)g(z, w)$$

(4.2) $$\mu(S)(x, y, z, w)=S(Jx, z)g(Jy, w)-S(Jy, z)g(Jx, w)+ \\ +S(Jx, y)g(Jz, w)-S(Jy, x)g(Jz, w).$$

λ and μ are $\mathfrak{U}(n)$—concomitants.

It is clear that $\lambda(S)$ and $\mu(S)$ do not belong to $\mathcal{K}(V)$, but we can prove

Theorem 4.1.

a) $(\lambda+\mu)(S)\in\mathcal{K}(V)$ *iff* $S(x, y)=S(Jx, Jy)$, *for all* x, $y\in V$;

b) $(\lambda-\mu)(S)\in\mathcal{K}(V)$ *iff* $S(x, y)=-S(Jx, Jy)$, *for all* x, $y\in V$.

Proof. a) It is easy to prove that $(\lambda+\mu)(S)(x, y, z, w)$ is antisymmetric with respect to x, y and satisfies the first Bianchi identity:

$$\underset{xyz}{\sigma}(\lambda+\mu)(S)(x, y, z, w)=0.$$

In order to have $(\lambda+\mu)(S)\in\mathfrak{K}(V)$, we also need :

(4.3) $\qquad (\lambda+\mu)(S)(x, y, z, w)=(\lambda+\mu)(S)(x, y, Jz, x, y, z, w\in V.$

Let now x, y, Jw be orthogonal unit vectors $(n\geq 2)$. Then (4.3) becomes

(4.4) $\qquad (S(x, z)-S(Jx, Jz))g(y, w)-(S(y, z)-S(Jy, Jz))g(x, w)=0;$

we see at once that (4.4) is satisfied if $S(x, z)=S(Jx, Jz)$. To prove the converse, choosing in particular $y=w$ in (4.4), we obtain $S(x, z)=S(Jx, Jz)$ for all $x, z\in V$.

b) can be proved in a similar way.

Definition 4.1. *Consider the following subspaces of* $\overset{2}{\otimes}V^*$ *(see also [25])*

$$V_1=\{S\in\overset{2}{\otimes}{}^*\,|\,S(x, y)=S(y, x),\ S(Jx, Jy)=S(x, y),\ \tau(S)=0\}.$$

$$V_2=\{S\in\overset{2}{\otimes}V^*\,|\,S(x, y)=S(y, x),\ S(Jx, Jy)=-S(x, y)\},$$

$$V_3=\{S\in\overset{2}{\otimes}V^*\,|\,S(x, y)=-S(y, x),\ S(Jx, Jy)=S(x, y),\ \bar{\tau}(S)=0\},$$

$$V_4=\{S\in\overset{2}{\otimes}V^*\,|\,S(x, y)=-S(y, x),\ S(Jx, Jy)=-S(x, y)\}.$$

where $x, y\in V$, $\tau(S)=\Sigma_{i=1}^{2n}S(e_i, e_i)$, $\bar{\tau}(S)=\Sigma_{i=1}^{2n}S(Je_i, e_i)$ *with* $\{e_i\}$, $i=1, 2, \ldots, 2n$, *an arbitrary orthonormal basis on* V.

It is well known that these subspaces are nutually orthogonal with respect to the inner product on $\overset{2}{\otimes}V^*$ given by ; $S\cdot\tilde{S}=\Sigma_{i,j=1}^{2n}S_{ij}\tilde{S}_{ij}$; we shall also prove in Section 6 that V_i, $i=1, 2, 3, 4$ is an irreducible space of $\overset{2}{\otimes}V^*$.

Definition 4.1. *We can define now the projections* P_i, *from* $\overset{2}{\otimes}V^*$ *in* V_i, $i=1, 2, 3, 4$:

$$P_1:\overset{2}{\otimes}V^*\to V_1,$$

$$P_1(S)(x, y)=\frac{1}{2}\{\mathcal{S}(S)(x, y)+\mathcal{S}(S)(Jx, Jy)\}-\frac{1}{2n}\tau(S)g(x, y);$$

$$P_2:\overset{2}{\otimes}V^*\to V_2,$$

$$P_2(S)(x, y)=\frac{1}{2}\{\mathcal{S}(S)(x, y)-\mathcal{S}(S)Jx, Jy)\};$$

$$P_3:\overset{2}{\otimes}V^*\to V_3,$$

$$P_3(S)(x, y)=\frac{1}{2}\mathcal{A}\{(S)(x, y)+\mathcal{A}(S)(Jx, Jy)\}+\frac{1}{2n}\bar{\tau}(S)g(Jx, y);$$

$$P_4:\overset{2}{\otimes}V^*\to V_4,$$

$$P_4(S)(x, y)=\frac{1}{2}\{A(S)(x, y)-A(S)(Jx, Jy)\}$$

where

(4.5) $\quad\left\{\begin{array}{l}\mathcal{S}:\overset{2}{\otimes}V^*\to\overset{2}{\otimes}V^*\\ S(x, y)\to\frac{1}{2}(S(x, y)+S(y, x))\end{array}\right.\qquad\left\{\begin{array}{l}\mathcal{A}:\overset{2}{\otimes}V^*\to\overset{2}{\otimes}V^*\\ S(x, y)\to\frac{1}{2}(S(x, y)-S(y, x))\end{array}\right.$

Theorem 4.2. *We have :*

i) $W_1=(\lambda+\mu)(V_1)$; ii) $W_2=(\lambda-\mu)(V_2)$; iii) $W_3=(\lambda+\mu)(V_3)$.

iv) $W_4=(\lambda-\mu)(V_4)$.

Moreover the subspaces W_i *are mutually orthogonal and invariant with respect to the action of* $\mathfrak{U}(n)$ *[4].*

The proof of this theorem partially derives from the following

Lemma 4.3.

a) If $K=(\lambda+\mu)(S)$ *with* $S\in V_1$, *we have* $\rho(K)=-2(n+1)(S)$, $\gamma(K)=-2S$, $\tau(K)=\bar{\tau}(K)=0$;

b) If $K=(\lambda-\mu)(S)$ *with* $S\in V_2$, *we have* $\rho(K)=-2(n-1)(S)$ $\gamma(K)=0$, $\tau(K)=\bar{\tau}(K)=0$;

c) If $K=(\lambda+\mu)(S)$ *with* $S\in V_3$, *we have* $\rho(K)=-2(n+1)(S)$, $\gamma(K)=-2S$, $\tau(K)=\bar{\tau}(K)=0$;

d) If $K=(\lambda-\mu)(S)$ *with* $S\in V_4$, *we have* $\rho(K)=-2(n+1)(S)$, $\gamma(K)=0$, $\tau(K)=\bar{\tau}(K)=0$.

Proof. From a straightforward computation.

Proof of Theorem 4.2. The definitions of λ, μ and V_i, $i=1, 2, 3, 4$, and Lemma 4.3 imply that $(\lambda+\mu)(V_1)\subset\mathfrak{K}^+\cap\mathfrak{K}^s_\rho$. On the other hand, it is easy to see that for $K\in\mathfrak{K}^s_\rho$ and $S\in V_1$ we have

$$\langle K, (\lambda+\mu)(S)\rangle=0$$

iff $K\in\mathfrak{K}^-\cap\mathfrak{K}^s_\rho$; so, being $\mathfrak{K}^s_\rho=(\mathfrak{K}^+\cap\mathfrak{K}^s_\rho)\oplus(K^-\cap\mathfrak{K}^s_\rho)$ (from Theorem 3.2), we obtain i). In the same way, we can prove ii), iii), iv). The proof of the orthogonality of W_i can be found by a straightforward computation (see also [4]). Finally W_i are invariant as images of invariant subspaces of $\overset{2}{\otimes}V^*$ by $\mathfrak{U}(n)$— concomitants.

Remark 4.4. We have : $\pi_1=(\lambda+\mu)(g)$ and $\pi_2=(\lambda+\mu)(\Omega)$, where $\Omega(x, y)=g(Jx, y)$.

Furthermore

$$\rho(\pi_1)=-2(n+1)g,\ \rho(\pi_2)=-2(n+1)\Omega,\ \gamma(\pi_1)=2(n+1)g,\ \gamma(\pi_2)=-2(n+1)\Omega,$$

$$\tau(\pi_1)=-4n(n+1),\quad \tau(\pi_2)=0,\quad \bar{\tau}(\pi_1)=0,\quad \bar{\tau}(\pi_2)=4n(n+1).$$

5. The decomposition of K_H.

Definition 5.1. *Let* $S\in\overset{2}{\otimes}V^*$. *Then we define the* $\mathfrak{U}(n)$— *concomitant* ϑ *from* $\overset{2}{\otimes}V^*$ *in* $\overset{4}{\otimes}V^*$ *given by :*

$$\vartheta(S)(x, y, z, w)=S(x, w)g(y, z)-S(y, w)g(x, z)+S(x, Jw)g(y, Jz)-$$
$$-S(y, Jw)g(x, Jz)-2S(z, Jw)g(x, Jy)$$

for all $x, y, z, w\in V$.

It holds the

Propositon 5.1. $\vartheta(S)\in\mathcal{R}(V)$ iff $S(Jx, Jy)=S(x, y)$ *for all* $x, y\in V$.

If we shall call ν and ζ the mappings from $\overset{2}{\otimes} V^*$ in $\mathcal{R}(V)$ defined by the equations

$$\nu(S)=(\lambda+\mu)(S)+(n+1)\vartheta(S), \tag{5.1}$$

$$\zeta(S)=(\lambda+\mu)(S)-(n+1)\vartheta(S) \tag{5.2}$$

for all $S\in V_1$ or $S\in V_3$ we obtain

Theorem 5.2. $W_7=\nu(V_1)$, $W_8=\zeta(V_3)$ *and they are mutually orthogonal, $\mathfrak{U}(n)$-invariant subspaces of* $\mathcal{R}(V)$.

To prove this theorem, we need the following

Lemma 5.3. a) *If* $K=\nu(S)$ *with* $S\in V_1$, *we have*

$$\rho(K)=0, \quad \gamma(K)=-2n(n+2)S, \quad \tau(K)=\bar{\tau}(K)=0\;;$$

b) *if* $K=\zeta(S)$ *with* $S\in V_3$, *we have*

$$\rho(K)=0, \quad \gamma(K)=2n(n+2)S, \quad \tau(K)=\bar{\tau}(K)=0.$$

Proof. The lemma is obtained from a direct computation.

Proof of Theorem 5.2. From the previous lemma we have: $\nu(V_1)\subset \mathcal{R}_H\cap\mathcal{R}_\gamma^\bullet$. Now, suppose $\tilde{\mathcal{R}}\in\mathcal{R}_H\cap\mathcal{R}_\gamma^\bullet$ orthogonal to $K=\nu(S)$ with $S\in V_1$. It follows

$$0=\langle K, \tilde{K}\rangle=-8(n+1)\sum_{j,k}\gamma(\tilde{K})_{jk}S_{jk}$$

which implies $\gamma(\tilde{K})=0$ (from (1.6). Then each tensor $\mathcal{R}\in\mathcal{R}_H\cap K_\gamma^\bullet$ orthogonal to K must belong also to $\mathcal{R}_0$, and from Definition 3.4, we see that $W_7=\nu(V_1)$. In a similar way we can prove that $W_8=\zeta(V_3)$, while the orthogonality of W_7 and W_8 derives from a straightforward computation. Finally, W_7 and W_8 are invariant, as ν, ζ are $\mathfrak{U}(n)$—concomitants.

In order to complete our work, we have still to decompose the space

$$\mathcal{R}_0=\{K\in\mathcal{R}(V)\,|\,\rho(K)=\gamma(K)=0\}.$$

With that aim in mind, we remark first of all that

$$\mathcal{R}_0=(\mathcal{R}^+\cap\mathcal{R}_0)\cup(\mathcal{R}^-\cap\mathcal{R}_0)$$

(Theorem 3.2); then we define in $\mathcal{R}^+\cap\mathcal{R}_0$ the following endomorphism [6]:

$$\chi(K)(x, y, z, w)=\frac{1}{4}\{K(x, y, z, w)+K(y, x, w, z)+ \tag{5.3}$$
$$+K(z, w, x, y)+K(w, z, y, x)\}.$$

Proposition 5.4. *For all* $K, \tilde{K}\in\mathcal{R}^+\cap\mathcal{R}_0$ *we have:*

(1) $\chi(K)\in K^+\cap K_0$ *i.e.:*

$$\chi(K)(x, y, z, w)=-\chi(K)(y, x, z, w),$$
$$\chi(K)(x, y, z, w)=\chi(K)(x, y, Jz, Jw)=\chi(K)(Jx, Jy, z, w),$$
$$\underset{xyz}{\sigma}\chi(K)(x, y, z, w)=0, \quad \rho(\chi(K))=0, \quad \gamma(\chi(K))=0\;;$$

(2) $$\chi(K)(x, y, z, w)=-\chi(K)(x, y, w, z)\;;$$

3) $\chi^2=\chi$;

4) χ *commutes with the action of* $\mathfrak{U}(n)$;

5) $\langle\chi(\tilde{K}), K\rangle=\langle\tilde{K}, \chi(K)\rangle$

and $\mathcal{R}^+\cap\mathcal{R}_0=\mathrm{Im}\chi\oplus Ker\ \chi$.

In order to decompose Ker χ, we introduce in that subspace the endomorphism ψ given by

$$\psi(K)(x, y, z, w)=\frac{1}{4}\{K(x, y, z, w)+K(x, y, w, z)- \tag{5.4.}$$
$$-K(Jz, w, Jx, y)+K(Jz, w, Jy, x)\}$$

From that formula, we can directly prove the following

Proposition 5.5. *Let* K, $\tilde{K}\in$ Ker χ. *Then*

1) $\psi(K)\in$ Ker χ.

2) $\psi(K)(x, y, z, w)=\psi(K)(x, y, w, z)$.

3) $\psi^2=\psi$.

4) ψ *commutes with the action of* $\mathfrak{U}(n)$.

5) $\langle\psi(K), \tilde{K}\rangle=(K, \psi(\tilde{K}))$

and, as a consequence, Ker $\chi=Im\psi\oplus$Ker ψ.

Finally, let's remark that $\mathcal{R}\in\mathcal{R}^-\cap \boldsymbol{R}_H$, $\boldsymbol{R}\neq 0$, cannot satisfy neither of the following properties

$$\boldsymbol{K}(x, y, z, w)=\boldsymbol{K}(x, y, w, z), \quad K(x, y, z, w)=-\boldsymbol{K}(x, y, w, z).$$

Thus, if we put $W_9=Im\chi$, $W_{10}=\mathrm{Im}\psi$, $W_{11}=\boldsymbol{K}\mathrm{er}\psi$ and $W_{12}=\mathcal{R}^-\cap\mathcal{R}_H$, we can conclude

Theorem 5.6. *If* $\dim V=2n$, $n>2$, *then*

$$\mathcal{R}_H=W_7\oplus\ldots.\oplus W_{12}$$

and, if $n=2$

$$\mathcal{R}_H=W_7\oplus\ldots\oplus W_{10}.$$

These subspaces are mutually orthogonal and invariant under the action of $\mathfrak{U}(n)$.

Proof,: The theorem follows from the previous results of this section. For $n=2$, the proof shall be completed in Section 7.

6. Quadratic invariants and irreducibility of the decomposition. It is known that an invariant subspace of $\overset{r}{\otimes} V^*$ is irreducible for the action of

$\mathfrak{A}(n)$ iff the space of its quadratic invariants is 1-dimensional ([2], [9]). All the quadratic invariants of a subsace of $\overset{r}{\otimes} V^*$ are restrictions of the quadratic invariants of $\overset{r}{\otimes} V^*$, which are given by

$$P(N)=\Sigma N(e_{i_1}, \dots, e_{i_r})N(e_{j_1}, \dots e_{j_r})p(e_{i_1}, \dots, e_{i_r}, e_{j_1}, \dots, e_{j_r}) \tag{6.1}$$

where $N\in\overset{r}{\otimes} V^*$ and p is an invariant tensor of $\overset{2r}{\otimes} V^*$. Iwahori's results in [10] allows us to compute the invariant tensors $p\in\overset{2r}{\otimes} V^*$. We already saw in Sect. 2 that every invariant tensor $p\in\overset{4}{\otimes} V^*$ is a linear combination of the elements $F_{\alpha_1\alpha_2}$ defined by (2.1). It is easy to see, after a direct computation, that the restrictions of the quadratic invariants of $\overset{2}{\otimes} V^*$ to V_i, $i=1, 2, 3, 4$, always are scalar multiples of $\|S\|^2=\Sigma_{i,j}S^2(e_i, e_j)$.

Lemma 6.1. *The subspaces V_i of $\overset{2}{\otimes} V^*$ are irreducible (see [25]).*

Proof. This follows from the above statements, taking into account that the dimensions of V_i are always different from zero for $n\geq 2$. In fact we have

$$\dim V_1=n^2-1,\quad \dim V_2=n(n+1),\quad \dim V_3=n^2-1,\quad \dim V_4= \tag{6.2}$$
$$=n(n-1).$$

Since the spaces $W_{i,i}=1, 2, 3, 4, 7, 8$ are images of the spaces V_i by $\mathfrak{A}(n)$—concomitants, the following result holds

Corollary 6.2. *The subspaces W_i, $i=1, \dots 8$, $i\neq 5, 6$ of $\mathfrak{K}(V)$ are irreducible.*

Proof. It is known that, if $\mathcal{G}$ is a group which acts on two vector spaces $\mathfrak{V}$ and $\mathfrak{S}$ every $\mathcal{G}$—concomitant between the two spaces takes irreducible subspaces of $\mathfrak{V}$ into irreducible subspaces of $\mathfrak{S}$. Moreover, in that case, the restriction of the $\mathcal{G}$— concomitant to these subspaces is an isomorphism [8] [25].

So, using 6.1 and the results of [10], from the symmetries of the tensors of $\mathfrak{K}(V)$ we obtain the following

Theorem 6.3. *The vector space of all quadratic invariants of $K\in\mathfrak{K}(V)$ is spanned by the following 20 invariants:*

$$\|R\|^2,\ \Sigma R(e_i, e_j, e_k, e_l)K(Je_i, Je_j, e_k, e_l),\ \Sigma K(e_i, e_j, e_k, e_l)K(e_i, e_j, e_l, e_k),$$
$$\Sigma K(e_i, e_j, e_k, e_l)K(e_i, e_j, e_l, Je_k),\ \Sigma K(e_i, e_j, e_k, e_l)K(e_k, e_l, e_i, e_j),\ \|\rho(K)\|^2,$$
$$\Sigma K(e_i, e_j, e_k, e_l)K(e_l, e_k, e_j, e_i).\ \Sigma K(e_i, e_j, e_k, e_l)K(e_l, e_k, Je_j, e_i),\ \Sigma K(e_i, e_j, e_k, e_l)$$
$$K(e_l, Je_j, Je_k, e_i),\Sigma K(e_i, e_j, e_k, e_l)K(e_l, Je_k, Je_j, e_i),\ \|\gamma(K)\|^2,\ \Sigma K(e_i, e_j, e_l, e_k)$$
$$K(e_l, e_k, e_l, e_j),\ \Sigma K(e_i, e_k, e_l, e_j)K(e_l, e_j, e_l, Je_k),\ \ \Sigma K(e_i, e_j, e_k, e_l)\, K(e_l, e_j, e_l, e_k),$$
$$\Sigma K(e_i, e_j, e_k, e_l)K(e_l, e_k, e_l, e_j),\ \Sigma K(e_i, e_j, e_k, e_l)K(e_l, Je_j, e_l, e_k),\ \Sigma K(e_i, e_j, e_k, e_l)$$
$$K(e_l,\ Je_k,\ e_l,\ e_j),\ (\tau(K))^2,\ (\bar{\tau}(K))^2,\ \ \tau(K)\bar{\tau}(K).$$

The last theorem implies

Corollary 6.4. *1) W_9, W_{10} are irreducible. 2) W_{11} and W_{12} are irreducible whenever $n>2$ and $W_{12}=W_{11}=\{0\}$ for $n=2$.*

Proof: From the properties which characterize the spaces W_i, $i=9, \dots 12$, the restrictions of all quadratic invariants to these subspaces are scalar multiples of $\|K\|^2$; this proves the corollary for $n>2$. We shall prove that $W_{11}=W_{12}=\{0\}$ when $n=2$ in the next section computing their dimensions.

7. Dimensions of the factors

Theorem 7.1. *The dimensions of the factors W_i, $i=1, 2, \dots, 12$ are given by the following table.*

dim V	4	$2n\geq 6$
W_1	3	n^2-1
W_2	6	$n(n+1)$
W_3	3	n^2-1
W_4	2	$n(n-1)$
$W_5=\mathfrak{L}(\pi_1)$	1	1
$W_6=\mathfrak{L}(\pi_2)$	1	1
W_7	3	n^2-1
W_8	3	n^2-1
W_9	5	$(1/4)n^2(n-1)(n+3)$
W_{10}	5	$(1/4)n^2(n-1)(n+3)$
W_{11}	0	$(1/2)(n-1)(n+1)(n-2)(n+2)$
W_{12}	0	$(2/3)n^2(n-2)(n+2)$

In particular,

$$\dim\mathfrak{K}(V)=\frac{n^2(n+1)(5n-2)}{3}.$$

From the definition of W_i, $i=1, \dots, 8$, $i\neq 5, 6$, the dimensions of these spaces are obtained directly from the dimensions of V_i (see Lemma 6.1 and Corollary 6.2). In order to compute the dimensions of the remaining spaces we need some further lemmas (see also [22]), [25]. Let $V^C=V\otimes C$; let us still denote as J the extension of the endomorphism J of V to V^C; and as V^+ and V^- the two eigenspaces which correspond to the eigenvalues $+i$ and $-i$ respectively. Given $K\in\mathfrak{K}(V)$, let K^C be the extension of K to V^C. K^C is a C— linear map in all its arguments and is also real:

$$K^C(\bar{x}, \bar{y}, \bar{z}, \bar{w})=\overline{K^C(x, y, z, w)}.$$

Then we have:

Lemma 7.2. $\mathfrak{K}^-$ *is isomorphic to the space of the C—multilinear mappings $F: V^+\times V^+\times V^+\times V^-\to C$ which satisfy the following relations*

$$F(x, y, z, \bar{w})=-F(y, x, z, \bar{w}), \tag{7.1}$$

$$\underset{xyz}{\sigma} F(x, y, z, w)=0 \qquad x, y, z, w\in V^C. \tag{7.2}$$

Proof. From the definition of the spaces $\mathfrak{K}(V)$ and $\mathfrak{K}^-$, $K^C(x, y, z, w)=0$ if $x\in V^+$ and $y\in V^-$ (or vice-versa) and if z and w both belong to V^+ (or to V^-).

On the other hand, from the symmetry properties of $K \in \mathcal{K}(V)$, which hold also for K^C, we obtain as only nonvanishing terms, those of type $K^C(x, y, z, \bar{w})$ and the ones obtained from these conjugation. The restriction of K^C to $V^+ \times V^+ \times \times V^+ \times V^-$ finally gives the required isomorphism.

Lemma 7.3. *The subspaces of* $\mathcal{K}^+$

$$A = \{K \in \mathcal{K}^+ | K(x, y, z, w) = -K(x, y, w, z)\},\ B = \{K \in \mathcal{K}^+ | K(x, y, z, w) = K(x, y, w, z)\}$$

are both isomorphic to the space of the C*-multilinear maps:*

$$F : V^+ \times V^- \times V^+ \times V^- \to C$$

such that

$$F(x, \bar{y}, z, \bar{w}) = F(z, \bar{y}, x, \bar{w}), \tag{7.3}$$

$$F(x, \bar{y}, z, \bar{w}) = F(x, \bar{w}, z, \bar{y}). \tag{7.4}$$

Proof. The isomorphism between A and the space of the maps F on $V^+ \times V^- \times V^+ \times V^-$ satisfying (7.3) and (7.4) was constructed by Sitaramayya in [22]. On the other hand, the mapping

$$M : A \to B,\ K(x, y, z, w) \to M(K)(x, y, z, w) = K(x, y, Jz, \bar{w})$$

is an isomorphism between A and B.

Proof of Theorem 7.1. Since W_{12} consists of the tensors K of for which $\rho(K) = 0$, Lemma 7.2 implies

$$\dim W_{12} = \frac{2}{3} n^2 (n-2)(n+2) \tag{7.5}$$

In fact there are exactly $n^2\binom{n}{3}C$-multilinear mappings which satisfy (7.1); the Bianchi identity and the condition $\rho(K) = 0$ give instead $n\binom{n}{3}$ and n^2 independent relations respectively. In particular we have

$$\dim \mathcal{K}^- = 2\left[n^2\binom{n}{2} - n\binom{n}{3}\right] = \frac{2}{3} n^2 (n-1)(n+1). \tag{7.6}$$

Using Lemma 7.3, we obtain directly [22]

$$\dim W_9 = \dim W_{10} = \frac{1}{4} n^2 (n-1)(n+3). \tag{77.}$$

Sitaramayya also proved that to every $K \in \mathcal{K}^+$ coresponds a tensor K^C of type $K^C(x, \bar{y}, z, \bar{w})$, $x, z \in V^+$, $\bar{y}, \bar{w} \in V^-$ or $K^C(x, y, z, w)$ (which is obtained by conjugation from the first one) such that

$$K^C(x, \bar{y}, z, \bar{w}) = K^C(z, \bar{y}, x, \bar{w}). \tag{7.8}$$

So, we have

$$\dim \mathcal{K}^+ = 2n^2 \binom{n+1}{2} = n^3 (n+1), \tag{7.9}$$

$$\dim K(V) = \frac{2}{3} n^2 (n-1)(n+1) + n^3 (n+1) = \frac{n^2 (n+1)(5n-2)}{3} \tag{7.10}$$

Since in $\mathcal{K}^+$ there are $n^3(n+1) - 2n^2 - 2(n^2+1)$ tensors such that $\rho(K) = \gamma(K) = 0$, we finally have

$$\dim W_{11} = \frac{1}{2}(n-1)(n+1)(n-2)(n+2). \tag{7.11}$$

8. Projections.

Theorem 8.1. *Given* $K \in \mathcal{K}(V)$ *and* $\dim V \geq 6$, *let us denote as* Q_i *the projections of* K *on the irreducible spaces* W_i, $i = 1, 2, \ldots, 11, 12$. *Then* Q_i *are described by the following equations*

$$Q_1(K) = -\frac{1}{2(n+1)}(\lambda + \mu)(P_1(\rho(K)), \quad Q_2(K) = -\frac{1}{2(n-1)}(\lambda - \mu)(P_2(\rho(K)),$$

$$Q_3(K) = -\frac{1}{2(n+1)}(\lambda + \mu)(P_3(\rho(K)), \quad Q_4(K) = -\frac{1}{2(n+1)}(\lambda - \mu)(P_4(\rho(K)),$$

$$Q_5(K) = -\frac{\tau(K)}{4n(n+1)}\pi_1, \quad Q_6(K) = \frac{\tilde{\tau}(K)}{4nn(+1)}\pi_2,$$

$$Q_7(K) = -\frac{1}{2n(n+2)}\nu\left\{P_1(\gamma(K)) + \frac{1}{n+1}P_1(\rho(K))\right\},$$

$$Q_8(K) = \frac{1}{2n(n+2)}\zeta\left\{P_3(\gamma(K)) - \frac{1}{n+1}P_3(\rho(K))\right\},$$

$$Q_9(K) = \frac{1}{2}\chi \circ (I + \Phi)(K_0), \quad Q_{10}(K) = \frac{1}{2}\psi \circ (I - \chi) \circ (I + \Phi)(K_0),$$

$$Q_{11}(K) = \frac{1}{2}(I - \psi) \circ (I - \chi) \circ (I + \Phi)(K_0), \quad Q_{12}(K) = \frac{1}{2}(I - \Phi)(K_0),$$

where I *is the identity transformation on* $K(V)$ *and* $K_0 = K - \Sigma_{i=1}^{8} Q_i(K)$.

Proof. We shall compute, for example $Q_5(K)$, $Q_1(K)$ and $Q_7(K)$. Supposing $K \in W_5 \oplus W_1 \oplus W_7$, from the definition of these spaces we have:

$$Q_5(K) = c\pi_1, \quad Q_1(K) = (\lambda + \mu)(S), Q_7(K) = \nu(\bar{S})$$

where $c \in \mathbf{R}$, S, $\tilde{S} \in V_1$ and $\tau(Q_1(K)) = \tau(Q_7(K)) = 0$ (see Theorem 4.2, Lemma 4.3 and Lemma 5.3). Then

$$\tau(K) = c\tau(\pi_1) = -4n(n+1)c$$

which implies

$$Q_5(K) = -\frac{\tau(K)}{4n(n+1)}\pi_1.$$

Now, using also Remark 4.4 and Definition 4.2, we have

$$\rho(K) = \rho(Q_5(K)) + \rho(Q_1(K)) = \frac{\tau(K)}{2n} g - 2(n+1)S \tag{8.1}$$

and

$$S = -\frac{1}{2(n+1)} P_1(\rho(K))$$

(8.2)
$$\gamma(K) = \gamma(Q_6(K)) + \gamma(Q_1(K)) + \gamma(Q_7(K)) =$$
$$= -\frac{\tau(K)}{2n} g - \frac{1}{n+1} P_1(\rho(K)) - 2n(n+2)\bar{S}$$

and

$$\bar{S} = -\frac{1}{2n(n+2)} \left\{ P_1(\gamma(K)) + \frac{1}{n+1} P_1(\rho(K)) \right\}.$$

Thus, from (8.1) and (8.2) we obtain

$$Q_1(K) = -\frac{1}{2(n+1)} (\lambda + \mu)) P_1(\rho(K)),$$

$$Q_7(K) = -\frac{1}{2n(n+2)} \nu \left\{ P_1(\gamma(K)) + \frac{1}{n+1} P_1(\rho(K)) \right\}.$$

The other projections have been obtained in a similar way.

9. Some applications to differential geometry. Let M be a $2n$-dimensional C^∞ differential manifold with an almost complex structure J and a Hermitian inner product g. Then, for all $X, Y \in X(M)$, the Lie algebra of C^∞ vector fields on M, we have

$$J^2 X = -X, \quad g(JX, JY) = g(X, Y)$$

It is known that ([27]) the existence on M of an arbitrary linear symmetric connection ∇ s.t. $\nabla J = 0$ is equivalent to the vanishing of the Nijenhuis tensor N_J defined by

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] \quad X, Y \in X(M).$$

For every $m \in M$, the tangent space $T_m M$ has a Hermitian structure given by $(J_{|m}, g_{|m})$. Now let $\mathfrak{K}(M)$ be the vector bundle on M with fibre $\mathfrak{K}(T_m M)$; the decomposition of $\mathfrak{K}(T_m M)$ gives rise to a decomposition of $\mathfrak{K}(M)$ into ortogonal subbundles with respect to the fibre metric induced by g on $\mathfrak{K}(M)$. We shall still denote the components of this decomposition by W_i, $i = 1, 2, \ldots 12$. If ∇ is an arbitrary linear torsion-free connection, the corresponding curvature tensor is a section of the vector bundle $\mathfrak{K}(M)$ and it is not difficult to check that its K_H-component in each point $m \in M$ gives the well-known holomorphical projective curvature tensor associated with ∇ (see [27], [4]); as a consequence, every subspace W_i, $i = 7, 8, \ldots, 12$, of the decomposition is projectively invariant. This problem will be studied by S. Nikčević together with some other applications of the above decomposition on Hermitian manifolds [19].

Furthermore P. Matzeu and V. Oproiu introduced in [15] a projective curvature tensor in normal almost contact manifolds. We recall that a $(2n+1)$-dimensional C^∞ differentiable manifold $\tilde{M}$ has an almost contact structure if a tensor field of type $(1, 1)\varphi$, a vector field ξ and a 1-form η such that

$$\varphi^2 = -I + \xi \otimes \eta, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0$$

exist on $\tilde{M}$. Then it is always possible [3] to consider on $\tilde{M}$ a Riemannian metric $\tilde{g}$ compatible with the structure, i.e.

$$\tilde{g}(\varphi\tilde{X}, \varphi\tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y}) - \eta(\tilde{X})\eta(\tilde{Y})$$

for all $\tilde{X}, \tilde{Y} \in X(\tilde{M})$, the C^∞ Lie algebra of vector field on $\tilde{M}$. The structure (φ, ξ, η) is normal iff the tensor N of type $(1, 2)$

$$N = N_\varphi + d\eta \otimes \xi$$

where N_φ is the Nijenhuis tensor of φ, vanishes on $\tilde{M}$. P. Matzeu proved in [13] that the normality of (φ, ξ, η) can be characterized by the existence on $\tilde{M}$ of special symmetric connection $\tilde{\nabla}$, the „adapted" connections, satisfying the following equations:

$$(\tilde{\nabla}_{\tilde{X}}\varphi)\tilde{Y} = \eta(\tilde{Y})h\tilde{X} + \frac{1}{2} d\eta(\varphi\tilde{X}, \tilde{Y})\xi, \quad (\tilde{\nabla}_{\tilde{X}}\eta)(\tilde{Y}) = \frac{1}{2} d\eta(\tilde{Y}, \tilde{X}), \quad \tilde{\nabla}_{\tilde{X}}\xi = \varphi\tilde{X},$$

where $h = -\varphi^2$ and $\tilde{X}, \tilde{Y} \in \mathfrak{X}(\tilde{M})$. Using this result, a curvature tensor W, invariant for C-projective transformation in $\tilde{M}$, has been associated to $\tilde{\nabla}$ [15]. In line with the already mentioned researches and taking into account the properties of W, considering a $(2n+1)$-dimensional real vector space V endowed with an almost contact metric structure (φ, ξ, η, g), the first author intends to study the $\mathfrak{U}(n) \times 1$-decomposition of the space of the tensors K over V which satisfy the following relations:

$$K(x, y, z, w) = -K(y, x, z, w), \quad \underset{xyz}{\sigma} K(x, y, z, w) = 0, \quad K(x, y, \varphi z, \varphi w) =$$
$$= K(x, y, z, w),$$

which implies

$$K(x, y, \xi, w) = 0 \text{ and } K(x, y, z, \xi) = 0$$

for all $x, y, z, w \in V$. Since for every $x \in V$, we have $x = hx + \eta(x)\zeta$ with $hx \in H = \operatorname{Ker}\eta$, considering the Hermitian structure $(\varphi_{|H}, g_{|H})$ of H, the decomposition studied in this paper plays also an important role in the almost contact case [7]. A detailed study of this last problem with some applications to normal almost contact manifolds will be found in [14].

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REGULAR VECTOR FIELDS AND CONNECTIONS ON COTANGENT BUNDLES

BY

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The tangent and cotangent bundles of a smooth manifold are isomorphic vector bundles when they are studied from the differential-topological point of view. However, there are some essential differences when one considers their differential geometry. The tangent bundle has a naturally defined integrable almost tangent structure and one can define the notion of a semispray (second order differential equation vector field) on it. From a semispray on the tangent bundle one can derive easily a (nonlinear) connection on this bundle. On the contrary, the cotangent bundle has a naturally defined symplectic structure but one cannot define naturally something similar to the notions of almost tangent structure and semispray.

In this paper we study some geometric properties of the cotangent bundle T^*M of the smooth manifold M, related to the notions of adapted almost tangent structure and regular vector fields. An adapted almost tangent structure on T^*M is essentialy determined by a nondegenerate M-tensor field of type (0, 2) (see [9] for the definition of M-tensor fields). From the integrability condition for an adapted almost tangent structure on T^*M we get a regular M-vector field on T^*M. This M-vector field defines naturally an M-bundle (local) diffeomorphism Φ from T^*M to the tangent bundle TM of M. Using this M-bundle diffeomorphism it is possible to transfer many results from the differential geometry of the tangent bundle to that of the cotangent bundle. A semispray on TM is transformed into a regular vector field on T^*M with the associated M-vector field defined by the components of the M-bundle diffeomorphism Φ. The natural almost tangent structure of TM is transformed into an integrable adapted almost tangent structure on T^*M. The connection derived from a semispray on TM is transformed into a connection on T^*M with many interesting geometric properties. In the case where the M-bundle diffeomorphism Φ is obtained as the (inverse of the) Legendre transformation defined by a regular Hamiltonian on T^*M i. e. a smooth function on T^*M with nondegenerate Hessian with respect to the cotangential coordinates) the regular vector field on T^*M corresponding to a semispray on TM defines a mechanical structure on T^*M. In this case there are another two symmetric connections on T^*M. If the regular vector field on T^*M is semi-Hamiltonian these three connections do coincide. Henceforth there are studied some geometric problems related to the connections defined on T^*M. It is shown that the Helmholtz conditions from the inverse