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ON SOME THEOREMS IN THE THEORY OF THERMOELASTIC POROUS MATERIALS *

BY

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1.Introduction. The theory of elastic materials with voids was studied in various papers (see for example [1,2]). The origin of the modern formulation of continuum theory of materials with voids goes back to papers of G o o d m a n and C o w i n [3] and N u n z i a t o and C o w i n [4,5]. In this theory the bulk density is written as the product of two fields, the matrix materials density field and the volume fraction field. The representation introduces an additional degree of kinematic freedom. The intended applications of the theory of elastic materials with voids are to geological materials (like rock and soils) and to manufactured porous materials (like ceramics and pressed powders). A theory of thermoelastic materials with voids was derived in [6] by using the Clausius-Duhem inequality. In [7], a linear theory of thermoelastic materials with voids was derived by discussing restrictions on constitutive equations with the help of an entropy production inequality proposed by G r e e n and L a w s [8] and G r e e n and L i n d s a y [9]. It is shown that the heat conduction tensor is symmetric and that the theory allows for "second sound effects".

The aim of this work is to extend some results established by I e s a n in [10] (for the classical dynamic theory of thermoelasticity) to the linear theory of centrosymmetric thermoelastic materials with voids established in [4,7]. First we establish a reciprocal theorem, avoiding the use of Laplace transform and the introduction of the initial conditions into the equations of motion. Subsequently we prove that this reciprocal theorem leads to reciprocity relation, uniqueness theorem and a minimum principle.

2.Basic Equations. Through this paper we consider a body with voids that at time t=0 occupies the bounded domain Ω of R^3. The motion of the body is referred to a fixed system of rectangular Cartesian axes $ox_i (i = 1, 2, 3)$. Vectors and

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tensors will have components denoted by Latin subscripts (ranging over (1,2,3)). Summation over repeated subscripts is implied. Letters in boldface stand for the tensors of an order $p \geq 1$ and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate. Superposed dot denotes the material derivative with respect to time.

We consider the linear theory of thermoelasticity for the porous materials established in [7]. The behaviour of a porous thermoelastic body is governed by the following local balances of momentum, equilibrated force and energy.

$$\begin{aligned} \nabla\cdot\mathbf{T}+\rho_0\mathbf{f}&=\rho_0\ddot{\mathbf{u}},\\ \nabla\cdot\mathbf{H}+g+\lambda&=\rho_0\chi\ddot{\varphi},\\ T_0\dot{\eta}&=\nabla\cdot\mathbf{q}+r, \end{aligned} \tag{1}$$

where $\mathbf{T}$ denotes the stress tensor, $\mathbf{H}$ is the equilibrated stress vector, $\mathbf{u}$ is the displacement vector, g is the intrinsic equilibrated body force, $\mathbf{q}$ is the heat flux vector and η is the specific entropy. Also, ρ_0 is the mass density, $\mathbf{f}$ is the external body force, λ is the extrinsic equilibrated body force, r is the extrinsic heat supply, χ is the equilibrated inertia, φ is the change in volume fraction with respect to the reference one, and T_0 is the absolute constant temperature in the reference configuration. The constitutive relations for the homogeneous and centrosymmetric materials with voids are

$$\begin{aligned} \mathbf{T}=\mathbf{C}\varepsilon+\mathbf{M}\varphi+\mathbf{A}(\vartheta+\alpha\dot{\vartheta}),&\qquad \mathbf{H}=\mathbf{D}\nabla\varphi,\\ g=-\mathbf{M}:\varepsilon-c\varphi-m(\vartheta+\alpha\dot{\vartheta}),&\\ \eta=a-\mathbf{A}:\varepsilon-m\varphi+d\vartheta+h\dot{\vartheta},&\qquad \mathbf{q}=T_0\mathbf{k}\nabla\vartheta. \end{aligned} \tag{2}$$

Here $\varepsilon=\frac{1}{2}[\nabla\mathbf{u}+(\nabla\mathbf{u})^T]$, $\mathbf{C}$ is a fourth-order tensor, $\mathbf{M}$, $\mathbf{A}$ and $\mathbf{D}$ are second-order tensors (in component form we have $(\mathbf{C}\varepsilon)_{ij}=C_{ijk\lambda}\varepsilon_{k\lambda}$, $(\mathbf{D}\nabla\varphi)_i=D_{ij}(\nabla\varphi)_j$, $\mathbf{M}:\varepsilon=M_{ij}\varepsilon_{ij}$, $etc.$). ϑ is the temperature measured from the constant absolute temperature $T_0(>0)$. The constitutive coefficients have the symmetry properties

$$C_{ijrs}=C_{rsij}=C_{jirs},\ A_{ij}=A_{ji},\ M_{ij}=M_{ji},\ K_{ij}=K_{ji} \tag{3}$$

The second law of thermodynamics implies that [7]

$$\alpha d-h\geq 0;\ K_{ij}\beta_i\beta_j\geq 0\ for\ all\ \beta\in\mathbb{R}^3 \tag{4}$$

We assume that $\mathbf{f}$,r and l are continuous in $\bar{\Omega}\times(0,\infty)$. The surface traction $\mathbf{t}$, surface equilibrated stress $\mathbf{H}$ and the heat flux $\mathbf{q}$ at regular points of $\partial\Omega\times(0,\infty)$ are defined

$$\mathbf{t}=\mathbf{T}\ \mathbf{n},\ H=\mathbf{H}\cdot\mathbf{n},\ q=\mathbf{q}\cdot\mathbf{n} \tag{5}$$

Let Σ_i $(i=1,2,\ldots,6)$ be subsets of $\partial\Omega$ so that $\overline{\Sigma}_i\cup\Sigma_{i+1}=\partial\Omega$ and $\overline{\Sigma}_i\cap\Sigma_{i+1}=\phi$. We consider the following initial conditions

$$\mathbf{u}=\mathbf{u}_0,\ \mathbf{u}=\mathbf{v}_0,\ \vartheta=\vartheta_0,\ \varphi=\varphi_0,\ \dot{\varphi}=\hat{\varphi}_0,\ \eta=\eta_0, \tag{6}$$

and boundary conditions

$$\begin{aligned} \mathbf{u}=\mathbf{u}^*\ on\ \overline{\Sigma}_1\times(0,\infty),&\quad \mathbf{T}\cdot\mathbf{n}=\mathbf{t}^*\ on\ \Sigma_2\times(0,\infty),\\ \varphi=\varphi^*\ on\ \overline{\Sigma}_3\times(0,\infty),&\quad \mathbf{H}\cdot\mathbf{n}=h^*\ on\ \Sigma_4\times(0,\infty),\\ \vartheta=\vartheta^*\ on\ \overline{\Sigma}_5\times(0,\infty),&\quad \mathbf{q}\cdot\mathbf{n}=q^*\ on\ \Sigma_6\times(0,\infty), \end{aligned} \tag{7}$$

where $\mathbf{n}$ is the outward unit normal of $\partial\Omega$ and the right-hand terms denote assigned fields. We assume that

(*i*) $\mathbf{u}_0,\ \mathbf{v}_0,\ \eta_0,\vartheta_0,\varphi_0$ *and* $\hat{\varphi}_0$ *are continuous on* Ω

(*ii*) $\mathbf{u}^*$ *is continuous on* $\overline{\Sigma}_1\times(0,\infty)$ *and* $\mathbf{t}^*$ *is piecewise regular on* Σ_2 *and continuous in time* ;

(*iii*) φ^* *is continuous on* $\overline{\Sigma}_3\times(0,\infty)$ *and* $\mathbf{h}^*$ *is piecewise regular on* Σ_4 *and continuous in time*

(*iv*) ϑ^* *is continuous on* $\overline{\Sigma}_5\times(0,\infty)$ *and* $\mathbf{q}^*$ *is piecewise regular on* Σ_6 *and continuous in time.*

3.Reciprocity Theorems. Let ω_1 and ω_2 be scalar fields on $\Omega\times(0,\infty)$, continuous with respect to time. We denote with $\omega_1*\omega_2$ the convolution in time of ω_1 and ω_2 , defined by

$$(\omega_1*\omega_2)(\mathbf{x},\mathbf{t})=\int_0^t\omega_1(\mathbf{x},t-\tau)\,\omega_2(\mathbf{x},\tau)\,d\tau, \tag{8}$$

We introduce the following functions

$$L(t)=1;s(t)=L(t)*L(t)=t. \tag{9}$$

Let F be a continuous function on $\Omega\times(0,\infty)$. We denote by $\tilde{F}$ the function on $\Omega\times(0,\infty)$ defined by

$$\tilde{F}(\mathbf{x},t)=(L*\mathbf{F})(\mathbf{x},t)=\int_0^t F(\mathbf{x},\tau)d\tau. \tag{10}$$

Furthermore, we define

$$W = \tilde{r} + T_0(\eta_0 - a). \tag{11}$$

We say that f(x, t) is of class $C^{M,N}$ on $\Omega \times (0,\infty)$ (with M and N non-negative integers) if **f** is continuous on $\Omega \times (0,\infty)$ and the functions

$$\frac{\partial^{n_2}}{\partial x_i \partial x_j \ldots \partial x_k}\left(\frac{\partial^{n_1} f}{\partial t^{n_1}}\right) \begin{cases} n_2 \in \{0,1,\ldots,M\};\, n_1 \in \{0,1,\ldots,N\} \\ n_1 + n_2 \le \max\{M, N\} \end{cases}$$

exist and are continuous on $\Omega \times (0,\infty)$.

Through (11) we have

Theorem 1. *The functions $\eta \in C^{0,1}$ and $q_i \in C^{1,0}$ satisfy the relation $(1)_3$ and the initial conditions, if and only if*

$$T_0(\eta - a) = \nabla \cdot \tilde{q} + W \text{ on } \bar{\Omega} \times (0,\infty) \tag{12}$$

The proof is an immediate consequence of the relations (10) and (11). Let **F** be a function of class $C^{0,1}$ on $\bar{\Omega} \times (0,\infty)$. We introduce the operators

$$P_1 F = F + \alpha \dot{F}, \qquad P_2 F = L * F + \alpha F. \tag{13}$$

Consider now two different sets of external data $D^{(i)} = \{\mathbf{f}^{(i)}, \mathbf{r}^{(i)}, \lambda^{(i)}, \mathbf{u}^{*(i)}, \vartheta^{*(i)}, \varphi^{*(i)}, \mathbf{q}^{*(i)}, \eta_0^{(i)}, \vartheta_0^{(i)}, \mathbf{u}_0^{(i)}, \mathbf{v}_0^{(i)}, \varphi_0^{(i)}, \hat{\varphi}_0^{(i)}, \mathbf{t}^{*(i)}, h^{*(i)}\}$ with the corresponding solutions $\Gamma^{(i)} = \{\mathbf{u}^{(i)}, \vartheta^{(i)}, \mathbf{T}^{(i)}, \eta^{(i)}, \mathbf{q}^{(i)}, \mathbf{H}^{(i)}, g^{(i)}, \varphi^{(i)}\}$. We introduce the following relations

$$\mathbf{t}^{(i)} = \mathbf{T}^{(i)}\mathbf{n}, \quad q^{(i)} = \mathbf{q}^{(i)} \cdot \mathbf{n}, \quad h^{(i)} = \mathbf{H}^{(i)} \cdot \mathbf{n} \tag{14}$$

$$W^{(i)} = \tilde{r}^{(i)} + T_0(\eta_0^{(i)} - a). \tag{15}$$

Theorem 2. *Let a thermoelastic body subjected to external data $D^{(1)}$ and $D^{(2)}$ and let $\Gamma^{(1)}$ and $\Gamma^{(2)}$ be the corresponding solutions.*

Let

$$\begin{aligned} \mathbf{I}_{P\gamma} = &\int_{\partial\Omega}\Big[\mathbf{t}^{(P)}(\mathbf{x},\tau_1)\cdot\mathbf{u}^{(\gamma)}(\mathbf{x},\tau_2) - \frac{1}{T_0}\tilde{q}^{(P)}(\mathbf{x},\tau_1)P_1\vartheta^{(\gamma)}(\mathbf{x},\tau_2) + \\ &+ h^{(P)}(\mathbf{x},\tau_1)\varphi^{(\gamma)}(\mathbf{x},\tau_2)\Big]d\sigma + \int_{\Omega}\Big[\mathbf{f}^{(P)}(\mathbf{x},\tau_1)\cdot\mathbf{u}^{(\gamma)}(\mathbf{x},\tau_2) - \\ &- \frac{1}{T_0}W^{(P)}(\mathbf{x},\tau_1)P_1\vartheta^{(\gamma)}(\mathbf{x},\tau_2) + \lambda^{(P)}(\mathbf{x},\tau_1)\varphi^{(\gamma)}(\mathbf{x},\tau_2)\Big]dV - \end{aligned} \tag{16}$$

$$\begin{aligned} &- \int_{\Omega}\Big[\rho_0\ddot{\mathbf{u}}_i^{(P)}(\mathbf{x},\tau_1)\cdot\mathbf{u}^{(\gamma)}(\mathbf{x},\tau_2) + \rho_0\chi\ddot{\varphi}^{(P)}(\mathbf{x},\tau_1)\varphi^{(\gamma)}(\mathbf{x},\tau_2) - \\ &- \alpha d\vartheta^{(P)}(\mathbf{x},\tau_1)\dot{\vartheta}^{(\gamma)}(\mathbf{x},\tau_2) - h\dot{\vartheta}^{(P)}(\mathbf{x},\tau_1)\vartheta^{(\gamma)}(\mathbf{x},\tau_2)\Big]dV + \\ &+ \frac{1}{T_0}\int_{\Omega}\tilde{\mathbf{q}}^{(P)}(\mathbf{x},\tau_1)\cdot P_1\nabla\vartheta^{(\gamma)}(\mathbf{x},\tau_2)dV. \end{aligned}$$

for all $\tau_1, \tau_2 \in (0,\infty)$. Then

$$\mathbf{I}_{P\gamma}(\tau_1,\tau_2) = \mathbf{I}_{\gamma P}(\tau_2,\tau_1), \tag{17}$$

for all $\tau_1, \tau_2 \in (0,\infty)$.

Proof. Let

$$\begin{aligned} S_{P\gamma}(\tau_1,\tau_2) = &\mathbf{T}^{(P)}(\tau_1):\varepsilon^{(\gamma)}(\tau_2) + \mathbf{H}^{(P)}(\tau_1)\cdot\nabla\varphi^{(\gamma)}(\tau_2) - \\ &- g^{(P)}(\tau_1)\varphi^{(\gamma)}(\tau_2) - [\eta^{(P)}(\tau_1) - a]P_1\vartheta^{(\gamma)}(\tau_2). \end{aligned} \tag{18}$$

It follows from constitutive equations and the relation (18) that

$$\begin{aligned} S_{ky}(\tau_1,\tau_2) = &\mathbf{C}\varepsilon^{(k)}(\tau_1):\varepsilon^{(y)}(\tau_2) + \mathbf{M}:[\varphi^{(k)}(\tau_1)\varepsilon^{(y)}(\tau_2) + \\ &+ \varepsilon^{(k)}(\tau_1)\varphi^{(y)}(\tau_2)] + \mathbf{D}\nabla\varphi^{(k)}(\tau_1)\cdot\nabla\varphi^{(y)}(\tau_2) + \\ &+ \mathbf{A}:[\varepsilon^{(k)}(\tau_1)P_1\vartheta^{(y)}(\tau_2) + \varepsilon^{(y)}(\tau_2)P_1\vartheta^{(k)}(\tau_1)] + \\ &+ m[\varphi^{(k)}(\tau_1)P_1\vartheta^{(y)}(\tau_2) + \varphi^{(y)}(\tau_2)P_1\vartheta^{(k)}(\tau_1)] + \\ &+ c\varphi^{(k)}(\tau_1)\varphi^{(y)}(\tau_2) - d\vartheta^{(k)}(\tau_1)\vartheta^{(y)}(\tau_2) - d\alpha\vartheta^{(k)}(\tau_1)\dot{\vartheta}^{(y)}(\tau_2) - \\ &- h\dot{\vartheta}^{(k)}(\tau_1)\vartheta^{(y)}(\tau_2) - h\alpha\dot{\vartheta}^{(k)}(\tau_1)\dot{\vartheta}^{(y)}(\tau_2) \end{aligned} \tag{19}$$

We now introduce the notation

$$\bar{S}_{k\xi}(\tau_1,\tau_2) = S_{k\xi}(\tau_1,\tau_2) + d\alpha\vartheta^{(k)}(\tau_1)\dot{\vartheta}^{(\xi)}(\tau_2) + h\dot{\vartheta}^{(k)}(\tau_1)\vartheta^{(\xi)}(\tau_2). \tag{20}$$

From (19), (20) and (3) we obtain

$$\bar{S}_{k\xi}(\tau_1,\tau_2) = \bar{S}_{\xi k}(\tau_1,\tau_2). \tag{21}$$

In view of the equations of motion, from (12) and (18), we have

$$S_{P\gamma}(\tau_1,\tau_2) = \nabla\cdot\Big[\mathbf{T}^{(P)}(\tau_1)\mathbf{u}^{(\gamma)}(\tau_2) + \mathbf{H}^{(P)}(\tau_1)\varphi^{(\gamma)}(\tau_2) - \tag{22}$$

$$- \frac{1}{T_0}\tilde{\mathbf{q}}^{(P)}(\tau_1)P_1\vartheta^{(\gamma)}(\tau_2)\Big] + \mathbf{f}^{(P)}(\tau_1)\cdot \mathbf{u}^{(\gamma)}(\tau_2) -$$

$$- \frac{1}{T_0}W^{(P)}(\tau_1)P_1\vartheta^{(\gamma)}(\tau_2)\Big] - \rho_0\ddot{\mathbf{u}}^{(P)}(\tau_1)\cdot \mathbf{u}^{(\gamma)}(\tau_2) -$$

$$- \rho_0\chi\ddot{\varphi}^{(P)}(\tau_1)\varphi^{(\gamma)}(\tau_2) - \frac{1}{T_0}\tilde{\mathbf{q}}^{(P)}(\tau_1)\cdot P_1\nabla\vartheta^{(\gamma)}(\tau_2) +$$

$$+ \lambda^{(P)}(\tau_1)\varphi^{(\gamma)}(\tau_2).$$

From (20), (21), (22) and the divergence theorem , we get

$$\int_\Omega \mathrm{S}_{P\gamma}(\tau_1,\tau_2)dV = \mathbf{I}_{P\gamma}(\tau_1,\tau_2). \tag{23}$$

From Theorem 2 we obtain

Theorem 3. *Let $\Gamma^{(i)}$ be o solution corresponding to external data system $D^{(i)}(i=1,2,3)$. Then*

$$\int_{\partial\Omega} s * \Big[t_i^{(1)} * u_i^{(2)} + h^{(1)} * \varphi^{(2)} - \frac{1}{T_0}q^{(1)} * P_2\vartheta^{(2)}\Big]d\sigma + \tag{24}$$

$$+ \int_\Omega \Big[\mathcal{F}_i^{(1)} * u_i^{(2)} - \mathcal{L}^{(1)} * \varphi^{(2)} - \frac{1}{T_0}L * \vartheta^{(2)} * P_2W^{(1)}\Big]dV -$$

$$- \frac{\alpha}{T_0}\int_\Omega L * q^{(1)} * R^{(2)}d\sigma - \frac{\alpha}{T_0}\int_\Omega W^{(1)} * R^{(2)}dV +$$

$$+ \int_\Omega \Big[(h-\alpha d)R^{(1)} * \vartheta^{(2)} + \alpha L * K_{ij}\vartheta_{,j}^{(1)} * R_{,j}^{(2)}\Big]dV =$$

$$= \int_{\partial\Omega} s * \Big[t_i^{(2)} * u_i^{(1)} + h^{(2)} * \varphi^{(1)} - \frac{1}{T_0}q^{(2)} * P_2\vartheta^{(1)}\Big]d\sigma +$$

$$+ \int_\Omega \Big[\mathcal{F}_i^{(2)} * u_i^{(1)} - \mathcal{L}^{(2)} * \varphi^{(1)} - \frac{1}{T_0}L * \vartheta^{(1)} * P_2W^{(2)}\Big]dV -$$

$$- \frac{\alpha}{T_0}\int_\Omega (L * q^{(2)}) * R^{(1)}d\sigma - \frac{\alpha}{T_0}\int_\Omega W^{(2)}) * R^{(1)}dV +$$

$$+ \int_\Omega \Big[(h-\alpha d)R^{(2)} * \vartheta^{(1)} + \alpha L * K_{ij}\vartheta_{,j}^{(2)} * R_{,i}^{(1)}\Big]dV,$$

where

$$\mathcal{F}_i^{(\beta)} = s * f_i^{(\beta)} + \rho_0(tv_{0i}^{(\beta)} + u_{0i}^{(\beta)}),$$

$$\mathcal{L}^{(\beta)} = s * l^{(\beta)} + \rho_0\chi(t\dot{\varphi}_0^{(\beta)} + \varphi_0^{(\beta)}), \tag{25}$$

$$\mathbf{R}^{(\beta)} = -t\vartheta_0.$$

Proof. If we put in (17) $\tau_1 = \tau$ and $\tau_2 = t-\tau$, integrating on (0,t), we obtain

$$\bar{\mathbf{I}}_{P\gamma}(t) = \int_0^t \mathbf{I}_{P\gamma}(\tau_1, t-\tau)d\tau \tag{26}$$

$$= \int_{\partial\Omega}\Big[t_i^{(P)} * u_i^{(\gamma)} + h^{(P)} * \varphi^{(\gamma)} - \frac{1}{T_0}L * q^{(P)}P_1\vartheta^{(\gamma)}\Big]d\sigma +$$

$$+ \int_\Omega \Big[f_i^{(P)} * u_i^{(\gamma)} + \lambda^{(P)} * \varphi^{(\gamma)} - \frac{1}{T_0}W^{(P)} * P_1\vartheta^{(\gamma)}\Big]dV -$$

$$- \int_\Omega \Big[\rho_0\ddot{u}_i^{(P)} * u_i^{(\gamma)} + \rho_0\chi\ddot{\varphi}^{(\gamma)} * \varphi^{(\gamma)} - h\dot{\vartheta}^{(P)} * \vartheta^{(\gamma)} - \alpha d\vartheta^{(P)} * \dot{\vartheta}^{(\gamma)}\Big]dV +$$

$$+ \frac{1}{T_0}\int_\Omega L * q_i^{(P)} * P_1\vartheta_{,i}^{(\gamma)}dV.$$

Taking into account the properties of convolution, we get

$$s * \mathbf{I}_{P\gamma}(t) = \int_{\partial\Omega} s * \Big[t_i^{(P)} * u_i^{(\gamma)} + h^{(P)} * \varphi^{(\gamma)} - \frac{1}{T_0}q^{(P)} * P_2\vartheta^{(\gamma)}\Big]d\sigma \tag{27}$$

$$+ \int_\Omega \Big[\mathcal{F}_i^{(P)} * u_i^{(\gamma)} - \mathcal{L}^{(P)} * \varphi^{(\gamma)} - \frac{1}{T_0}L * \vartheta^{(P)} * P_2W^{(\gamma)}\Big]dV -$$

$$- \int_\Omega \Big[\rho_0u_i^{(P)} * u_i^{(\gamma)} - (h+\alpha d)L * \vartheta^{(P)} * \vartheta^{(\gamma)} + \rho_0\chi\varphi^{(P)}\varphi^{(\gamma)}\Big]dV +$$

$$+ \int_\Omega s * K_{ij}\vartheta_{,j}^{(P)} * (L * \vartheta_{,i}^{(\gamma)} + \alpha\vartheta_{,i}^{(\gamma)})dV -$$

$$- \frac{\alpha}{T_0}\int_{\partial\Omega} L * q^{(P)} * R^{(\gamma)}d\sigma - \frac{\alpha}{T_0}\int_\Omega W^{(P)} * R^{(\gamma)}dV +$$

$$+ \int_\Omega \Big[hR^{(P)}) * \vartheta^{(\gamma)} + \alpha dR^{(P)}\vartheta^{(\gamma)} + \alpha L * K_{ij}\vartheta_{,j}^{(P)} * R_{,i}^{(\gamma)}\Big]dV.$$

From the condition (3), and the relations (17), (26) and (27), we obtain the desired result.

Clearly from the Theorem 3 we can recover the reciprocity relations established in [6, 11 - 15].

4. Uniqueness Theorem. *We show that the Theorem 2 leads to a uniqueness result*

Theorem 4. *Let*

$$\Lambda(\tau_1,\tau_2)=\int_\Omega[\mathbf{f}(\tau_1)\cdot\mathbf{u}(\tau_2)-\frac{1}{T_0}W(\tau_1)P_1\vartheta(\tau_2)+ \tag{28}$$

$$+\lambda(\tau_1)\varphi(\tau_2)]dV+\int_{\partial\Omega}[\mathbf{t}(\tau_1)\cdot\mathbf{u}(\tau_2)+H(\tau_1)\varphi(\tau_2)-$$

$$-\frac{1}{T_0}\tilde{q}(\tau_1)P_1\vartheta(\tau_2)]d\sigma,$$

for all $\tau_1,\tau_2\in(0,\infty)$. *Then*

$$\frac{d}{dt}\Big\{\int_\Omega[\rho_0\mathbf{u}^2+\rho_0\chi\varphi^2+\alpha K_{ij}\tilde{\vartheta}_{,i}\tilde{\vartheta}_{,j}]dV+ \tag{29}$$

$$+\int_0^t\int_\Omega[(\alpha d-h)\vartheta^2+K_{ij}\tilde{\vartheta}_{,i}\tilde{\vartheta}_{,j}]d\tau_2dV\Big\}=$$

$$=\int_0^t[\Lambda(t-\tau_2,t+\tau_2)-\Lambda(t+\tau_2,t-\tau_2)]d\tau_2+$$

$$+\int_\Omega\Big\{\rho_0\Big[\dot{\mathbf{u}}(2t)\cdot\mathbf{u}(0)+\dot{\mathbf{u}}(0)\cdot\mathbf{u}(2t)+\chi\dot{\varphi}(2t)\varphi(0)+\dot{\varphi}(0)\varphi(2t)\Big]+$$

$$+(\alpha d-h)\vartheta(0)\vartheta(2t)+\alpha K_{ij}\tilde{\vartheta}_{,i}(2t)\tilde{\vartheta}_{,j}(0)\Big\}dV.$$

Proof. By Theorem 2 we obtain

$$\int_0^t\mathbf{I}_{11}(t+\tau_2,t-\tau_2)d\tau_2=\int_0^t\mathbf{I}_{11}(t-\tau_2,t+\tau_2)d\tau_2. \tag{30}$$

We apply this relation to the process $(\mathbf{u},\vartheta,\varphi,\mathbf{T},\mathbf{H},\eta,\mathbf{q})$. From (16) and (28), we get

$$\int_0^t\mathbf{I}_{11}(t+\tau_2,t-\tau_2)d\tau_2=\int_0^t\Lambda(t+\tau_2,t-\tau_2)d\tau_2- \tag{31}$$

$$-\int_0^t\int_\Omega\Big[\rho_0\ddot{\mathbf{u}}(t+\tau_2)\cdot\mathbf{u}(t-\tau_2)+\rho_0\chi\ddot{\varphi}(t+\tau_2)\varphi(t-\tau_2)-$$

$$-h\dot{\vartheta}(t+\tau_2)\vartheta(t-\tau_2)-\alpha d\vartheta(t+\tau_2)\dot{\vartheta}(t-\tau_2)\Big]dV+$$

$$+\int_0^t\int_\Omega\mathbf{K}\nabla\tilde{\vartheta}(t+\tau_2)\cdot\Big[\nabla\dot{\tilde{\vartheta}}(t-\tau_2)+\alpha\nabla\ddot{\tilde{\vartheta}}(t-\tau_2)\Big]d\tau_2dV,$$

and

$$\int_0^t\mathbf{I}_{11}(t-\tau_2,t+\tau_2)d\tau_2=\int_0^t\Lambda(t-\tau_2,t+\tau_2)d\tau_2- \tag{32}$$

$$-\int_0^t\int_\Omega\Big[\rho_0\ddot{\mathbf{u}}(t-\tau_2)\cdot\mathbf{u}(t+\tau_2)+\rho_0\chi\ddot{\varphi}(t-\tau_2)\varphi(t+\tau_2)-$$

$$-h\dot{\vartheta}(t-\tau_2)\vartheta(t+\tau_2)-\alpha d\vartheta(t+\tau_2)\dot{\vartheta}(t+\tau_2)\Big]dV+$$

$$+\int_0^t\int_\Omega\mathbf{K}\nabla\tilde{\vartheta}(t-\tau_2)\cdot\Big[\nabla\dot{\tilde{\vartheta}}(t+\tau_2)+\alpha\nabla\ddot{\tilde{\vartheta}}(t+\tau_2)\Big]d\tau_2dV.$$

Integrating by parts, from (31) and (32), we have

$$\int_0^t\mathbf{I}_{11}(t+\tau_2,t-\tau_2)d\tau_2=\int_0^t\Lambda(t+\tau_2,t-\tau_2)d\tau_2- \tag{33}$$

$$-\int_\Omega\Big\{\rho_0\Big[\dot{\mathbf{t}}(2t)\cdot\mathbf{u}(0)-\dot{\mathbf{u}}(t)\cdot\mathbf{u}(t)+\chi\dot{\varphi}(2t)\varphi(0)-\chi\dot{\varphi}(t)\varphi(t)\Big]-$$

$$-h\Big[\vartheta(2t)\vartheta(0)-\vartheta^2(t)\Big]+\alpha d\Big[\vartheta^2(t)-\vartheta(0)\vartheta(2t)\Big]\Big\}dV-$$

$$-\int_0^t\int_\Omega\Big[\rho_0\dot{\mathbf{u}}(t+\tau_2)\cdot\dot{\mathbf{u}}(t-\tau_2)+\rho_0\chi\dot{\varphi}(t-\tau_2)\dot{\varphi}(t-\tau_2)-$$

$$-h\vartheta(t+\tau_2)\dot{\vartheta}(t-\tau_2)-\alpha d\dot{\vartheta}(t+\tau_2)\vartheta(t-\tau_2)\Big]d\tau_2dV+$$

$$+\int_0^t\int_\Omega\mathbf{K}\nabla\dot{\tilde{\vartheta}}(t+\tau_2)\cdot\nabla\tilde{\vartheta}(t-\tau_2)+\alpha\mathbf{K}\nabla\vartheta(t+\tau_2)\cdot\nabla\vartheta(t-\tau_2)\Big]d\tau_2dV+$$

$$+\int_\Omega\Big[\mathbf{K}\nabla\tilde{\vartheta}\cdot(\nabla\tilde{\vartheta}+\alpha\nabla\dot{\tilde{\vartheta}})-\alpha\mathbf{K}\nabla\tilde{\vartheta}(2t)\cdot\nabla\vartheta(0)\Big]dV,$$

and

$$\int_0^t\mathbf{I}_{11}(t-\tau_2,t+\tau_2)d\tau_2=\int_0^t\Lambda(t-\tau_2,t+\tau_2)d\tau_2- \tag{34}$$

$$-\int_\Omega\left\{\rho_0\left[\dot{\mathbf{u}}(t)\cdot\mathbf{u}(t)-\dot{\mathbf{u}}(0)\cdot\mathbf{u}(2t)+\chi\dot\varphi(t)\dot\varphi(t)-\dot\varphi(0)\varphi(2t)\right]dV-\right.$$

$$-\int_0^t\int_\Omega\left[\rho_0\dot{\mathbf{u}}(t-\tau_2)\cdot\dot{\mathbf{u}}(t+\tau_2)+\rho_0\chi\dot\varphi(t-\tau_2)\dot\varphi(t+\tau_2)-\right.$$

$$\left.-h\dot\vartheta(t-\tau_2)\vartheta(t+\tau_2)-\alpha d\vartheta(t-\tau_2)\dot\vartheta(t+\tau_2)\right]d\tau_2 dV+$$

$$+\int_0^t\int_\Omega \mathbf{K}\nabla\tilde\vartheta(t-\tau_2)\cdot\nabla\dot{\tilde\vartheta}(t+\tau_2)d\tau_2 dV+$$

$$+\int_0^t\int_\Omega \alpha\mathbf{K}\nabla\tilde\vartheta(t+\tau_2)\cdot\nabla\tilde\vartheta(t-\tau_2)d\tau_2 dV-\int_\Omega\alpha\mathbf{K}\nabla\dot{\tilde\vartheta}\cdot\nabla\tilde\vartheta dV.$$

Taking into account the properties (3), from (30), (33) and (34) we obtain the desired result.

Theorem 5.(Uniqueness). *Assume that*

$$\rho_0,\chi,\alpha>0\ ,\ K_{ij}\xi_i\xi_j\geq 0, \tag{35}$$

for all ξ_i. Then the boundary-initial value problem has at most one solution

Proof. We consider two solutions. Then their difference ($\mathbf{u}$,φ, ϑ) corresponds to null data and the relation (29) reduces to

$$\int_\Omega\left[\rho_0\mathbf{u}^2+\rho_0\chi\varphi^2+\alpha\mathbf{K}\nabla\tilde\vartheta\cdot\nabla\tilde\vartheta\right]dV+ \tag{36}$$

$$+\int_0^t\int_\Omega\left[(\alpha d-h)\vartheta^2+\mathbf{K}\nabla\tilde\vartheta\cdot\nabla\tilde\vartheta\right]d\tau_2 dV=0\ for\ t\ \in[0,\infty).$$

Through (4) and (35), from (36) we obtain $\mathbf{u}=0$, $\vartheta=0$, $\varphi=0$.

5.Minimum Principle. In this section we establish a minimum principle of Reiss type (see [10,16,17]). Let Π_1 be the set of all functions γ on $\bar\Omega\times(0,\infty)$ such that

(i) *γ is of class C^2 on $\Omega\times(0,\infty)$ and of class $C^{1,0}$ on $\bar\Omega\times(0,\infty)$;*

(ii) $\gamma,\gamma_{,i},\dot\gamma,\gamma_{,ij},\dot\gamma_{,i}$ **and** $\ddot\gamma$ **are bounded at** ∞.

Let Π_2 be a set of all continuous functions on $\bar\Omega\times(0,\infty)$ that are bounded at ∞. We introduce (see also [16,17]) the set Π_3 of the admissible weight functions $\lambda: t\in(0,\infty)\rightarrow\lambda(t)\in R^+$ such that

(i) λ is the Laplace transform of some non-negative smooth function G having compact support in R^+. Thus ,

$$\lambda(t)=\int_0^\infty G(s)\exp(-st)ds, \tag{37}$$

(ii) the integrals

$$\int_0^\infty\int_0^\infty\lambda(t+s)dtds\ ,\ \int_0^\infty\int_0^\infty\lambda'(t+s)dtds\ ,\ \int_0^\infty\int_0^\infty\lambda''(t+s)dtds\ ,$$

exist. We note that the assumption (i) is more general then the corresponding assumption considered in [16], (see [18, 19]).

Let u, w $\in\Pi_2$. We introduce the notation

$$[u,w]_\lambda=[w,u]_\lambda=\int_0^\infty\int_0^\infty\int_\Omega\lambda(t+s)u(\mathbf{x},t)w(\mathbf{x},s)dtdsdV, \tag{38}$$

for any $\lambda\in\Pi_3$. Let A be a linear operator from Π_1 into Π_2 and D_A its domain of definition. The operator A is λ- symmetric if

$$[Au,w]_\lambda=[u,Aw]_\lambda, \tag{39}$$

for any u, v $\in D_A$. A λ- symmetric operator A is called positive if

$$[Au,u]_\lambda\geq 0, \tag{40}$$

for any u $\in D_A$. Let f $\in\Pi_2$. We consider the equation

$$Au=f \tag{41}$$

Let Φ_λ be the functional on D_A defined by

$$\Phi_\lambda\{u\}=[Au,w]_\lambda-2[u,f]_\lambda, \tag{42}$$

for every u $\in D_A$. We now state the following well-known result.

Theorem 6. *Let A a λ - positive operator, and let Φ_λ be the functional on D_A defined by (42). Further, let u be a solution of the equation (41). Then*

$$\Phi_\lambda\{u\}\leq\Phi_\lambda\{u^*\}, \tag{43}$$

for every $u^*\in D_A$. By (37) and (38), we get

$$[u,w]_\lambda=\int_0^\infty\int_\Omega G(s)\hat u(\mathbf{x},s)\hat w(\mathbf{x},s)dsdV, \tag{44}$$

where $\hat{\mathbf{f}}$ is the Laplace transform of the function $\mathbf{f}$.

From (1), (2) and (12) we obtain the field equations expressed in terms of $\mathbf{u}$, φ and ϑ.

$$C_{ijrs}u_{r,sj} + M_{ij}\varphi_{,j} + A_{ij}(\vartheta + \alpha\dot{\vartheta}) + \rho_0 f_i = \rho_0 \ddot{u}_i,$$

(45) $$D_{ij}\varphi_{,ji} - M_{ij}u_{i,j} - c\varphi - m(\vartheta + \alpha\dot{\vartheta}) + \lambda = \rho_0\chi\ddot{\varphi},$$

$$T_0(d\dot{\vartheta} + h\dot{\vartheta} - m\varphi - A_{ij}u_{i,j}) - T_0 K_{ij}\alpha\tilde{\vartheta}_{,ji} = W,$$

on $\bar{\Omega} \times (0,\infty)$. To the system (45) we add the following boundary and initial conditions

(46) $$\mathbf{u} = 0 \text{ on } \overline{\Sigma}_1 \times (0,\infty) \; ;$$

$$\mathbf{Tn} = [\mathbf{C}\epsilon + \mathbf{M}\varphi - \mathbf{A}(\vartheta + \alpha\dot{\vartheta})]\mathbf{n} = 0 \text{ on } \Sigma_2 \times (0,\infty),$$

$$\varphi = 0 \text{ on } \overline{\Sigma}_3 \times (0,\infty) \; ; \; \mathbf{H}\cdot\mathbf{n} = [\mathbf{D}\nabla\varphi]\cdot\mathbf{n} = 0 \text{ on } \Sigma_4 \times (0,\infty),$$

and

(47) $$\mathbf{u}(\mathbf{x},0) = 0 \; , \; \dot{\mathbf{u}}(\mathbf{x},0) = 0 \; , \; \varphi(\mathbf{x},0) = 0$$

$$\dot{\varphi}(\mathbf{x},0) = 0 \; , \; \vartheta(\mathbf{x},0) = 0 \; , \; \dot{\vartheta}(\mathbf{x},0) = 0 \quad for \; \mathbf{x} \in \Omega.$$

It follows from (11) and (47) that

(48) $$W(\mathbf{x},0) = 0 \text{ on } \bar{\Omega}$$

We say that $\mathrm{U} = (\mathbf{u},\vartheta,\varphi)$ is an admissible five-dimensional vector if $\mathbf{u},\vartheta$ and $\varphi \in \Pi_1$ and if ϑ is of class $C^{1,1}$ on $\bar{\Omega} \times (0,\infty)$. A solution of the problem is an admissible five-dimensional vector field $\mathrm{U} = (\mathbf{u},\vartheta,\varphi)$ that satisfies the Eqs.(45), the boundary conditions (46) and the initial conditions (47).

Let $\mathcal{H}$ be the set of all admissible five-dimensional vector fields U that satisfy the relations (46) and (47). The Eqs.(45) can be written in the form

(49) $$ZU = \mathcal{F}$$

where the five-dimensional vectors U, ZU and $\mathcal{F}$ are defined by

(50) $$U = (u_1,u_2,u_3,\vartheta,\varphi) \; , \mathcal{F} = (-\rho_0 f_1, -\rho_0 f_2, -\rho_0 f_3, -\lambda, -\frac{1}{T_0}W),$$

$$Z_i U = -\rho_0\ddot{u}_i + C_{ijrs}u_{r,sj} + M_{ij}\varphi_{,j} + A_{ij}(\vartheta + \alpha\dot{\vartheta})_{,j} \quad (i = 1,2,3),$$

$$Z_4 U = -\rho_0\chi\ddot{\varphi} + D_{ij}\varphi_{,ji} - M_{ij}u_{i,j} - c\varphi - m(\vartheta + \alpha\dot{\vartheta}),$$

$$Z_5 U = -K_{ij}\tilde{\vartheta}_{,ji} + h\dot{\vartheta} + d\vartheta - m\varphi - A_{ij}u_{i,j}$$

for every $\mathrm{U} = (\mathbf{u},\vartheta,\varphi) \in \mathcal{H}$..

We define the operator A on $\mathcal{H}$ such that

(51) $$A_i U = Z_i, U \; (i = 1,2,3,4) \; , \quad A_5 U = P_1 Z_5 U \; ,$$

and consider the equation

(52) $$AU = F \; ,$$

where

(53) $$F = (-\rho_0 f_1, -\rho_0 f_2, -\rho_0 f_3, -\lambda, -\frac{1}{T_0}W)$$

We consider an admissible five-dimensional vector field $\mathrm{U} = (\mathbf{u},\vartheta,\varphi)$ that satisfies the Eqs.(52) and the boundary and initial conditions (46) and (47). Clearly, if $\mathrm{U} = (\mathbf{u},\vartheta,\varphi)$ satisfies the Eqs.(45) with the conditions (46) and (47) then U satifies also the Eqs.(52) and the boundary and initial conditions. If U is a solution of the problem (52), (46) and (47) then U satisfies the equations

(54) $$Z_i U = -\rho_0 f_i \qquad (i = 1,2,3).$$

$$Z_4 U = -\lambda \; , \qquad P_1 Z_5 U = -\frac{1}{T_0}W \; ,$$

From (47) and (50) we obtain

(55) $$\left[Z_5 U + \frac{1}{T_0}\right](\mathbf{x},0) = 0,$$

and, consequently, we have

(56) $$\left[Z_5 U + \frac{1}{T_0}w\right](\mathbf{x},t) = 0 \qquad \text{on } \Omega \times (0,\infty)$$

Thus if $\mathrm{U} = (\mathbf{u},\vartheta,\varphi)$ is a solution of the problem (52), (46) and (47) , then U is a solution of the problem (45), (46) and (47).

Let $\mathrm{U} = (\mathbf{u},\vartheta,\varphi)$ and $\mathrm{V} = (\mathbf{w},\vartheta_1,\varphi_1)$ be five-dimensional vector fields on $\bar{\Omega} \times (0,\infty)$. We consider the relation

(57) $$[U,V]_\lambda = \Sigma^3_{1i}[u_i,w_i]_\lambda + [\varphi,\varphi_1]_\lambda + [\vartheta,\vartheta_1]_\lambda.$$

From the reciprocity relation we obtain

Theorem 7. *Let $\Gamma^{(i)}$ be a solution corresponding to the external data system* $D^{(i)} = \{\mathbf{f}^{(i)}, r^{(i)}, l^{(i)}, \mathbf{u}^{*(i)} = 0, \varphi^{*(i)} = 0, \mathbf{t}^{*(i)}, \vartheta^{*(i)} = 0, \mathbf{q}^{*(i)} = 0, h^{*(i)} = 0, \eta_0{}^{(i)} = 0, \vartheta_0{}^{(i)} = 0, \mathbf{u}_0^{(i)} = 0, \mathbf{v}_0^{(i)} = 0, \varphi_0^{(i)} = 0, \dot{\varphi}_0^{(i)} = 0\}$ *Then*

(58) $$[\mathbf{f}^{(1)},\mathbf{u}^{(2)}]_\lambda + [\lambda^{(1)},\varphi^{(2)}]_\lambda - \frac{1}{T_0}[W^{(1)},P_1\vartheta^{(2)}]_\lambda \; =$$

$$= [\mathbf{f}^{(2)}, \mathbf{u}^{(1)}]_\lambda + [\lambda^{(2)}, \varphi^{(1)}]_\lambda - \frac{1}{T_0}[W^{(2)}, P_1\vartheta^{(1)}]_\lambda ,$$

and

(59)
$$[\mathbf{f}^{(1)}, \mathbf{u}^{(1)}]_\lambda + [\lambda^{(1)}, \varphi^{(1)}]_\lambda - \frac{1}{T_0}[W^{(1)}, P_1\vartheta^{(1)}]_\lambda =$$

$$= \int_0^\infty \int_0^\infty \lambda(t+s)\{\mathbf{C}\epsilon^{(1)}(t) : \epsilon^{(1)}(s) + \mathbf{M} : [\varphi^{(1)}(t)\epsilon^{(1)}(s) +$$
$$+ \epsilon^{(1)}(t)\varphi^{(1)}(s)] + \mathbf{D}\nabla\varphi^{(1)}(t) \cdot \nabla\varphi^{(1)}(s) +$$
$$+ A : [\epsilon^{(1)}(t)P_1\vartheta^{(1)}(s) + \epsilon^{(1)}(s)P_1\vartheta^{(1)}(t)] +$$
$$+ m[\varphi^{(1)}(t)P_1\vartheta^{(1)}(s) + \varphi^{(1)}(s)P_1\vartheta^{(1)}(t)] +$$
$$+ c\varphi^{(1)}(t)\varphi^{(1)}(s) - d\vartheta^{(1)}(t)\vartheta^{(1)}(s) - d\alpha\vartheta^{(1)}(t)\dot{\vartheta}^{(1)}(s) +$$
$$+ h\dot{\vartheta}^{(1)}(t)\vartheta^{(1)}(s) - h\alpha\dot{\vartheta}^{(1)}_{(t)}\dot{\vartheta}^{(1)}(s)\}dtdsdV +$$
$$+ [\rho_0\dot{\mathbf{u}}^{(1)}, \dot{\mathbf{u}}^{(1)}]_\lambda + [\rho_0\chi\dot{\varphi}^{(1)}, \chi\dot{\varphi}^{(1)}]_\lambda + [\mathbf{K}\nabla\tilde{\vartheta}^{(1)}, P_1\nabla\tilde{\vartheta}^{(1)}]_\lambda .$$

Proof. We introduce the notation

(60)
$$\mathcal{P}_{P\gamma} = \int_0^\infty \int_0^\infty \lambda(t+s)\mathbf{I}_{P\gamma}(t,s)dtds.$$

From (17) we obtain

(61)
$$\mathcal{P}_{P\gamma} = \mathcal{P}_{\gamma P}$$

If $\mathbf{u}^{(i)}, \varphi^{(i)}$ and $\vartheta^{(i)}$ correspond to null boundary and initial data, from (16) and (60), considering that $[\ddot{\mathbf{u}}, \mathbf{u}]_\lambda = [\dot{\mathbf{u}}, \dot{\mathbf{u}}]_\lambda, [\rho_0\chi\ddot{\varphi}, \varphi]_\lambda = [\rho_0\chi\dot{\varphi}, \dot{\varphi}]_\lambda$ we have

(62)
$$\mathcal{P}_{P\gamma} = [\mathbf{f}^{(P)}, \mathbf{u}^{(\gamma)}]_\lambda + [\lambda^{(P)}, \varphi^{(\gamma)}]_\lambda - [\rho_0\dot{\mathbf{u}}^{(P)}, \dot{\mathbf{u}}^{(\gamma)}]_\lambda - [\rho_0\chi\dot{\varphi}^{(P)}, \dot{\varphi}^{(\gamma)}]_\lambda -$$
$$- \frac{1}{T_0}[W^{(P)}, P_1\vartheta^{(\gamma)}]_\lambda + h[\dot{\vartheta}^{(P)}, \vartheta^{(\gamma)}]_\lambda +$$
$$+ \alpha d[\vartheta^{(P)}, \dot{\vartheta}^{(\gamma)}]_\lambda + [\mathbf{K}\nabla\tilde{\vartheta}^{(P)}, P_1\nabla\vartheta^{(\gamma)}]_\lambda .$$

Taking into account the relations (3), (61), (62) and the initial conditions, we obtain (58). From (19), (20), (23) and (60) we have

(63)
$$\mathcal{P}_{11} = \int_0^\infty \int_0^\infty \int_\Omega \lambda(t+s)\{\mathbf{C}\epsilon^{(1)}(t) : \epsilon^{(1)}(s) + \mathbf{M} : [\varphi^{(1)}(t)\epsilon^{(1)}(s) +$$
$$+ \epsilon^{(1)}(t)\varphi^{(1)}(s)] + \mathbf{D}\nabla\varphi^{(1)}(t) \cdot \nabla\varphi^{(1)}(s) +$$

$$+ \mathbf{A} : [\epsilon^{(1)}(t)P_1\vartheta^{(1)}(s) + \epsilon^{(1)}(s)P_1\vartheta^{(1)}(t)] +$$
$$+ m[\varphi^{(1)}(t)P_1\vartheta^{(1)}(s) + \varphi^{(1)}(s)P_1\vartheta^{(1)}(t)] +$$
$$+ c\varphi^{(1)}(t)\varphi^{(1)}(s) - d\vartheta^{(1)}(t)\vartheta^{(1)}(s) - d\alpha\dot{\vartheta}^{(1)}(t)\dot{\vartheta}^{(1)}(s)\}dtdsdV.$$

By (62) and (63) we obtain the relation (59).

Let $\mathrm{U} = (\mathbf{u}, \vartheta, \varphi) \in \mathcal{H}$. Then we define the following functional

(64)
$$\Psi(U, \lambda) = \int_0^\infty \int_0^\infty \lambda(t+s) \int_\Omega \{\mathbf{K}\nabla\tilde{\vartheta}(t) \cdot \nabla\vartheta(s) + \alpha\mathbf{K}\nabla\tilde{\vartheta}(t) \cdot \nabla\dot{\vartheta}(s) +$$
$$+ (\alpha d - h)\vartheta(t)\dot{\vartheta}(s) + h[\dot{\vartheta}(t)\vartheta(s) + \dot{\vartheta}(s)\vartheta(t)] +$$
$$- c\varphi(t)\varphi(s) + d\vartheta(t)\vartheta(s) + \alpha h\dot{\vartheta}(t)\dot{\vartheta}(s) - \rho_0\dot{\mathbf{u}}(t) \cdot \dot{\mathbf{u}}(s) -$$
$$- \rho_0\chi\dot{\varphi}(t)\dot{\varphi}(s)\mathbf{A} : [\epsilon(t)P_1\vartheta(s) + \epsilon(s)P_1\vartheta(t)] -$$
$$- m[\varphi(t)P_1\vartheta(s) + \varphi(s)P_1\vartheta(t)] - \mathbf{C}\epsilon(t) : \epsilon(s) -$$
$$- \mathbf{M} : [\varphi(t)\epsilon(s) + \varphi(s)\epsilon(t)] - \mathbf{D}\nabla\varphi(t) \cdot \nabla\varphi(s)\}dtdsdV =$$
$$= \int_0^\infty \int_\Omega G(s)\Big\{\frac{1}{s}\mathbf{K}\nabla\hat{\vartheta} \cdot \nabla\hat{\vartheta} + \alpha\mathbf{K}\nabla\hat{\vartheta} \cdot \nabla\hat{\vartheta} + (\alpha d - h)s\hat{\vartheta}^2 +$$
$$+ 2hs\hat{\vartheta}^2 + d\hat{\vartheta}^2 + \alpha hs\hat{\vartheta}^2 - \rho_0\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}s^2 - \rho_0\chi s^2\hat{\varphi}^2 -$$
$$- 2\mathbf{A} : [\hat{\epsilon}\hat{\vartheta} + \alpha\hat{\epsilon}\hat{\vartheta}] - m[\hat{\varphi}\hat{\vartheta} + s\alpha\hat{\varphi}\hat{\vartheta}] - \mathbf{C}\hat{\epsilon} : \hat{\epsilon}$$
$$- 2\mathbf{M} : \hat{\varphi}\hat{\epsilon} - \mathbf{D}\nabla\hat{\varphi} \cdot \nabla\hat{\varphi} - c\hat{\varphi}^2\Big\}dsdV,$$

where $\hat{\vartheta}, \hat{\varphi}, \hat{\mathbf{u}}, \hat{\epsilon}$ are the Laplace transforms of $\vartheta, \varphi, \mathbf{u}$ and ϵ.

Let $\mathcal{H}^+$ be the set of all five-dimensional vector fields $U \in \mathcal{H}$ such that

(65)
$$\Psi(U, \lambda) \geq 0.$$

We contract the operator A by replacing its domain of definition $\mathcal{H}$ with the more restricted domain $\mathcal{H}^+$. We denote this contracted operator by A^+.

Theorem 8. *The operator A^+ is λ -positive.*

Proof. Let $\Gamma^{(i)}$ be a solution corresponding to external data system $D^{(i)}$. Let $\mathrm{U} = (\mathbf{u}, \vartheta, \varphi)$ and $\mathrm{V} = (\mathbf{u}_1, \vartheta_1, \varphi_1)$ be solutions of the problem (52) , (46) and (47) corresponding to $D^{(1)}$ and $D^{(2)}$. Then

$$A^+U = F^{(1)} , \qquad A^+V = F^{(2)} ,$$

where
$$F^{(i)} = (-\rho_0 f_1^{(i)} , -\rho_0 f_2^{(i)} , -\rho_0 f_3^{(i)} , -\lambda^{(i)} , -\frac{1}{T_0}W^{(i)}).$$

By (38), (47) and (48), we obtain

$$(66)\qquad [W^{(P)}, P_1\vartheta^{(\gamma)}]_\lambda = [P_1 W^{(P)}, \vartheta^{(\gamma)}]_\lambda$$

It follows, from (58) and (59) that

$$(67)\qquad [A^+U, V]_\lambda = [U, A^+V]_\lambda\,, \quad [A^+U, U]_\lambda = \Psi[U, \lambda]_\lambda$$

for every U ,V $\in \mathcal{H}^+$.

We consider the following functional on $\mathcal{H}^+$.

$$(68)\qquad \Lambda_\lambda\{U\} = [A^+U, U]_\lambda - 2[U, F]_\lambda$$

Clearly

$$(69)\qquad \begin{aligned} \Lambda[\mathbf{u}, \varphi, \vartheta; \lambda] = \Lambda_\lambda\{U\} = \int_0^\infty \int_0^\infty \int_\Omega \lambda(t+s)\Big\{&\mathbf{K}\nabla\tilde{\vartheta}(t)\cdot\nabla\vartheta(s) + \\ &+ \alpha\mathbf{k}\nabla\tilde{\vartheta}(t)\cdot\nabla\dot{\vartheta}(s) + (\alpha d - h)\vartheta(t)\dot{\vartheta}(s) + h[\dot{\vartheta}(t)\vartheta(s) + \dot{\vartheta}(s)\vartheta(t)] + \\ &+ d\vartheta(t)\vartheta(s) - c\varphi(t)\varphi(s) + \alpha h\dot{\vartheta}(t)\dot{\vartheta}(s) - \rho_0\dot{\mathbf{u}}(t)\cdot\dot{\mathbf{u}}(s) - \\ &- \rho_0\chi\dot{\varphi}(t)\dot{\varphi}(s) - \mathbf{A} : [\epsilon(t)P_1\vartheta(s) + \epsilon(s)P_1\vartheta(t)] - \\ &- m[\varphi(t)P_1\vartheta(s) + \varphi(s)P_1\vartheta(t)] - \mathbf{C}\epsilon(t) : \epsilon(s) - \\ &- \mathbf{M} : [\varphi(t)\epsilon(s) + \varphi(s)\epsilon(t)] - \mathbf{D}\nabla\varphi(t)\cdot\nabla\varphi(s) + \\ &+ 2\mathbf{F}(t)\cdot\mathbf{u}(s) + 2\rho_0\chi\varphi(t)\lambda(s) + \frac{2}{T_0}\vartheta(s)P_1W(t)\Big\}dtdsdV. \end{aligned}$$

By the Theorems 6 and 7, we obtain

Theorem 9. *Let Λ_λ be the functional on $\mathcal{H}^+$ defined by (69), and $U = (\mathbf{u}, \vartheta, \varphi) \in \mathcal{H}^+$ be a solution of the equations (45). Then*

$$\Lambda_\lambda\{U\} \le \Lambda_\lambda\{U^*\}$$

for every $U^* \in \mathcal{H}^+$.

A similar result can be established when $\Psi[U, \lambda] \le 0$.

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