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TWO BODY PROBLEM IN A NON-INERTIAL FRAME OF REFERENCE AND THE ADVANCE OF PERIHELION

BY

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In a short paper [1] the two body problem in a non-inertial frame of reference was studied and a plausible explanation for the appearance of the tails in the comentary motions was given. In this paper an explanation will be given for the planetary advance of perihelion.

Let us consider two particles S and P (for example the Sun and a planet) which are free to move in space under the influence of mutual forces of attraction in conformity with the Newton's law.

With respect to an inertial frame of reference S_0 the equation of motion are

$$m_1\ddot{\mathbf{r}}_1 = f\frac{m_1m_2}{r^2}\mathbf{u}\ , m_2\ddot{\mathbf{r}}_2 = -f\frac{m_1m_2}{r^2}\mathbf{u}\ ,$$

where by $\mathbf{r}_1, \mathbf{r}_2$ we denote the position vectors of the particles, by m_1, m_2 their masses and by r the distance between this two particles ; $\mathbf{u}$ is the unit vector of $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{SP}$.

The centre of gravity G moves, in S_0, uniformly in a straight line. Then, there is an inertial frame of reference S_G whose origin is at G. In S_G the particles S and P are acted on by central forces. Consequently , the particles will always remain in a fixed plane of S_G. Therefore, the normal to the plane of the trajectories is fixed.

Let us consider a frame of reference $Sx_1x_2x_3$ with its origin S and Sx_1x_2 in coincidence with the plane of trajectories. This frame of reference is non-inertial. The rotation of this frame of reference with respect to S_G is of the form $\mathbf{w} = \omega(\mathrm{t})$ $\mathbf{e}_3$, where $\mathbf{e}_3$ is the unit vector of Sx_3 and $\omega(\cdot) \in C^1$ is a function of the time t.

It follows that the equation of motion for the particle P in $Sx_1x_2x_3$ is given by

$$m_2\ddot{\mathbf{r}} = -f\frac{m_1m_2}{r^2}\mathbf{u} - m_2\left[f\frac{m_2}{r^2}\mathbf{u} + \dot{\mathbf{w}}\times\mathbf{r} + \mathbf{w}\times(\mathbf{w}\times\mathbf{r}) + 2\mathbf{w}\times\mathbf{v}\right],$$

where $\mathbf{r}$ and $\mathbf{v}$ are the position and the velocity of P in $Sx_1x_2x_3$. If the position of P is defined by polar coordinates r,θ of the plane Sx_1x_2 , we have the equations

$$\ddot{r} - r\dot{\theta}^2 = -f\frac{m_1+m_2}{r^2} + \omega^2 r + 2r\omega\dot{\theta}\ , \tag{1}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -r\dot{\omega} - 2\dot{r}\omega\ ,$$

where θ is the angle between the moving axis Sx_1 and the position vector $\mathbf{r}$.

The equations (1) posses the integrales

$$r^2(\dot{\theta}+\omega) = C\ , \tag{2}$$

$$\dot{r}^2 = \frac{2hr^2 + 2\mu r - C^2}{r^2}\ , \mu = f(m_1+m_2)\ ,$$

where C and h are arbitrary constants to be determined by the initial conditions of the motion.

The apsidal distances are given by the roots of the equation

$$2hr^2 + 2\mu r - C^2 = 0\ . \tag{3}$$

From (2) we get

$$dt = \frac{rdr}{\pm\sqrt{2hr^2+2\mu r - C^2}}\ , \tag{4}$$

$$d\theta = \frac{(C-\mu r^2)dr}{\pm\sqrt{2hr^2+2\mu r - C^2}}\ . \tag{5}$$

With the help of integrals (2) we will examine two special cases.

Case 1. Let us suppose $h < 0$. The trajectory of P is comprised between two concentric cercles whose centres are at S.

Taking the roots of (3) in the form

$$r_1 = a(1-e), r_2 = a(1+e),\ 0 \le e < 1, a > 0\ ,$$

we find

$$C^2 = \mu a(1-e^2),\ -2h = \frac{\mu}{a} \tag{6}$$

and $r_1 \le r \le r_2$ for $h < 0, C \neq 0$.

The function $r(\cdot)$ is a periodical function and its period is given by

$$T = 2\int_{r_1}^{r_2}\frac{rdr}{\sqrt{2hr^2+2r-C^2}}.$$

If we introduce a new function $\mathbf{u}$ defined by the equation

$$r = a(1 - e\cos u) \tag{7}$$

and we take into account that

$$(r-r_1)(r-r_2) = -a^2e^2(1-\cos^2 u) \tag{8}$$

we have

$$T = \frac{2a}{\sqrt{-2h}}\int_0^{\pi}(1-e\cos u)du = 2\pi\sqrt{\frac{a^3}{\mu}}$$

Therefore the period T has the same expression as in the classic theory.

Taking into account (7), (8), (6), at the last, the equation (4) can also be written in the form

$$(1-e\cos u)du = \sqrt{\frac{\mu}{a^3}}dt \tag{9}$$

Integrating, we have

$$u - e\sin u = \sqrt{\frac{\mu}{a^3}}(t - t_\alpha) \tag{10}$$

and this is the Kepler's equation, which has the same form as in the inertial frame of reference; t_α is a time at which the particle goes through the pericentre (perihelion).

In the last century it has been shown by Le Verrier and by Newcomb that there are differences between the observed and calculated values of the motions of the planetary perihelions (for the perihelion of Mercury this difference is near to 43" per century).

Let us prove that in our problem there exists an advance of pericentre (perihelion).

From the equation (5) we obtain

$$d\theta = \frac{\sqrt{1-e^2}du}{1-e\cos u} - \sqrt{\frac{a^3}{\mu}}(1-e\cos u)du.$$

Supposing $u \in [0, 2\pi)$, the variation $\delta\theta$ of θ is given by

$$\delta\theta = \sqrt{1-e^2}\int_0^{2\pi}\frac{du}{1-e\cos u} - \sqrt{\frac{a^3}{\mu}}\int_0^{2\pi}\omega(1-e\cos u)du - 2\pi.$$

Using (9), we have

$$\delta\theta = -\int_{t_\alpha}^{t_\alpha+T}\omega dt = -\omega(t^*)T\ , t^* \in (t_\alpha, t_\alpha+T). \tag{11}$$

For any rotation of P around S we have (11) but with another t^*.
When $\omega = $ const the formula (11) becomes

$$\delta\theta = -\omega T.$$

Maybe, using the above results we will arrive to explain the advance of perihelions in our solar system, inside of the Newtonian theory of gravitation.

Case 2. If we suppose $h > 0$ the roots of the equations (3) have different signs and can be taken in the form

$$r_1 = a(e-1)\ , r_2 = -a(e+1)\ ,\ e > 1, a > 0.$$

We find

$$2h = \frac{\mu}{a}\ ,\ C^2 = \mu a(e^2 - 1) \tag{12}$$

and $r_1 \leq r < \infty$ for $h > 0, C \neq 0$.
Let u be a new function defined by

$$r = a(e\ \mathrm{ch}u - 1). \tag{13}$$

Then we have

$$2h(r - r_1)(r - r_2) = \mu a e^2 \mathrm{sh}^2 u \tag{14}$$

and from (4) we obtain

$$(e\mathrm{ch}u - 1)du = \sqrt{\frac{\mu}{a^3}}dt. \tag{15}$$

It follows

$$e\mathrm{sh}u - u = \sqrt{\frac{\mu}{a^3}}(t - t_\alpha), \tag{16}$$

where the constant t_α has the same meaning as in (10).
Using (13), (14), the equation (5) can be written

$$d\theta = \frac{Cdu}{\sqrt{a\mu}(e\mathrm{ch}u - 1)} - \sqrt{\frac{a^3}{\mu}}\omega(e\ \mathrm{ch}u - 1)du,$$

and introducing (15) we find

$$d\theta = \frac{\sqrt{e^2 - 1}du}{e\ \mathrm{ch}u - 1} - \omega dt. \tag{17}$$

This last equation may be used in order to determinate the deflection of light at the limb of the Sun.

Some remarks:

- It seems to be, to a certain extent, a resemblance between the experience of F o u c a u l t with his pendulum and the result obtained in *case 1*. Perhaps we may assimilate the solar system by a gigantic Foucault's pendulum. the advance of perihelion may depend on the inclination (noted by i in celestinian mechanics) of the orbital plane of planet to the ecliptic.
- When $\omega = $ const , the advance of perihelion is the same for each planet with same i.
- For the time being, this model of two body problem points out that exists an advance of pericentre (perihelion); but, only a new theory of perturbations for this model can decide if it is useful for the solar system.
- The second term of the right of (17) do not depends of the attractive centre; maybe, using the occultation of a star by the Sun (or by the Moon !?) with (17) we will arrive to determine the distance at the star.
- Until to a complet proof of the application of this model to the solar system, in order to explain its discrepancies, this model remains, beyond any doubt, an expectation.

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