ANALELE STIINTIFICE ALE UNIVERSITATII "AL.I.CUZA" IASI
Tomul XXXVIII, s.I a, Matematica, 1992,f1

EXISTENCE FOR NONLINEAR TIME DEPENDENT VOLTERRA EQUATIONS IN HILBERT SPACES

BY

VELI-MATTI HOKKANEN

1.Introduction. We consider the following nonlinear Volterra equation, for each $t \in [0,T]$.

$$A(t)u(t) + \int_0^t a(t-s)B(s)u(s)ds + \int_0^t b(t-s)C(s)u(s)ds \ni f(t), \tag{1.1}$$

where the values of $u(.)$ belongs to a real reflexive Banach space, $a(.)$ and $b(.)$ are scalar functions, $A(t)$ and $B(t)$ are subdifferentials, $A(t)$ is or is not compact and $C(t)$ is Lipschitzian compact operator. If $a \equiv b \equiv 1$, then the problems (1.1) and (1.2),

$$\frac{d}{dt}A(t)u(t) + B(t)u(t) + C(t)u(t) \ni f'(t), \ \ for\ a.e.\ t \in]0,\infty[\tag{1.2a}$$

$$A(0)u(0) \ni f(0), \tag{1.2b}$$

are equivalent. The classical theory for the Cauchy problem ([2],[6]) can not be applied to the problem (1.2), but there is works on (1.2) in the case where A and B do not depend on time and $C \equiv 0$ ([4] and [9]); in the case where A depend on time and C is not zero, ([5]), and in the case where even B depend on time, ([10]). V.B a r b u has studied the problem (1.1) in the case where A and B do not depend on time and $C \equiv 0$, ([1]).

This work differs from [11] by following closer the methods of B a r b u ([1]) and by including results for the case of degenerated $A(t)$. The problem (1.1) is obtained as the limit $\varepsilon \to 0+$ of the Hilbert space problem where the term $\varepsilon B(0)u(t) + \varepsilon I + A(t)u(t)$ is instead of $A(t)u(t)$; I is the identity of the Hilbert space. The Hilbert space problem is regularized by the Yosida approximate $B_\lambda(t)$ instead of $B(t)$. A corollary of the fixed point theorem implies that this problem has a solution. By taking the limits $\lambda \to 0+$ and $\varepsilon \to 0+$ and utilizing the maximal monotocity and compacteness we shall prove that the problem (1.1) has a solution.

In Chapter 2 there is some useful lemmas. The hypotheses and the theorems are presented in Chapter 3; the proofs can be found in Chapters 4, 5, 6 and 7. In Chapter 8 there is the remarks on concrete applications of the problem (1.1).

2.Some useful lemmas. We denote by X a real Banach space and by X^* its dual. If $x \in X$ and $x^* \in X$, then $\langle x, x^* \rangle_X = x^*(x)$. Let H be a real Hilbert space with the scalar product $\langle .,. \rangle_H$. The range of operator A or function f is denoted by $R(A)$ or $R(f)$, respectively.

Let $S \subset \mathbf{R}$ be an interval, $T > 0, p \in [1, \infty]$ and $k \in \mathbf{N}$. We utilize the ordinary spaces $C(S;X), C_0^\infty(S;X), L^p(S;X), L^p(0,T;X)$ and $L^p_{loc}(S;X))$ and the spaces

$$W^{k,p}(S;X) = \{f \in L^p(S;X) \mid \textit{distrib. derivatives } f', f'', ..., f^{(k)} \in L^p(S;X)\},$$

$$\hat{W}^{k,p}(S;X) = \{f \in W^{k,p}(S;X) \mid f^{(k-1)} \textit{ is absolutely continuous}\},$$

and $H^k(S;X) = \hat{W}^{k,2}(S;X)$.

Lemma 2.1. *Let $a \in \hat{W}^{1,1}_{loc}([0,\infty[;\mathbf{R})$ and $w \in L^1_{loc}([0,\infty[;X)$. If we denote*

$$a * w(t) = \int_0^t a(t-s)w(s)ds \textit{ and } F(t) = \int_0^t w(s)ds \ \forall\, t \geq 0,$$

then

$$a * w(t) = a' * F(t) + a(0)F(t) \ \forall\, t \geq 0.$$

Proof. It suffices to integrate by parts.

Lemma 2.2 *Let $a \in H^1_{loc}([0,\infty[;\mathbf{R})$ and $w \in L^2_{loc}([0,\infty[;X)$. Then $a * w \in H^1_{loc}([0,\infty[;X)$ and*

$$\frac{d}{dt} a * w(t) = a' * w(t) + a(0)w(t) \textit{ a.e. on } [0,\infty[.$$

Proof. For each $T > 0$,

$$\lim_{h \to 0+} \int_0^{T-h} \left\|\frac{a * w(t+h) - a * w(t)}{h} - a' * w(t) - a(0)w(t)\right\|_X^2 dt = 0$$

by [6,pp. 143, 145, 154]. Hence, by [6, p.154], we obtain Lemma 2.2.

Lemma 2.3 *Let $M, T \in]0,\infty[, a_1, a_2 \in L^2(0,T)$ and $K_1, K_2 : [0,T] \times X \mapsto X$ be such that*

$$\|K_i(t)x - K_i(t)y\|_X \leq M\|x - y\|_X \ \forall\, t \in [0,T], x, y \in X, i = 1,2. \tag{2.1}$$

$$t \mapsto K_i(t)x \textit{ belongs to } L^2(0,T;X) \ \forall\, x \in X, i = 1,2. \tag{2.2}$$

If $f \in L^2(0,T;X)$, then there exists unique $y \in L^2(0,T;X)$ which satisfies

$$y(t) + a_1 * K_1 y(t) + a_2 * K_2 y(t) = f(t) \textit{ a.e. on } [0,T]. \tag{2.3}$$

Proof: It suffices to apply the fixed point theorem on the subintervals of [0,T].

Lemma 2.4. *Let $a \in C([0,\infty[)$ such that $a(0) > 0$ and a' is of bounded variation on $[0,\infty[$. If $T > 0$ and $d_1, d_2 \in L^\infty(0,T)$ are increasing and nonnegative, then there exists $K_T > 0$ such that every $v \in L^2(0,T;H)$, satifying*

$$\int_0^t \langle a * v(s), v(s)\rangle_H ds \leq d_1(t) + \tag{2.4}$$

$$+ d_2(t) \max_{0 \leq s \leq t} \left\|\int_0^s v(\tau)d\tau\right\|_H \ \forall t \in [0,T],$$

satisfies also the inequality

$$\left\|\int_0^t v(s)ds\right\|_H \leq K_T(\sqrt{d_1(t)} + d_2(t)), \textit{ for each } t \in [0,T]. \tag{2.5}$$

Proof. Lemma 2.4 follows from [7, p. 703-704].

Lemma 2.5. *(Gronwall's inequality). Let $T \in]0,\infty], m \in L^1(0,T)$ with $m \geq 0$, and $d \in L^\infty(0,T)$ increasing such that $d \geq 0$. If $\phi \in L^\infty(0,T)$ satisfies*

$$\phi(t) \leq d(t) + \int_0^t m(s)\phi(s)ds, \textit{ for a.e. } t \in]0,T[, \tag{2.6}$$

then

$$\phi(t) \leq d(t) \exp \int_0^t m(s)ds, \textit{ for a.e. } t \in]0,T[. \tag{2.7}$$

Proof. This form of Gronwall's inequality follows from [6,p.156].

Lemma 2.6. ([2,p.42]) *Let $A \subset X \times X^*$ be a maximal monotone operator in X. If there exists a sequence $((x_n, y_n))$ satisfying $(x_n, y_n) \in A, x_n \to x$ weakly in $X, y_n \to y$ weakly star in X^*, as $n \to \infty$, and $\lim \inf_{n\to\infty} \langle x_n, y_n\rangle_W \leq \langle x, y\rangle_W$, then $(x,y) \in A$.*

3.The hypotheses and the theorems. For our first two theorems, let us list the following hypotheses. Let $M, T, \alpha \in]0,\infty[, g \in C(\mathbf{R})$ increasing, $\eta_1 \in L^1(0,T)$ and $\eta_2 \in L^2(0,T)$ positive. For the convenience, when we mean constants $M_1, M_2, ...$ or functions $\eta_{1,1}, \eta_{1,2}, ...$ and so on, we shortly write M, η_1, η_2 or g.

Let V and W be real reflexive Banach space and H be a real Hilbert space such that

$$V \hookrightarrow W \hookrightarrow H = H^* \hookrightarrow W^* \hookrightarrow V^*, \tag{3.1}$$

by linear continuous injections.

If B is maximal monotone operator in W, we define maximal monotone operator B_H in H as the maximal monotone continuation in H of the monotone operator $\hat{B}$, defined by $\hat{B}x = Bx \cap H$, $\forall\, x \in D(B)$, [6, p.23]. Thus one can define the Yosida approximate of B_H which even approximates B.

The functions $\phi : [0,T] \times V \to]-\infty, \infty]$ and $\psi : [0,T] \times W \to \mathbf{R}$ satisfy the conditions

(3.2) *$\phi(t,.)$ and $\psi(t,.)$ are proper convex and lower semicontionuous (l.s.c), for each $t \in [0,T]$;*

(3.3) *for a.e. $t \in [0,T]$, $\mathbf{dom}\,\phi(t,.)$ does not depend on t, $0 \in \mathbf{dom}\,\phi(t,.)$ and $\phi_t(t,x)$ exists, for each $x \in \mathbf{dom}\,\phi(t,.)$;*

(3.4) *for a.e. $t \in]0,T[$ and for each $x, y \in \mathbf{dom}\,\phi(t,.) \in L^1(0,T)$ and $| \phi_t(t,x) - \phi_t(t,y) | \leq \eta_1(t) g(\|x\|_V + \|y\|_V) \|x-y\|_V$;*

(3.5) *for each $\delta \in \mathbf{R}$, there is $\eta \in L^2(0,T)$ such that $\|x\|_W \leq \eta(t) + g(\|x\|_H)$ if $\psi(x,t) \leq \delta$;*

(3.6) *$t \mapsto \psi(t,x)$ is measurable, for each $x \in W$.*

Moreover, the operators $A(t) = \partial\phi(t,.)$ and $B(t) = \partial\psi(t,.)$ satisfy:

(3.7) *$\langle x_1 - x_2, y_1 - y_2 \rangle_{V \times V^*} \geq \alpha \|x_1 - x_2\|_V^2$, for each $(x_1,y_1),(x_2,y_2) \in A(t), t \in [0,T]$;*

(3.8) *$R((I + B_H(.))^{-1} S)$ is separable in W, for each $S \subset V$ bounded, (I is the identity of H);*

(3.9) *$\langle y, B_\lambda(t)x \rangle_H \geq -\eta_1(t)(1 + \|x\|_V^2)$, for each $(x,y) \in A(t) \cap (H \times H)$ and $t \in [0,T]$; $B_\lambda(t)$ is the Yosida approximate of $B_H(t)$;*

(3.10) *there is $w^* \in L^2(0,T;H)$ such that $w^*(t) \in B_H(t)0$, for a.e. $t \in]0,T[$.*

On the operators $C(t) : V \mapsto H, t \in [0,T]$, we assume that

(3.11) *$t \mapsto C(t)x$ is measurable in H and $\|C(t)x\|_H \leq \eta_2(t) + M\|x\|_V$, for each $x \in V$;*

(3.12) *$\|C(t)x - C(t)y\|_{V^*} \leq M\|x-y\|_V$, for each $x,y \in V$.*

Theorem 3.1 *If, in addition to the hypotheses above, there exist $a, b \in H^1(0,T)$ such that $a'' \in (0,T)$ and $a(0) > 0$, and $f \in H^1(0,T;H)$ and if it holds that*

(3.13) *$V \hookrightarrow W$ compactly, and*

(3.14) *for each $x \in W$ and for a.e. $t \in]0,T[, \psi(t,x) \leq \eta_2(t) g(\|x\|_W)$,*

then there exist $u \in H^1(0,T;V), v \in H^1(0,T;W^*)$ and $w \in L^2(0,T;W^*)$ satisfying

$$v(t) + a * w(t) + b * Cu(t) = f(t), \quad \text{for each } t \in [0,T], \tag{3.15}$$

$$v(t) \in A(t)u(t), \quad w(t) \in B(t)u(t), \quad \text{for a.e. } t \in]0,T[. \tag{3.16}$$

Theorem 3.2 *If, in addition to the hypotheses (3.1)-(3.12) with $V = W = H$, there exist $a, b \in H^1(0,T)$ such that $a''' \in L^1(0,T), a(0) > 0$ and $a'(0) \leq 0$, and $f \in H^1(0,T;H)$, and if it holds that*

(3.17) *for each $\delta > 0$ the set $\{u \in H \mid \phi(t,u) \leq \delta, 0 \leq t \leq T\}$ is compact in H and*

(3.18) *for each $t \in [0,T], D(A(t)) \subset D(B(t))$ and for each $u \in D(B(t))$ there is $r_{u,t} > 0$ and $g_{u,t} \in C(\mathbf{R})$ increasing such that*

$$\|B^0(t)x\|_H \leq g_{u,t}(\|A^0(t)x\|_H) \ \forall\, x \in \{y \in D(A(t)) \mid \|y-u\|_H \leq r_{u,t}\}$$

then there exist $u, v \in H^1(0,T;H)$ and $w \in L^2(0,T;H)$ satisfying the equations (3.15) and (3.16).

In the case where V, W and H are Sobolev spaces on the domain $\Omega \subset \mathbf{R}^n$ and $A(t)$ and $B(t)$ are partial differential operators, Theorems 3.1 and 3.2 roughly correspond the situation where the order of $A(t)$ is greater than the order of $B(t)$. For the case where $B(t)$ dominates $A(t)$, i.e. the order of $B(t)$ is greater than the order of $A(t)$, we formulate Theorems 3.3, 3.4 and 3.5. Let us list their hypotheses.

Let W be a real reflexive Banach space and H a real Hilbert space such that

$$W \hookrightarrow H = H^* \hookrightarrow W^* \tag{3.19}$$

compactly and densely, by linear injections.

The functions $\phi : [0,T] \times H \mapsto]-\infty, \infty]$ and $\psi : [0,T] \times W \mapsto \mathbf{R}$ satisfy , for each $t \in [0,T]$:

(3.20) *$\phi(t,.)$ is convex proper l.s.c. and $\psi(t,.)$ is convex continuous.*

(3.21) *$0 \in \mathbf{int\ dom}\,\phi(0,.), \mathbf{dom}\,\phi(t,.)$ does not depend on t and $\phi_t(t,x)$ exists for each $x \in \mathbf{dom}\phi(t,.)$*

(3.22) *$\phi_t(t,0) \in L^1(0,T)$ and, for each $x, y \in \mathbf{dom}\,\phi(t,.), | \phi_t(t,x) - \phi_t(t,y) | \leq \eta_1(t) g(\|x\|_H + \|y\|_H)\|x-y\|_H$.*

(3.23) *$\psi(t,x) \geq \alpha\|x\|_W^2 - M\|x\|_H^2 - M$* for each *$x \in W$.*

(3.24) *$t \mapsto \psi(t,x)$ is measurable, for each $x \in W$.*

Moreover, the operators $A(t) = \partial\phi(t,.), B(t) = \partial\psi(t,.), C_\lambda(t) : H \mapsto H$ and $C(t) : W \mapsto H$ satisfy, for each $\lambda \in]0,1]$ and $t \in [0,T]$:

(3.25) *$R((I + \lambda B_H(.))^{-1} S)$ is separable in S, for each $S \subset H$ bounded.*

(3.26) *$\langle y, B_\lambda(t)x \rangle_H \geq -\eta_1(t)(1 + \|x\|_H^2)$, for each $(x,y) \in A(t)$.*

(3.27) *$\langle B_\lambda(t)x, y \rangle_H \geq -\eta_1(t)(1 + \|x\|_H^2)$, for each $(x,y) \in B_H(0)$, and $\langle y, z \rangle_H \geq -\eta_1(t) - M\|x\|_W^2$, for each $(x,z) \in B_H(t), y \in A(t)x + \lambda B_H(0)x$.*

(3.28) *$\langle x_1 - x_2, y_1 - y_2 \rangle_H \geq \alpha\|x_1 - x_2\|_H^2$, for each $x \in H$ and $(x_1,y_1),(x_2,y_2) \in A(t)$*

(3.29) *$\langle x_1 - x_2, y_1 - y_2 \rangle_W \geq \alpha\|x_1 - x_2\|_W^2 - M\|x_1 - x_2\|_H^2$, for each (x_1,y_1) $(x_2,y_2) \in B(0)$.*

(3.30) *$D(B_H(0)) \subset D(B_H(t))$ and $\|B_H^0(t)x\|_H \leq g(\|B_H^0(0)x\|_H)$, for each $x \in D(B_H(0))$.*

(3.31) *$t \mapsto C_\lambda(t)x$ is measurable in H and $\|C_\lambda(t)x\|_H \leq \eta_2(t) + M_\lambda\|x\|_H$ for each $x \in H$.*

(3.32) *$\|C_\lambda(t)x - C_\lambda(t)y\|_H \leq M_\lambda\|x-y\|_H$, for each $x,y \in H$, where $M_\lambda \in \mathbf{R}$.*

(3.33) *$\|C_\lambda(t)x\|_H \leq \eta_2(t) + M\|x\|_W$, for each $x \in W$.*

(3.34) *If (x_{λ_n}) converges weakly to x in $L^2(0,T;W)$ and strongly in $L^2(0,T;H)$, then it has a subsequence $(x_{\lambda_n'})$ such that $C_{\lambda_n'}x_{\lambda_n'}$ converges to Cx weakly in $L^2(0,T;W^*)$.*

Theorem 3.3 *If the hypotheses (3.19) - (3.34) are valid and if there is $a, b \in H^1(0,T)$ and $f \in H^1(0,T;H)$ satisfying $a(0) > 0, a'(0) \le 0, a''' \in L^1(0,T)$, and if*

$$\|y\|_{W^*} \le \eta_2(t) + M\|y\|_W, \text{ for each } (x,y) \in B(t),\ t \in [0,T], \tag{3.35}$$

then there exist $u \in L^2(0,T;W)$, $v \in H^1(0,T;W^*) \cap C(0,T;W)$ and $w \in L^2(0,T;W^*)$ which satisfy the equations

$$v(t) + a * w(t) + b * Cu(t) = f(t), \text{ for each } t \in [0,T] \tag{3.36}$$

$$u(t) \in A(t)u(t), w(t) \in B(t)u(t), \text{ for a.e. } t \in]0,T[\tag{3.37}$$

The assumption (3.35) on the affine boundness of $B(t)$ can be avoided:

Theorem 3.4. *If the hypotheses (3.19)-(3.34) are valid, if $a, b \in H^1(0,T)$ with $a(0) > 0, a'(0) \le 0, a''' \in L^1(0,T), f \in H^1(0,T;H)$ and*

(3.38) *For each $\delta \in]0,\infty[$, there is $\eta \in L^2(0,T)$ everywhere finite such that $\|y\|_{W^*} \le \eta(t)$ if $\|x\|_W \le \delta$ and $(x,y) \in B(t)$ and $t \in [0,T]$;*

(3.39) *if $y \in H^1(0,T;H), x \in H^1(0,T;H)$ and $y(t) \in A(t)x(t)$ a.e. on $[0,T]$, then $\langle x'(t), y'(t)\rangle_H \ge \alpha\|x'(t)\|_H^2 - \eta_1(t)\|x(t)\|_W^2 - \eta_1(t)$ a.e. on $[0,T]$;*

(3.40) *$\psi(.,x) \in W^{1,1}(0,T)$ and $\psi_t(t,x) \le \eta_1(t)(1 + \|x\|_W^2)$, for each $x \in W$ and a.e. $t \in]0,T[$;*

(3.41) *$f(0) \in A(0)W$;*

then there exist $u \in L^\infty(0,T;W), v \in H^1(0,T;W^*) \cup C(0,T;H)$ and $w \in L^2(0,T;W^*)$ which satisfy the equations (3.36)- (3.37).

For the case of degenerate $A(t)$ we state

Theorem 3.5 *If the conditions of Theorem 3.3 are satisfied, maybe except (3.28), and if it holds, for each $t \in [0,T]$,*

(3.42) *$b \in H^2(0,T)$ and $b(0) = 0$,*

(3.43) *if $x_n, x \in L^2(0,T;W), n = 1,2,\dots$, satisfy, $x_n \to x$ weakly in $L^2(0,T;W)$, as $n \to \infty$ and $x_n(t), x(t) \in \mathrm{dom}\phi(t,.)$, for a.e. $t \in]0,T[$, then, for a subsequence,*

$$\liminf_{n\to\infty} \int_0^T \phi_t(t, x_n(t))dt \ge \int_0^T \phi_t(t, x(t))dt,$$

(3.44) *$\langle x, y\rangle_W \ge \alpha\|x\|_W^2 - M$ for each $(x,y) \in B(t)$.*

(3.45) *the conditions (3.34) holds without the strong limit in $L^2(0,T;H)$,*

then there exist $u \in L^2(0,T;W), v \in H^1(0,T;W^*) \cap C(0,T;H)$ and $w \in L^2(0,T;W^*)$ which satisfy the equations (3.36)-(3.37).

4.The proof of Theorem 3.1. The following Lemmas are valid

Lemma 4.1. *We can assume that $\psi(t,0) = 0$ and $0 \in B_H(t)0$, for each $t \in [0,T]$.*

Proof: Define $\tilde\psi(t,x) = \psi(t,x) + \psi(t,0) - \langle x, w^*(t)\rangle_W, \tilde f(t) = f(t) - b * w^*(t)$.

Lemma 4.2 *The operators $A^{-1}(t) : V^* \mapsto V$ and $B_\lambda(t) : H \mapsto H, t \in [0,T]$, are Lipschitzian with constants $\frac{1}{\alpha}$ and $\frac{1}{\lambda}$, respectively.*

Proof. By [2, p.28] $B_\lambda(t)$ is Lipschitzian. The Lipschitzianity of $A^{-1}(t)$ follows from the strong monotonicity of $A(t)$.

Lemma 4.3. *For each $t \in [0,T], D(B(t)) = W$.*

Proof: By (3.14) and Lemma 4.1

$$\|B(t)x\|_{W^*} = \sup_{\|\xi\|_W \le 1} \langle \xi, B(t)x\rangle_W \le \sup_{\|\xi\|_W \le 1} (\psi(t, \xi + x) - \psi(t,x)) \le \eta_2(t)M,$$

if $\|x\|_W \le M$. Hence $B(t)$ is bounded and thus $D(B(t)) = W$.

Lemma 4.4. *The operators*
$A = \{(x,y) \in L^2(0,T;V) \times L^2(0,T;V^*) \mid (x(t), y(t)) \in A(t)$ a.e. on$[0,T]\}$,
$B = \{(x,y) \in L^2(0,T;W) \times L^2(0,T;W^*) \mid (x(t), y(t)) \in B(t)$ a.e. on$[0,T]\}$,
are maximal monotone in $L^2(0,T;V)$ and $L^2(0,T;W)$ respectively.

Proof: The monotonicity of A and B is clear.

By the Asplund's renorming theorem [2, p. 14] there exist equivalent norms to $\|.\|_V$ and $\|.\|_{V^*}$ with which V and V^* are strictly convex. Thus we may assume that V and V^* are strictly convex.

Let us prove that A is *m.m.o.* By [2, p.39] it suffices to show that $R(F_V + A) = L^2(0,T;V^*)$ where $F_V :\mapsto V^*$ is the duality mapping of V.

Let $y \in L^2(0,T;V^*)$. Define $x(t) = (F_V + A(t))^{-1}y(t)$. By the strong monotonicity of $A(t), (F_V + A(t))^{-1}$ is Lipschitzian with the constant $\frac{1}{\alpha}$. By [2, p.39], $R(F_V + A(t)) = V^*$. Thus $x(t)$ is well define and $y(t) \in (F_V + A(t))x(t)$. The function $\tilde\phi$, defined by $\tilde\phi(t,x) = \frac{1}{2}\|x\|_V^2 + \phi(t,x)$ satisfies the conditions of Lemma 4.6, and $(F_V + A(t))^{-1} = \partial\tilde\phi^*(t,.)$. Thus $(F_V + A(.))^{-1}z \in \hat W^{1,2}(0,T;V) \subset L^2(0,T;V)$ *for all $z \in V^*$*. By the continuity of $(F_V + A(t))^{-1}$, we conclude that x is measurable in V. Finally, $x \in L^2(0,T;V)$, since

$$\|x(t)\|_V \le \frac{1}{\alpha}\|y(t)\|_{V^*} + \|(F_V + A(t))^{-1}0\|_{V^*} \le \eta_2(t).$$

Hence A is *m.m.o.*

By Lemma 4.3 $B(t)$ is bounded and everywhere defined. Hence its maximal monotonicity can be proved as the maximal monotonicity of G in [11, Lemma 4.4].

Lemma 4.5. *If $y \in H^1(0,T;V^*)$ and $(x,y) \in A$, then $\phi^*(.,y(.)) \in W^{1,1}(0,T)$ and*

$$\frac{d}{dt}\phi^*(t, y(t)) = -\phi_t(t, x(t)) + \langle x(t), y'(t)\rangle_V \text{ , for a.e. } t \in]0,T[. \tag{4.1}$$

Proof: The proof is the same as the proof of Lemma 4.7 in [11]. Notice that the assumption on the time independence of **dom** $\phi(t,.)$ is essential for the proof.

Lemma 4.6. *If $z \in \hat{W}^{1,p}(0,T;V^*)$ and $p=1,2$, then $t \mapsto A(t)^{-1}z(t)$ belongs to $W^{1,p}(0,T;V)$.*

Proof: $\phi^*(t,.)$ is finite and continuous, by the continuity of $A(t)^{-1}$, since $\phi^*(t,x) = \langle A(t)^{-1}x, x\rangle_V - \phi(t, A(t)^{-1}x)$ and $\phi^*(t,.)$ are *l.s.c.*. Otherwise the proof is same as the proof of Lemma 4.6 in [11].

Lemma 4.7. *If $\psi_H(t,.): H \mapsto]-\infty,\infty]$ is defined by*

$$\psi_H(t,x) = \begin{cases} \psi(t,x) & , \text{ if } x \in W, \\ \infty & , \text{ otherwise} \end{cases} \tag{4.2}$$

then $\psi_H(t,.)$ is proper convex l.s.c., and $\partial\psi_H(t,.) = B_H(t)$, for each $t \in [0,T]$.

Proof: Let $t \in [0,T]$. Clearly $\psi_H(t,.)$ is proper and convex. Let $\lambda \in \mathbf{R}$, $x_n \in S_\lambda, n = 1,2,\ldots, S_\lambda = \{y \in H \mid \psi_H(t,y) \le \lambda\}$ and $x \in H$ be such that $\|x_n - x\|_H \to 0$, as $n \to \infty$. Then $x_n \in W$ and (x_n) is bounded in W by (3.5). Thus, for a subsequence, $x_n \to \tilde{x}$ weakly in $W, \tilde{x} \in W$. But $\tilde{x} = x$ by the embedding. Since $\psi(t,.)$ is *l.s.c.* even in the weak topology of W,

$$\psi_H(t,x) = \psi(t,\tilde{x}) \le \liminf_{n\to\infty} \psi(t,x_n) \le \lambda.$$

Thus $x \in S_\lambda$, and S_λ is closed. Hence $\psi_H(t,.)$ is *l.s.c.*

Let $(x,y) \in \partial\psi_H(t,.)$. Then $(x,y) \in \partial\psi(t,.)$ and $y \in H$. Thus $(x,y) \in B_H(t)$. Hence $\partial\psi_H(t,.) \subset B_H(t)$. Since $\partial\psi_H(t,.)$ is *m.m.o* as the subdifferential of a proper *l.s.c.* function, we conclude the equality.

Let $\lambda > 0$ and consider the problem

$$v_\lambda(t) + a * B_\lambda u_\lambda(t) + b * Cu_\lambda(t) = f(t), \text{ for each } t \in [0,T]. \tag{4.3}$$

$$v_\lambda(t) \in A(t)u_\lambda(t), \text{ for each } t \in [0,T]. \tag{4.4}$$

By the following lemma and Lemma 2.3 there is $u_\lambda \in L^2(0,T;V), v_\lambda \in L^2(0,T;V^*)$ satifying (4.3)-(4.4). Moreover, by (3.11), $v_\lambda \in H^1(0,T;H)$.

Lemma 4.8. *The operators $K_1(t) = B_\lambda(t)A(t)^{-1}, K_2(t) = C(t)A(t)^{-1}$: $V^* \mapsto V^*$ satisfy the assumptions of Lemma 2.3.*

Proof: Lemma 4.2 and (3.12) imply the Lipschitzianity of $K_1(t)$ and $K_2(t)$. Let $x \in V$. By Lemma 4.6 and (3.11) $t \mapsto K_2(t)x$ belongs to $L^2(0,T;V^*)$.

Let $x \in V$. In order to show that $t \mapsto K_1(t)x$ is measurable in V^* it suffices to show that $t \mapsto J_\lambda(t)y$ is measurable in W^* for each $y \in V$ since $A(.)^{-1}x$ is continuous in V by Lemma 4.6 and $B_\lambda(t)$ is continuous in H, and since $\lambda B_\lambda(t) = I - J_\lambda(t)$. But $J_\lambda(t)y$ is the Frechet differential of

$$z \mapsto \sup_{\xi\in H}(\langle\xi,z\rangle_H - \frac{1}{2}\|\xi\|_H^2 - \lambda\psi_H(t,\xi)) = \sup_{\xi\in W}(\langle\xi,z\rangle_H - \frac{1}{2}\|\xi\|_H^2 - \lambda\psi(t,\xi)),$$

which is measurable as the supremum of measurable functions. Hence $J_\lambda(.)y$ is weakly measurable in W^*. By Pettis'theorem, $J_\lambda(.)y$ is measurable in W^*, since

$$\|J_\lambda(t)y\|_H \le \|J_\lambda(t)y - J_\lambda(t)(I+\lambda B_H(t))0\|_H \le \|y-0\|_H = M_y,$$

and so $R(J_\lambda(.)y)$ is in a compact set of W^*. Finally, $\|K_1(.)x\|_{V^*} \in L^2(0,T)$, since by Lemma 4.6

$$\|K_1(t)x\|_{V^*} \le \|B_\lambda(t)A(t)^{-1}x - B_\lambda(t)0\|_H \le \frac{1}{\lambda}\|A(t)^{-1}0\|_H \le M_{\lambda,x}.$$

Lemma 4.9. *There is a constant $M \in]0,\infty[$ independent on λ satisfying*

(a) $\|u_\lambda\|_{L^\infty(0,T;V)} \le M$, *(b)* $\|J_\lambda u_\lambda\|_{L^2(0,T;W)} \le M$,

(c) $\|v_\lambda\|_{L^\infty(0,T;H)} \le M$, *(d)* $\|v'_\lambda\|_{L^2(0,T;W^*)} \le M$,

(e) $\|B_\lambda u_\lambda\|_{L^2(0,T;W^*)} \le M$, *(f)* $\langle B_\lambda u_\lambda, v'_\lambda\rangle_{L^2(0,T;H)} \le M$,

Proof: Denote

$$F_\lambda(t) = \int_0^t B_\lambda(s)u_\lambda(s)ds. \tag{4.5}$$

The equation (4.3) implies that

$$\int_0^t \langle a * B_\lambda(s)u_\lambda(s), B_\lambda(s)u_\lambda(s)\rangle_H ds =$$

$$= \int_0^t \langle f(s) - v_\lambda(s) - b * Cu_\lambda(s), B_\lambda(s)u_\lambda(s)\rangle_H ds$$

$$\le_{(i)} \Big(\|f(t)\|_H + \|b * Cu_\lambda(t)\|_H + \|f'\|_{L^1(0,T;H)}$$

$$+ \mid b(0) \mid \int_0^t \|C(s)u_\lambda(s)\|_H ds + \int_0^t \| \int_0^s b'(s-\tau)C(\tau)u_\lambda(\tau)d\tau\|_H ds\Big)$$

$$\times \max_{s\in[0,t]} \|F_\lambda(s)\|_H + \int_0^t \eta_1(s)(1+\|u_\lambda(s)\|_V^2)ds \le \max_{s\in[0,t]}\|F_\lambda(s)\|_H$$

$$\times(M + M(\int_0^t \|u_\lambda(s)\|_V^2 ds)^{\frac{1}{2}}) + M + \int_0^t \eta_1(s)\|u_\lambda(s)\|_V^2 ds,$$

where (3.11) and the Hölder's inequality are exploited; in the step (i) we integrated by parts and applied (3.9). Hence it follows from Lemma 2.4 that, for each $t \in [0,T]$ and $\lambda > 0$

$$\|F_\lambda(t)\|_H \le M\Big(1 + (\int_0^t \eta_1(s)\|u_\lambda(s)\|_V^2 ds)^{\frac{1}{2}}\Big). \tag{4.5}$$

The equations (4.3) implies also that

$$\langle u_\lambda(t), v_\lambda(t)\rangle_V = \langle u_\lambda(t), f(t) - a * B_\lambda u_\lambda(t) - b * Cu_\lambda(t)\rangle_H \leq$$

$$\leq \Big(M + (\|a'\|_{L^2(0,T)} + a(0))\|F_\lambda\|_{L^\infty(0,T;H)} +$$

$$(4.6) \qquad + \|b\|_{L^2(0,T)}\Big(\int_0^t \|C(s)u_\lambda(s)\|_H^2 ds\Big)^{\frac{1}{2}}\Big)\|u_\lambda(t)\|_H \leq$$

$$\leq \Big(M + \Big(\int_0^t \eta_1(s)\|u_\lambda(s)\|_V^2 ds\Big)^{\frac{1}{2}}\Big)\|u_\lambda(t)\|_H,$$

where (3.11) and (4.5) are applied. By Lemma 4.2

$$\|u_\lambda(t)\|_V \leq \|u_\lambda(t) - A(t)^{-1}0\|_V + \|A(t)^{-1}0\|_V.$$

Thus by Lemma 4.6 and (3.7)

$$(4.7) \qquad \|u_\lambda(t)\|_V^2 \leq \frac{1}{\alpha}\langle u_\lambda(t), v_\lambda(t)\rangle_V + M\|v_\lambda(t)\|_V + M.$$

It follows from (4.3) and (4.5) that v_λ is measurable in H and

$$\|v_\lambda(t)\|_H \leq \|f(t)\|_H + \|b\|_{L^\infty(0,T)}\|Cu_\lambda\|_{L^1(0,T;H)} +$$

$$+ \|a' * F_\lambda(t) + a(0)F_\lambda(t)\|_H \leq M + M\int_0^t \|u_\lambda(s)\|_V ds +$$

$$+ \|a'\|_{L^2(0,T)}\|F_\lambda\|_{L^2(0,T;H)} + a(0)\|F_\lambda(t)\|_H \leq$$

$$\leq M + M\Big(\int_0^t \eta_1(s)\|u_\lambda(s)\|_V^2 ds\Big)^{\frac{1}{2}}.$$

Thus we obtain from (4.6), (4.7) and Gronwall's inequality that (a) holds. The inequality (c) follows from (4.3), (4.5), (3.11) and (a).

By [3, p.121], (a), (b) and Lemma 4.1

$$\psi_H(t, J_\lambda(t)u_\lambda(t)) \leq \psi_H(t, u_\lambda(t)) \leq \psi_H(t,0) + \langle u_\lambda(t), v_\lambda(t)\rangle_H \leq M.$$

Thus, by (3.5) and (a), we obtain that $J_\lambda(t)u_\lambda(t) \in W$ and $\|J_\lambda(t)u_\lambda(t)\|_W \leq \eta_2(t)$. We prove above that $J_\lambda(.)u_\lambda(.)$ is measurable in H. Thus it is weakly measurable in W, by Pettis' theorem and the dense embedding $H \hookrightarrow W^*$. By (3.8) and Pettis'

theorem $J_\lambda(.)u_\lambda(.)$ is measurable in W. Hence $J_\lambda(.)u_\lambda(.) \in L^2(0,T;W)$ and (b) holds.

The inequality (e) is proved by [3, p.121], (a), (3.14) and by the calculation

$$\|B_\lambda(t)u_\lambda(t)\|_{W^*} = \sup_{\|\xi\|_W \leq 1} \langle \xi, B_\lambda(t)u_\lambda(t)\rangle_W = \sup_{\|\xi\|_W \leq 1} \langle \xi, B_\lambda(t)u_\lambda(t)\rangle_H \leq$$

$$\leq \sup_{\|\xi\|_W \leq 1}\Big(\psi_\lambda(t, \xi + u_\lambda(t)) - \psi_\lambda(t, u_\lambda(t))\Big) \leq$$

$$\leq \sup_{\|\xi\|_W \leq 1}\Big(\psi_H(t, \xi + u_\lambda(t)) - \psi_H(t, J_\lambda(t))u_\lambda(t))\Big) \leq \eta_2(t) - 0,$$

where Lemma 4.1 is applied. Notice that $B_\lambda(.)u_\lambda(.)$ is measurable in W^*, by the proof of Lemma 4.8. By differentiating (4.3) we obtain thus

$$\|v'_\lambda(t)\|_{W^*} \leq \|f'(t)\|_{W^*} + \|a'\|_{L^2(0,T)}\|B_\lambda u_\lambda\|_{L^2(0,T;W^*)} +$$

$$+ a(0)\|B_\lambda(t)u_\lambda(t)\|_{W^*} + \|b'\|_{L^2(0,T)}\|Cu_\lambda\|_{L^2(0,T;W^*)} +$$

$$+ \mid b(0) \mid (\eta_2(t) + M\|u_\lambda(t)\|_V) \leq \eta_2(t).$$

where (a) and (3.11) are applied. Thus (d) holds.

By differentiating (4.3) we see also that

$$\|v'_\lambda(t)\|_H^2 + a(0)\langle B_\lambda(t)u_\lambda(t), v'_\lambda(t)\rangle_H =$$

$$= \langle f'(t) - a'' * F_\lambda(t) - a'(0)F_\lambda(t) - b' * Cu_\lambda(t) -$$

$$- b(0)C(t)u_\lambda(t), v'_\lambda(t)\rangle_H \leq \eta_1(t) + \frac{1}{2}\|v'_\lambda(t)\|_H^2,$$

where (a), (4.5) and (3.11) are exploited. Thus we conclude by (d) the estimate (f) since $B_\lambda(.)u_\lambda(.)$ is measurable in H, and hence $v'_\lambda(.)$ is measurable in H.

Lemma 4.10. *There exist subsequences and functions $u \in C(0,T;V), v \in C(0,T;V^*)$ and $w \in L^2(0,T;W^*)$ such that, as $\lambda \to 0+$, for a subsequence,*

(4.8) $\quad v_\lambda \to v$ *uniformly in V^* on $[0,T]$,*
(4.9) $\quad u_\lambda \to u$ *uniformly in v on $[0,T]$,*
(4.10) $\quad J_\lambda u_\lambda \to u$ *weakly in $L^2(0,T;W)$;*
(4.11) $\quad B_\lambda u_\lambda \to w$ *weakly in $L^2(0,T;W^*)$;*
(4.12) $\quad J_\lambda u_\lambda \to u$ *in $L^2(0,T;H)$;*
(4.13) $\quad Cu_\lambda \to Cu$ *uniformly in V^* on $[0,T]$.*

Hence the limits u, v and w satisfy the equation (3.15). By $v \in H^1(0,T;W^*)$ and Lemma 4.6, $u \in H^1(0,T;V)$. Since A is $m.m.o.$ in $L^2(0,T;V)$ and $v_\lambda \in Au_\lambda$, we obtain from (4.8), (4.9) and Lemma 2.6 that $v \in Au$.

By (f) of Lemma 4.9 and (4.3), we write

$$a(0)\int_0^T \langle J_\lambda(t)u_\lambda(t), B_\lambda(t)u_\lambda(t)\rangle_W dt = -\int_0^T \langle J_\lambda(t)u_\lambda(t), v'_\lambda(t)\rangle_W dt+$$

$$+\int_0^T \langle J_\lambda(t)u_\lambda(t), f'(t) - a' * B_\lambda(t)u_\lambda(t) - b' * Cu_\lambda(t) - b(0)C(t)u_\lambda(t)\rangle_W dt =$$

$$= \langle J_\lambda u_\lambda, f'\rangle_{L^2(0,T;W)} + \langle J_\lambda u_\lambda - u, a'' * F_\lambda + a'(0)F_\lambda\rangle_{L^2(0,T;W)} -$$

$$- \langle u, a' * B_\lambda u_\lambda\rangle_{L^2(0,T;W)} - \langle u, b' * Cu_\lambda - b(0)Cu_\lambda\rangle_{L^2(0,T;W)} -$$

$$- \langle J_\lambda u_\lambda - u, b' * Cu_\lambda - b(0)Cu_\lambda\rangle_{L^2(0,T,H)} + \langle \lambda B_\lambda u_\lambda - u_\lambda, v'_\lambda\rangle_{L^2(0,T;H)}.$$

The limit superior $\lambda \to 0+$ of the right hand side is by Lemmas 4.10, 2.2, 4.5 and 4.9 less or equal to

$$\langle u, f' - a' * w\rangle_{L^2(0,T;W)} - \langle u, b' * Cu - b(0)Cu\rangle_{L^2(0,T;W)} +$$

$$+ \limsup_{\lambda\to 0+}\left(\lambda\langle B_\lambda u_\lambda, v'_\lambda\rangle_{L^2(0,T;H)} - \int_0^T (\frac{d}{dt}\phi^*(t, v_\lambda(t)) - \phi_t(t, u_\lambda(t)))dt\right) \le$$

$$\le \langle u, f - a' * w - b' * Cu - b(0)Cu\rangle_{L^2(0,T;W)} +$$

$$+ \limsup_{\lambda\to 0+} -\phi^*(T, v_\lambda(T)) + \phi^*(t, f(0)) + \limsup_{\lambda\to 0+}\int_0^T \phi_t(t, u_\lambda(t))dt \le$$

$$\le a(0)\langle u, w\rangle_{L^2(0,T;W)} + \langle u, v'\rangle_{L^2(0,T;W)} - \phi^*(t, v(T)) +$$

$$+ \phi^*(t, f(0)) + \int_0^T \phi_t(t, u_\lambda(t))dt = a(0)\langle u, w\rangle_{L^2(0,T;W)},$$

by (3.3) and the Lebesgue dominated convergence theorem and (3.15). Thus $w \in Bu$ by Lemma 2.6 since we have $J_\lambda u_\lambda \to u$ weakly in $L^2(0,T;W)$, $B_\lambda u_\lambda \to w$ weakly in $L^2(0,T;W^*)$, $B_\lambda(t)u_\lambda(t) \in B_H(t)J_\lambda(t)u_\lambda(t) \subset B(t)J_\lambda(t)u_\lambda(t)$ and B is maximal monotone.

The proof of Lemma 4.10: The limit (4.10) follows from (b) of Lemma 4.9 and the reflexivity of $L^2(0,T;W)$. Similarly (4.11) follows from (e) of Lemma 4.9 as well as the existence of $u \in L^2(0,T;V)$ with

$$u_\lambda \to u \textit{ weakly in } L^2(0,T;V) \tag{4.14}$$

By (c) of Lemma 4.9 and (3.13), $\cup_{\lambda>0}R(v_\lambda(.))$ is a compact set of V^*. By (d) of Lemma 4.9 the functions v_λ are equicontinuous in $W^* \subset V^*$. Thus, by Ascoli's theorem there exist subsequence and $v \in C(0,T;V^*)$ which satisfy (4.8). The limit

(4.9) follows from (4.8) and the Lipschitz continuity of $A(t)^{-1}$ if we redefine u to be continuous and notice that $v(t) \in A(t)u(t)$ by the maximality of A.

The limit (4.13) follows from (3.12) and (4.11). Since

$$\|J_\lambda(t)u_\lambda(t) - u(t)\|_H \le \|J_\lambda(t)u_\lambda(t) - J_\lambda(t)u(t)\|_H +$$

$$+ \|J_\lambda(t)u_\lambda(t) - u(t)\|_H \le \|u_\lambda(t) - u(t)\|_H + 2\|u(t)\|_H,$$

we obtain from the boundness of $u_\lambda(t)$ in H, [3, p.71], (4.9) and from the dominated convergence theorem that (4.12) holds, i.e.

$$\lim_{\lambda\to 0+}\int_0^T \|J_\lambda(t)u_\lambda(t) - u(t)\|_H^2 dt = \int_0^T \lim_{\lambda\to 0+}\|J_\lambda(t)u_\lambda(t) - u(t)\|_H^2 dt = 0$$

The proof is complete.

5. The proof of theorem 3.2. The proof is similar to the proof of Theorem 3.1, only few modifications are necessary. We deal with the approximate problem (4.3)-(4.4) with the same notation as above.

The Lemmas from 4.1 up to 4.8 except 4.3 turn out directly to be valid; the maximal monotonicity of B in the Hilbert space $L^2(0,T;H$ is shown as in [11, Lemma 4.4]. So, there exist $u_\lambda \in L^2(0,T;H)$ and $v_\lambda \in H^1(0,T;H)$ satisfying the equations (4.3)-(4.4). Moreover, (a), (b), (c) of Lemma 4.9 are valid. Let us prove (e). By (a) and (c),

$$\phi(t, u_\lambda(t)) = \langle u_\lambda(t), v_\lambda(t)\rangle_H - \phi^*(t, v_\lambda(t)) \le M - \phi(t,0) \le M$$

where Lemma 4.4 and (3.4) are exploited. Hence by (3.17), $S = \cup_{\lambda>0}R(u_\lambda(.))$ is relatively compact in H. It is covered by the union of open balls

$$\cup_{t\in[0,T]}\cup_{x\in S} B(x, r_{x,t}),$$

where $r_{x,t}$ are determined by (3.18). By the compactness S is covered by a finite number of $B(x, r_{x,t})'$ s. Thus there is continuous and increasing g with

$$\|B_\lambda(t)u_\lambda(t)\|_H \le \|B^0(t)u_\lambda(t)\|_H \le g(\|A^0(t)u_\lambda(t)\|_H) \le g(\|v_\lambda(t)\|_H) \le M.$$

So, the inequality (e) holds, even as pointwise. Finally, (d) and (f) are proved as above.

Let us prove the conclusions of Lemma 4.9. The family of functions u_λ is equicontinuous by (d) and the proof of Lemma 4.6. By Ascoli's theorem there is $u \in C(0,T;H)$ and a subsequence such that, as $\lambda \to 0+$

$$u_\lambda \to u \textit{ uniformly in } H \textit{ on } [0,T]. \tag{5.1}$$

This gives (4.12) and (4.13) , as in the proof of Lemma 4.9. Of course, now (4.10) is a special case of (4.12).

Since $(B_\lambda u_\lambda), (v_\lambda)$ and (v'_λ) are bounded in the Hilbert space $L^2(0,T;H)$, there is $w, v, v' \in L^2(0,T;H)$ and subsequences satisfying, as $\lambda \to 0+$,

$$B_\lambda u_\lambda \to w, v_\lambda \to v, v'_\lambda \to v' \text{ weakly in } L^2(0,T;H). \tag{5.2}$$

Moreover, we can redefine v a.e. on $[0,T]$ to belong to $H^1(0,T;H)$. Then, as $\lambda \to 0+$

$$v_\lambda \to v \text{ weakly in } H, \text{ for each } [0,T]. \tag{5.3}$$

The conclusions of Lemma 4.9 are now proved with the modification (5.3), instead of (4.8).

Since the convex lower semicontinuous function $\phi^*(T,.)$ is lower semicontinuous even with respect to the weak topology of H, we can finish the proof of Theorem 3.2 exactly as the proof of Theorem 3.1.

The proof of theorem 3.3. Let $\varepsilon \in]0, \frac{\alpha}{M}[$ and consider the problem

$$v_\varepsilon(t) + a * w_\varepsilon(t) + b * C_\varepsilon u_\varepsilon(t) = f(t), \text{ for each } t \in [0,T] \tag{6.1}$$

$$v_\varepsilon(t) \in \partial(\phi(t,.) + \varepsilon\psi_H(0,.))u_\varepsilon(t), \text{ for a.e. } t \in [0,T]. \tag{6.2}$$

Lemma 6.1. *We can assume that $0 \in B_H(t)0$ and $\psi_H(t,0) = 0$ for each $t \in [0,T]$.*

Proof: Define $\tilde{\psi}(t,x) = \psi(t,x) - \psi(t,0) - \langle x, w^* \rangle_W$ and $\tilde{f}(t) = f(t) - a * w^*(t)$.

Lemma 6.2. *The functions $\tilde{\phi}_\varepsilon : [0,T] \times H \mapsto]-\infty,\infty], \tilde{\phi}_\varepsilon(t,x) = \phi(t,x) + \varepsilon\psi_H(0,x), \psi_H$ and $\tilde{C}_\varepsilon(t)$ satisfy the conditions on ϕ, ψ and $C(t)$ of Theorem 3.2, respectively.*

Proof: The proof is a straightforward checking.

So there is $u_\varepsilon, v_\varepsilon \in H^1(0,T;H)$ and $w_\varepsilon \in L^2(0,T;H)$ satisfying (6.1)-(6.2).

Lemma 6.3. *$u_\varepsilon \in L^\infty(0,T;W)$.*

Proof: We have the inequalities

$$\phi(t,u_\varepsilon(t)) + \varepsilon\psi_H(0,u_\varepsilon(t)) \le \phi(t,0) + \varepsilon\psi_H(0,0) + \langle u_\varepsilon(t), v_\varepsilon(t)\rangle_H \le M_\varepsilon.$$

Hence $u_\varepsilon(t) \in W$ and $\|u_\varepsilon(t)\|_W \le M_\varepsilon$, by (3.23), (3.21) and Lemma 6.1.

Since $\mathbf{intdom}\phi(t,.) \cap \mathbf{dom}\ \psi_H(0,.) \neq 0$ then $A(t) + \varepsilon B_H(0) = \partial\phi_\varepsilon(t,.)$, by [3, p.128]. Thus $v_\varepsilon(t) \in (A(t) + \varepsilon B_H(0))u_\varepsilon(t)$, and so $u_\varepsilon(t) \in (A(t) + \varepsilon B(0))^{-1}\ v_\varepsilon(t)$. The coercitivity of $B(0)$ imply that $(A(t) + \varepsilon B(0))^{-1}$ is Lipschitzian.

Hence u_ε is measurable in W since $t \mapsto (A(t) + \varepsilon B(0))^{-1}x$ turns out to belong even to $H^1(0,T;W)$, for each $x \in W^*$, by the proof of Lemma 4.6.

Lemma 6.4 *If $y \in H^1(0,T;W^*)$ and $(x,y) \in A$, then $\phi^*(.,y(.)) \in W^{1,1}(0,T)$ and*

$$\frac{d}{dt}\phi^*(t,y(t)) = -\phi_t(t,x(t)) + \langle x(t), y'(t)\rangle_W \text{ a.e. on } [0,T].$$

Proof: The proof is similar to the proof of Lemma 4.5.

Lemma 6.5 *The realizations of $A(t) \subset H \times H$ and $B(t) \subset W \times W^*$ in $L^2(0,T;H)$ and $L^2(0,T;W)$, respectively, are maximal monotone.*

Proof: The proof follows the lines of the proof of Lemma 4.4.

Lemma 6.6. *For each $t \in [0,T]$ and $x \in H, \phi(t,x) \ge \frac{\alpha}{8}\|x\|_H^2 - M$.*

Proof: By (3.28), (3.21), the definition for the subdifferential and $y \in A(t)\frac{x}{2}$,

$$\phi(t,x) \ge \phi(t,\frac{x}{2}) + \langle x - \frac{x}{2}, y\rangle_H \ge \phi(t,0) + 2\langle \frac{x}{2} - 0, A(t)^{-1}0\rangle_H +$$

$$+ \langle \frac{x}{2} - 0, y - A(t)^{-1}\rangle_H \ge -M - \|\frac{x}{2}\|_H M + \alpha\|\frac{x}{2}\|_H^2 \ge \frac{\alpha}{8}\|x\|_H^2 - M,$$

by (3.22), since we can prove, as Lemma 4.6, that $A(.)^{-1}0 \in H^1(0,T;H)$.

Lemma 6.7 *There is a constant $M \in]0,\infty[$ independent on ε satisfying*

(a) $\|u_\varepsilon\|_{L^2(0,T;W)} \le M$, *(b)* $\|F_\varepsilon\|_{L^\infty(0,T;H)} \le M$,
(c) $\|v_\varepsilon\|_{L^\infty(0,T;H)} \le M$, *(d)* $\|v'_\varepsilon\|_{L^2(0,T;W^*)} \le M$,
(e) $\|w_\varepsilon\|_{L^2(0,T;W^*)} \le M$, *(f)* $\|u_\varepsilon\|_{L^\infty(0,T;H)} \le M$,

Proof: Denote $F_\varepsilon(t) = \int_0^t w_\varepsilon(s)ds$. It follows from (6.1), by integrating by parts and by (3.27) and (3.33), that

$$\int_0^t \langle a * w_\varepsilon(s), w_\varepsilon(s)\rangle_H ds = \int_0^t \langle f(s) - v_\varepsilon(s) - b * C_\varepsilon u_\varepsilon(s), w_\varepsilon(s)\rangle_H ds \le$$

$$\le \Big(\|f(t)\|_H + (\|b\|_{L^\infty(0,T)} + |\,b(0)\,|)\int_0^t \|C_\varepsilon(s)u_\varepsilon(s)\|_H ds +$$

$$+ \|f'\|_{L^1(0,T;H)} + \|b'\|_{L^2(0,T)}\|C_\varepsilon u_\varepsilon\|_{L^2(0,T;H)}\Big)\max_{s\in[0,t]}\|F_\varepsilon(s)\|_H +$$

$$+ M + M\int_0^t \|u_\varepsilon(s)\|_W^2 ds \le M + M\int_0^t \|u_\varepsilon(s)\|_W^2 ds +$$

$$+ \Big(M + M(\int_0^t \|u_\varepsilon(s)\|_W^2 ds)^{\frac{1}{2}}\Big)\max_{s\in[0,t]}\|F_\varepsilon(s)\|_H.$$

Hence by Lemma 2.4 we obtain, for each $t \in [0,T]$.

$$\|F_\varepsilon(t)\|_H \le M + M\Big(\int_0^t \|u_\varepsilon(s)\|_W^2 ds\Big)^{\frac{1}{2}}. \tag{6.3}$$

From (6.1) we obtain

$$\phi_\varepsilon(t,u_\varepsilon(t)) - \phi_\varepsilon(t,0) \le \langle v_\varepsilon(t), u_\varepsilon(t)\rangle_H =$$

$$= \langle f(t) - a * w_\varepsilon(t) - b * C_\varepsilon u_\varepsilon(t), u_\varepsilon(t)\rangle_H \leq$$

$$\leq \left(M + M\left(\int_0^t \|u_\varepsilon(s)\|_W^2 ds\right)^{\frac{1}{2}}\right)\|u_\varepsilon(t)\|_H,$$

where (6.3), (3.33) and the definition for the subdifferential are aplied. Hence, by Lemma 6.6, we conclude, for each $t \in [0, T]$,

$$\|u_\varepsilon(s)\|_H^2 \leq M + M \int_0^t \|u_\varepsilon(s)\|_W^2 ds. \tag{6.4}$$

It follows from (6.1) by differentiating also that

$$a(0) \int_0^t \langle u_\varepsilon(s), w_\varepsilon(s)\rangle_H ds =$$

$$= \int_0^t \langle f'(s) - v'_\varepsilon - a' * w_\varepsilon(s) - b' * C_\varepsilon u_\varepsilon(s) - b(0)C_\varepsilon(s)u_\varepsilon(s), u_\varepsilon(s)\rangle_H ds =$$

$$= \int_0^t \Big(-\frac{d}{ds}\phi_\varepsilon^*(s, v_\varepsilon(s)) + \phi_{\varepsilon,t}(s, u_\varepsilon(s)) + \langle f'(s) - a'' * F_\varepsilon(s) -$$

$$- a'(0)F_\varepsilon(s) - b' * C_\varepsilon u_\varepsilon(s) - b(0)C_\varepsilon(s)u_\varepsilon(s), u_\varepsilon(s)\rangle_H\Big) ds \leq$$

$$\leq -\phi_\varepsilon^*(t, v_\varepsilon(t)) + \phi_\varepsilon^*(0, v_\varepsilon(0)) + \int_0^t \phi_t(s, u_\varepsilon(s)) ds +$$

$$+ \int_0^t \Big\|f'(s)\Big\|_H + M\|F_\varepsilon\|_{L^\infty(0,s;H)} + M\|C_\varepsilon u_\varepsilon\|_{L^2(0,s;H)} +$$

$$+ M\|C_\varepsilon(s)u_\varepsilon(s)\|_H\Big)\|u_\varepsilon(s)\|_H ds \leq \phi_\varepsilon(t,0) + M +$$

$$+ \int_0^t \left(\eta_1(s)\|u_\varepsilon(s)\|_H^2 + M\|F_\varepsilon\|_{L^\infty(0,s;H)} + \frac{\alpha a(0)}{2}\|u_\varepsilon(s)\|_W^2 + M\|u_\varepsilon(s)\|_H^2\right) ds \leq$$

$$\leq M + \int_0^t \left(\eta_1(s) \int_0^s \|u_\varepsilon(\tau)\|_W^2 d\tau + \frac{\alpha a(0)}{2}\|u_\varepsilon(s)\|_W^2\right) ds,$$

since by Lemma 6.6, [3, p.96] and (6.1), $\phi_\varepsilon^*(, 0, v_\varepsilon(0)\|_H^2 = M\|v_\varepsilon(0)\|_H^2 + M = M$; in the last step above, (3.21), (3.22), (6.3) and (6.4) are applied. By (3.23), $\langle u_\varepsilon(t), w_\varepsilon(t)\rangle_H = \langle u_\varepsilon(t), w_\varepsilon(t)\rangle_W \geq \psi(t, u_\varepsilon(t)) - \psi(t, 0) \geq \alpha\|u_\varepsilon(t)\|_W^2 - M\|u_\varepsilon(t)\|_H^2 - M$. Hence, by Gronwall's inequality and (6.4), we obtain the inequality (a).

The inequality (b) follows now from (6.3); (e) follows from affine boundness of $B(t)$; (c) and (d) follow from (6.1) (a) and (e). The inequality (f) follows from (a) and (6.4).

Lemma 6.8 *There exist subsequences and $v \in C(0, T; W^*) \cap L^\infty(0, T; H)$, $u \in L^2(0, T; W) \cap L^\infty(0, T; H)$ and $w \in L^2(0, T; W^*)$ such that, as $\varepsilon \to 0+$*

(6.5) $v_\varepsilon \to v$ *uniformly in* W^*

(6.6) $v_\varepsilon \to v$ *weakly in* $L^2(0, T; H)$;

(6.7) $u_\varepsilon \to u$ *weakly in* $L^2(0, T; W)$;

(6.8) $u_\varepsilon \to u$ *in* $L^2(0, T; H)$;

(6.9) $w_\varepsilon \to w$ *weakly in* $L^2(0, T; W^*)$;

(6.10) $Cu_\varepsilon \to Cu$ *weakly in* $L^2(0, T; W^*)$.

Proof: The weak limits (6.6), (6.7) and (6.9) follow from Lemma 6.7 and the reflexivities of the spaces.

By (d) of Lemma 6.7, the family $v_\varepsilon : [0, T] \mapsto W^*$, is equicontinuous. So, by Ascoli's theorem, the limit (6.5) holds.

Let us prove that $v \in Au$. Since $\text{intdom}\, \phi(t, .) \cap \text{dom}\, \psi_h(0, .) \neq \emptyset$, then $\partial\phi_\varepsilon(t, .) = \partial\phi(t, .) + \partial\psi_H(0, .)$, by [3, p.128]. Hence $v_\varepsilon(t) - \varepsilon B_H^0(0)u_\varepsilon(t) \in A(t)u_\varepsilon(t)$. Since (6.5) and (a) hold and

$$| B_H^0(0)u_\varepsilon(t)\|_{W^*} = \sup_{\|\xi\|_W^* \leq 1} \langle \xi, B_H^0(0)u_\varepsilon(t)\rangle_W \leq$$

$$\leq \sup_{\|\xi\|_{W^*} \leq 1} (\psi_H(0, u_\varepsilon(t) + \xi) - \psi_H(0, u_\varepsilon(t))) \leq$$

$$\leq \sup_{\|\xi\|_{W^*} \leq 1} (\psi(0, u_\varepsilon(t) + \xi) \leq M + M\|u_\varepsilon(t)\|_W,$$

by the definition for the subdifferential, $0 \in B_H^0(0)0$ and the boundness of $B(t)$, we obtain that $v_\varepsilon \to v$ in $L^2(0, T; W^*)$. By the maximality of A, (6.7) and Lemma 2.6, we conclude that $v \in Au$, since

$$\limsup_{\varepsilon \to 0+} \langle u_\varepsilon, w_\varepsilon\rangle_{L^2(0,T;H)} = \limsup_{\varepsilon \to 0+} \langle u_\varepsilon, w_\varepsilon\rangle_{L^2(0,T;W)} = \langle u, w\rangle_{L^2(0,T;H)}.$$

The limit (6.8) is implied by (3.28) and

$$\alpha\|u_\varepsilon - u\|_{L^2(0,T;H)}^2 \leq \langle u_\varepsilon - u, v_\varepsilon - \varepsilon B_H^2(0)u_\varepsilon - v\rangle_{L^2(0,T;W)} \to 0,$$

as $\varepsilon \to 0+$. Finally, (6.10) follows from (6.34), (6.7) and (6.8).

Lemma 6.8 and (6.1) imply that the equation (3.36) is valid. It remains to prove that $w \in Bu$.

The equation (6.1) imply that

$$a(0)\langle u_\varepsilon, w_\varepsilon\rangle_{L^2(0,T;W)} = \langle u_\varepsilon f' - v_\varepsilon - a' * w_\varepsilon - b' * C_\varepsilon u_\varepsilon - b(0)C_\varepsilon u_\varepsilon\rangle_{L^2(0,T;W)} =$$

$$= \langle u_\varepsilon, f'\rangle_{L^2(0,T;W)} - \langle u, a' * w_\varepsilon\rangle_{L^2(0,T;W)} -$$

$$-\langle u_\varepsilon - u, a' * w_\varepsilon\rangle_{L^2(0,T;H)} - \langle u, b' * C_\varepsilon u_\varepsilon - b(0)C_\varepsilon u_\varepsilon\rangle_{L^2(0,T;W)} -$$

$$-\langle u_\varepsilon - u, b' * C_\varepsilon u_\varepsilon - b(0)C_\varepsilon u_\varepsilon\rangle_{L^2(0,T;H)} - \langle u_\varepsilon, v'_\varepsilon\rangle_{L^2(0,T;H)}$$

By Lemmas 6.7 and 6.8 and (3.33), we conclude that

$$\limsup_{\varepsilon\to 0+} a(0)\langle u_\varepsilon, w_\varepsilon\rangle_{L^2(0,T;W)} \le \limsup_{\varepsilon\to 0+} -\langle u_\varepsilon, v'_\varepsilon\rangle_{L^2(0,T;H)} +$$

$$+ \langle u, f' - a' * w - b' * Cu - b(0)Cu\rangle_{L^2(0,T;W)}.$$

Since ϕ_ε satisfies the conditions on ϕ of Lemma 6.4, and $\phi_\varepsilon^* \le \phi^*$ and $-\phi_\varepsilon^* \le -\phi_\lambda^*$ by $\phi \le \phi_\varepsilon$ and $\phi_\varepsilon \le \phi_\lambda$, for each $\lambda > \varepsilon$, it holds that

$$\limsup_{\varepsilon\to 0+} -\langle v'_\varepsilon, u_\varepsilon\rangle_{L^2(0,T;H)} =$$

$$= \limsup_{\varepsilon\to 0+}\left(-\phi_\varepsilon^*(T, v_\varepsilon(T)) + \phi_\varepsilon^*(0, v_\varepsilon(0)) - \int_0^T \phi_{\varepsilon,t}(t, u_\varepsilon(t))dt\right) \le$$

$$\le \limsup_{\varepsilon\to 0+} -\phi_\lambda^*(T, v_\varepsilon(T)) + \phi^*(0, f(0)) + \limsup_{\varepsilon\to 0+}\int_0^T -\phi_t(t, u_\varepsilon(t))dt \le$$

$$\le -\phi_\lambda^*(T, v(T)) + \phi^*(0, v(0)) - \int_0^T -\phi_t(t, u(t))dt \le$$

$$\le -\phi^*(T, v(T)) + \lambda\psi_H(0, u(T)) + \phi^*(0, v(0)) - \int_0^T -\phi_t(t, u(t))dt,$$

by the lower semicontinuity of ϕ_λ^*, (6.5), (6.8) and the dominated convergence theorem with (f) and (3.22). Since $\lambda > 0$ is arbitrary and $\psi_H(0, u(T))$ can be taken to be finite, we conclude by (3.36) and Lemma 6.4 that

$$\limsup_{\varepsilon\to 0+} -\langle u_\varepsilon, w_\varepsilon\rangle_{L^2(0,T;W)} = \langle u, w\rangle_{L^2(0,T;W)}$$

Hence the maximality of B implies that $w \in Bu$.

The proof is complete.

7.The proof of theorems 3.4 and 3.5. The proof of theorem 3.4: The proof of Theorem 3.3. applies almost as such. Only few modifications have to be done.

We can prove the chain rule

$$\frac{d}{dt}\psi(t, u_\varepsilon(t)) = \langle w_\varepsilon(t), u'_\varepsilon(t)\rangle_H + \psi_t(t, u_\varepsilon(t))$$

as we proved Lemma 6.4. Then , by (6.1), (3.39), (3.40), (6.3) and (3.33),

$$a(0)\psi(t, u_\varepsilon(t)) \le a(0)\psi(0, u_\varepsilon(0)) + a(0)\int_0^t \psi_t(s, u_\varepsilon(s))ds +$$

$$+ \int_0^t \langle u'_\varepsilon(s), f' - v'_\varepsilon - a' * w_\varepsilon(s) - b' * C_\varepsilon u_\varepsilon(s) - b(0)C_\varepsilon(s)u_\varepsilon(s)\rangle_H ds \le$$

$$= \le a(0)\psi(0, u_\varepsilon(0)) + M + \int_0^t \eta_1(s)\|u_\varepsilon(s)\|_W^2 ds,$$

which gives, by Gronwall's inequality, (3.23) and (f) of Lemma 6.7,

$$\|u_\varepsilon(t)\|_W^2 \le M\psi(0, u_\varepsilon(0)) + M.$$

Let $u_0 = A(0)^{-1}f(0) \subset W$, (cf.(3.41)). Then, by (6.2)

$$0 \in A(0)u_\varepsilon(0) - A(0)u_0 + \varepsilon B_H(0)u_\varepsilon(0).$$

and hence, by the monotocity of $A(0)$,

$$\psi_H(0, u_\varepsilon(0)) \le \psi_H(0, u_0) + \langle u_\varepsilon(0) - u_0, B_H^0(0)u_\varepsilon(0)\rangle_H \le \psi_H(0, u_0) = M.$$

Thus even $\|u_\varepsilon\|_{L^\infty(0,T;W)} \le M$ and hence $\|w_\varepsilon\|_{L^\infty(0,T;W^*)} \le M$.

In the other critical points the boundness of $B(t)$ is enought.

The proof of theorem 3.5: The proof of Theorem 3.3 applies with the following modifications. The inequalities (6.4) and (f) are not needed, since the last estimate in the proof of Lemma 6.7 is shortly

$$a(0)\int_0^t \langle u_\varepsilon(s), w_\varepsilon(s)\rangle_H ds \le M\int_0^t \eta_1(s)\int_0^s \|u_\varepsilon(\tau)\|_W^2 d\tau ds,$$

which implies (a) by Gronwall's inequality and (3.44). Since we have (3.46), we do not need limit $u_\varepsilon \to u$ in $L^2(0,T;H)$, as $\varepsilon \to 0+$, and so we have only to prove that

$$\limsup_{\varepsilon\to 0+} -\langle u_\varepsilon, w_\varepsilon\rangle_{L^2(0,T;W)} \le \langle u, w\rangle_{L^2(0,T;W)} \tag{7.1}$$

By Ascoli's theorem, (b), (e), (6.9) and (3.45), we obtain, that for a subsequence, as $\varepsilon \to 0+$,

$$F_\varepsilon \to F \text{ and } b' * C_\varepsilon u_\varepsilon \to b'Cu \text{ in } W^* \text{ uniformly on}[0,T], \tag{7.2}$$

where $F(t) = \int_0^t w(s)ds$. From (6.1) it follows that

$$a(0)\langle u_\varepsilon, w_\varepsilon\rangle_{L^2(0,T;W)} = \langle u_\varepsilon, f'\rangle_{L^2(0,T;W)} - \langle u_\varepsilon, a'' F_\varepsilon + a'(0)F_\varepsilon\rangle_{L^2(0,T;W)} -$$

$$- \langle u_\varepsilon, b' * C_\varepsilon u_\varepsilon\rangle_{L^2(0,T;W)} - \langle u_\varepsilon, v'_\varepsilon\rangle_{L^2(0,T;H)}$$

Hence we obtain (7.1), by (7.2) and (6.7), since the limit of the last term is estimated as in the proof of Theorem 3.3, applying (3.43) and (6.7).

8.Applications. For the applications we refer to the examples of [1], [4], [11] and of [10]. By the definitions

$$V \subset W = H^1(\Omega), H = L^2(\Omega), \Omega \subset \mathbf{R}^n, n = 1, 2, ...$$

our theorems cover a large class of parabolic partial differential equations with time dependent coefficients.

Let $t \in [0,T]$. The first order partial differential operators from W to H of type $z \mapsto \nabla \cdot \mathbf{g}(\mathbf{x}, t, z(\mathbf{x}))$ can be represented by $C(t) : W \mapsto H$,

$$\langle y, C(t)z \rangle_H = \int_\Omega y(\mathbf{x}), \nabla \cdot \mathbf{g}(\mathbf{x}, t, z((\mathbf{x}))d^n\mathbf{x}.$$

The approximating conditions (3.31) - (3.34) are satisfied by $C_\lambda(t) : H \mapsto H$,

$$\langle y, C_\lambda(t)z \rangle_H = \int_\Omega y(\mathbf{x}), \nabla \cdot \int_{\mathbf{R}^n} \rho_\lambda(\xi - \mathbf{x})g(\xi, t, z(\xi))d^n\xi d^n\mathbf{x},$$

where $\rho_\lambda \in C_0^\infty(\mathbf{R}^n)$ is the dimensional Friedrich's mollifier which satisfies

$$\rho_\lambda \leq 0, \text{ supp } \rho_\lambda \subset B(0, \lambda) \text{ and } \int_{\mathbf{R}^n} \rho_\lambda(\mathbf{x})d^n\mathbf{x} = 1,$$

for each $\lambda > 0$.

REFERENCES

1. Barbu, V. – *Existence for nonlinear Volterra equations in Hilbert spaces* , SIAM J. Math. Anal. 10 (1979) , 552-569.
2. Barbu, V. – *Non-linear semigroups and differential equations in Banach spaces* , Editura Academiei and Noordhoff International Publishing, Bucharest and Leyden, Netherlands, 1976.
3. Barbu, V. et Precupanu, Th. – *Convexity and optimization in Banach spaces*, second ed., Editura Academiei and Sijthoff& Noordhoff International publishers, Bucharest and Alphen aan de Rijn, Netherlands, 1986.
4. Di Benedetto, E. and Showalter, R.E. – *Implicit degenerate evolution equations and applications*, SIAM J. Math. Anal. **12:5** (1981), 731-751.
5. Bermudez, A., Durany, J. and Saquez, C. – *An existence theorem for an implicit evolution equation* , Coll. math. Universidad de Barcelona XXXV (1984).
6. Brezis, H. – *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert* , North-Holland, Amsterdam, 1973.
7. Crandall, M.G.,Londen, S.O. and Nohel, J.A. – *An abstract nonlinear Volterra integrodifferential equation* , J. Math. Anal. Appl. **64** (1978) 701-735.

8. Ekeland, I. and Temam, R. – *Analyses convexe et problèmes variationnels* , Dunod, Gauthier-Villars, Paris, 1974.
9. Grange, O. and Mignot, F. – *Sur la résolution d'une équation et d'une inéquation paraboliques non-linéaires* , J. Funct. Anal **11** (1972), 77-92.
10. Hokkanen, V.-M. – *An implicit non-linear time dependent equation has a solution* , J. Math. Anal. Appl. (to apear).
11. Hokkanen, V.-M. – *On nonlinear Volterra equation in Hilbert spaces*, Diff. Int. Eq. (to appear).
12. Saguez, C. – *Controle optimal de sytèmes á frontière libre* , Ph. D. thesis, Ecole polytecnique de Compiegne, 1980.
13. Yosida, K. – *Functional analysis* , fifth edition, Springer - Verlag, Berlin , 1978.

Received 15.V.1991

Department of Mathematics
University Jyväskylä
P.O.Box 35 SF - 40351
Jyväskylä, Finland