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A CLASS OF CONTROL PROBLEMS GOVERNED BY MULTIVALUED DIFFERENTIAL EQUATIONS

BY

CATALINA DAVIDEANU

In this paper necessary optimality conditions (maximum principle) for a class of control problems are established (theorem 2). A generalization of a multivalued differential equation presented in [4] is studied. In theorem 1 the existence and uniqueness of the solution to this type of equation is presented.

We consider the following control problem

(P) Minimize

$$H(y,v)=\int_0^T (g(t,y(t))+h(v(t)))dt+\phi_0(y(T)), 0<T<\infty,$$

over the set of all functions $y\in W^{1,2}(0,T;\mathbf{R}^n)$ satisfying the multivalued differential equation

$$(GE) \qquad y'(t)+F(y(t),u(t))+\beta(y(t))\ni f(t)\ a.e.\ t\in]0,T[,$$

with the initial conditions

$$(IC) \qquad y(0)=y_0\in D(\beta)$$

where $u\in W^{1,2}(0,T,\mathbf{R}^m)$ is the unique solution of the equation

$$(C) \qquad u'(t)+G(u(t))=v(t)\ a.e.\ t\in]0,T[,$$

with the initial condition

$$(IC_1) \qquad u(0)=u_0\in\mathbf{R}^m\ for\ v\in L^2(0,T;\mathbf{R}^m).$$

The above elements have the following properties:

(i) $F : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ is locally Lipschitz function in the pair (y,u) i.e. for every $M \in \mathbf{R}_+^*$ there exists $L_M \in R_+^*$ such that

$$(1) \qquad \|F(y,u) - F(z,v)\| \le L_M(\|y-z\|_n + \|u-v\|_m)$$

for every $y, z \in \mathbf{R}^n$ with $\|y\|, \|z\| \le M$ and $u, v \in \mathbf{R}^m$ with $\|u\|, \|v\| \le M$;

(ii) there exist $D_i \in \mathbf{R}_+^*, i = \overline{1,3}$, so that

$$(2) \qquad < F(y,u), y > \ge -D_1\|y\|^2 - D_2\|u\|^2 - D_3$$

for every $y \in \mathbf{R}^n$ and $u \in \mathbf{R}^m$;

(iii) $\beta : D(\beta) \subset \mathbf{R}^n \to \mathbf{R}^n$ is a maximal monotone operator with $0 \in D(\beta)$;

(iv) $f \in W^{1,1}(0,T;\mathbf{R}^n)$;

(v) $v \in L^2(0,T;\mathbf{R}^m)$;

(vi) $G : \mathbf{R}^m \to \mathbf{R}^m$ is a locally Lipschitz function which satisfies the condition: there exist $A_i \in \mathbf{R}_+^*, i = 1,2$, so that

$$(1)' \qquad < G(u), u > \ge -A_1\|u\|^2 - A_2$$

for every $u \in \mathbf{R}^m$;

(vii) the function $g : [0,T] \times \mathbf{R}^n \to \mathbf{R}_+$ is measurable in t, $g(t,\cdot) \in L^\infty(0,T)$, $\phi_0 : \mathbf{R}^n \to \mathbf{R}_+$ and for every $\delta > 0$ there exists $L_\delta > 0$, independent of t, so that :

$$(3) \qquad | g(t,y) - g(t,z) | + | \phi_0(y) - \phi_0(z) | \le L_\delta\|y-z\|_n$$

for all $t \in [0,T]$ and $\|y\| + \|z\| \le \delta$;

(viii) $h : \mathbf{R}^m \to \mathbf{R}$ is a convex lower semicontinuous function and there exist $b_1 \in \mathbf{R}_+^*$ and $b_2 \in \mathbf{R}$ so that

$$(4) \qquad h(v) \ge b_1\|v\|_m^2 - b_2 \ for\ all\ v \in \mathbf{R}^m.$$

Remark. (a) The multivalued differential equation

$$(E) \qquad y'(t) + (F+\beta)(y(t)) \ni Bu(t) + f(t) \ a.e.\ t \in]0,T[,$$

with the initial condition (IC) is a particular case of the equation (GE) and was studied in [4].

(b) Condition (2) can be replaced by a stronger one:

$$(2)' \qquad \|F(y,u)\|_n \le C(\|y\|_n + \|u\|_m + 1)$$

for every $y \in \mathbf{R}^n$ and $u \in \mathbf{R}^m$.

(c) If F is a Lipschitz function then (2)' is true and therefore F satisfies conditions (i) and (ii).

One can easily prove that for every $u_0 \in \mathbf{R}^m$ and $v \in L^2(0,T;\mathbf{R}^m)$ the differential equation (C) with the initial condition (IC_1) has a unique solution $u \in W^{1,2}(0,T;\mathbf{R}^m)$.

Definition 1. *The function $y : [0,T] \to \mathbf{R}^n$ is called the strong solution to the equation (GE) with the initial condition (IC) if the following properties are satisfied :*

(a)' *y is an absolutely continuous function on every compact subset of the interval* $]0,T[$,

(b)' $y(t) \in D(\beta)$ *a.e.* $t \in]0,T[$,

(c)' $f(t) - y'(t) - F(y(t),u(t)) \in \beta(y(t))$ *a.e.* $t \in]0,T[$,

(d)' $y(0) = y_0 \in D(\beta)$.

The difficulties encountered in the study of the equation (GE) arise from the fact that generally β is not equal with the subdifferential of a proper, convex and lower semicontinuous function.

We shall prove the existence and uniqueness for the strong solution to the equation (GE) in

Theorem 1. *Under the assumptions (i) – (v) for every $y_0 \in D(\beta)$ and $(u,f) \in W^{1,2}(0,T;\mathbf{R}^m) \times W^{1,1}(0,T;\mathbf{R}^n)$ the equation (GE) with the condition (IC) has a unique solution*

$$y = y(.,y_0,u,f) \in W^{1,\infty}(0,T;\mathbf{R}^n).$$

Moreover, y is differentiable from the right at every point of the interval $[0,T[$ and satisfies the relations

$$(5) \qquad \frac{d^+y}{dt} = (f(t) - F(y(t),u(t)) - z(t))^0 \ for\ every\ 0 \le t \le T,$$

$$(6) \qquad \|\frac{d^+y}{dt}(t)\| \le \|f(0) - F(y_0,u_0) - z_0\| + \int_0^t \|\frac{df}{ds}(s)\|ds \ , 0 \le t < T,$$

where $(f(t) - F(y(t),u(t)) - z(t))^0$ represents the element of minimum norm of the set $(f(t) - F(y(t),u(t)) - \beta(y(t)))$; $z_0 \in \beta(0)$, $z(t) \in \beta(y(t))$.

If y^1 and y^2 are the strong solutions to the equation for the elements (y_0^1,u^1,f^1) , $(y_0^2,u^2,f^2) \in D(\beta) \times W^{1,2}(0,T;\mathbf{R}^m) \times W^{1,1}(0,T;\mathbf{R}^n)$ then inequality

$$(7) \qquad \|y^1(t) - y^2(t)\| \le M(\|y_0^1 - y_0^2\| + \int_0^t \|u^1(s) - u^2(s)\|ds + \int_0^t \|f^1(s) - f^2(s)\|ds)$$

holds for every $0 \le t \le T$.

The proof is similar with that presented in [4].

The control problem (P) has, at least , one optimal pair. The maximum principal for problem (P) is as follows.

Theorem 2. *Let $(y^*, v^*) \in W^{1,2}(0,T;\mathbf{R}^n) \times L^2(0,T;\mathbf{R}^m)$ be an arbitrary optimal pair for the problem (P).*

Then there are a function $p \in BV(0,T;\mathbf{R}^n)$ (the dual extremal arc) and a measure $\mu \in ((L^\infty(0,T))^*)^n$ that satisfy the relations

$$(8) \qquad p'(t) - \left(\frac{\partial F}{\partial y}(y^*(t),u^*(t))\right)^* p(t) - \mu \in \partial g(t, y^*(t)), \ a.e. \ t \in]0,T[,$$

$$(9) \qquad p(T) + \partial\varphi_0(y^*(T)) \ni 0,$$

$$(10) \qquad \int_t^T (U_v^{-1}(t))^* U_v^*(\tau)\left(\frac{\partial F}{\partial u}(y^*(\tau),u^*(\tau))\right)^* p(\tau)d\tau \in \partial h(v^*(t)) \ a.e. \ t \in]0,T[,$$

where U_v is fundamental matrix given by

$$\begin{cases} U'_v(t) + \nabla G(u(t))U_v(t) = O_m & , \ a.e. \ t \in]0,T[, \\ U_v(0) = I_m. \end{cases}$$

Here is denoted by I_m the m -dimensional unit matrix.

Proof. The idea of the proof consists in
(a) approximating the control problem (P) by a family of smooth problems (P^ε) (for which the necessary optimality conditions are immediate)
and
(b) passing to the limit in the approximate optimality conditions (see[1]).

This method can be used for control problems with nonconvex and nondifferentiable cost functional and nonlinear state equation.

For every $\varepsilon > 0$ we construct the approximate control problem
(P^ε) Minimize

$$H^\varepsilon(y,v) = \int_0^T (g^\varepsilon(t,y(t)) + h(v(t)) + \frac{1}{2}\|v(t) - v^*(t)\|^2 dt + \varphi_0^\varepsilon(y(T)), \quad 0 < T < \infty,$$

over the set of all functions $y \in W^{1,2}(0,T;\mathbf{R}^n)$ that satisfy the approximate differential equation

$$(GE^\varepsilon) \qquad y'(t) + F^\varepsilon(y(t),u(t)) + \beta^\varepsilon(y(t)) = f(t), \ a.e. \ t \in]0,T[.$$

with the initial condition (IC), where $u \in W^{1,2}(0,T;\mathbf{R}^m)$ is the unique solution to the equation

$$(C^\varepsilon) \qquad u'(t) + G^\varepsilon(u(t)) = v(t) \ a.e. \ t \in]0,T[,$$

with the initial condition (IC_1) for $v \in L^2(0,T;\mathbf{R}^m)$.

The elements from $(\mathrm{GE}^\varepsilon)$ and (C^ε) are

$$F^\varepsilon(y,u) = \int_{\mathbf{R}^n} F(y - \varepsilon^2\theta, u - \varepsilon^2\theta)\rho(\theta)d\theta$$

$$\beta^\varepsilon(y) = \int_{\mathbf{R}^n} (\beta_\varepsilon(y - \varepsilon^2\theta) - \beta_\varepsilon(-\varepsilon^2\theta))\rho(\theta)d\theta + \beta_\varepsilon(0)$$

with

$$\beta_\varepsilon = \varepsilon^{-1}(I - (I + \varepsilon\beta)^{-1}).$$

$$G^\varepsilon(u) = \int_{\mathbf{R}^m} G(u - \varepsilon^2\theta)\rho(\theta)d\theta$$

$$g^\varepsilon(t,y(t)) = \int_{\mathbf{R}^n} g(t, y(t) - \varepsilon^2\theta)\rho(\theta)d\theta$$

$$\varphi_0^\varepsilon(y) = \int_{\mathbf{R}^n} \varphi_0(y - \varepsilon^2\theta)\rho(\theta)d\theta$$

Lemma 1. *The following convergences are true:*

$v^\varepsilon \to v^*$	*weakly in* $L^2(0,T;\mathbf{R}^m)$;
$u^\varepsilon \to u^* = u(.,u_0,v^*)$	*strongly in* $C([0,T];\mathbf{R}^m)$ *and*
	weakly in $W^{1,2}(0,T;\mathbf{R}^m)$;
$y^\varepsilon \to y^* = y(.,y_0,u^*)$	*strongly in* $C([0,T];\mathbf{R}^n)$ *and*
	weakly in $W^{1,2}(0,T;\mathbf{R}^n)$;

For the proof see [1].

If $(y^\varepsilon, v^\varepsilon)$ is an optimal pair for the problem (P^ε) there exists the inequality

$$(11) \qquad H^\varepsilon(y^\varepsilon, v^\varepsilon) \le H^\varepsilon({y_\lambda}^\varepsilon, v^\varepsilon + \lambda\bar{v})$$

for every $\varepsilon > 0$ and $\bar{v} \in L^2(0,T;\mathbf{R}^m)$.

We have denoted $y_\lambda^\varepsilon(t) = y(t, y_0, u_\lambda^\varepsilon)$ the unique solution to the differential equation $(\mathrm{GE}^\varepsilon)$ corresponding to the unique solution $u^\varepsilon(t) = u(t,u_0,v^\varepsilon + \lambda\bar{v})$ to the equation (C^ε).

The inequality (11) reads

$$(11)' \qquad \int_0^T ((\nabla g^\varepsilon(t,y^\varepsilon(t)), \bar{y}^\varepsilon(t)) + h'(v^\varepsilon, \bar{v}) + < v^\varepsilon - v^*, \bar{v}>)dt +$$

$$+ < \nabla\varphi_0^\varepsilon(y^\varepsilon(T)), \bar{y}^\varepsilon(T) > \ \ge 0, \quad \text{for all } \bar{v} \in L^2(0,T;\mathbf{R}^m).$$

Let $p^\varepsilon \in BV([0,T];\mathbf{R}^m)$ be the unique solution to the problem

$$(12) \qquad p^{\varepsilon\prime}(t) - (\nabla_y F^\varepsilon(y^\varepsilon, u^\varepsilon) + \nabla\beta^\varepsilon(y^\varepsilon))^* p^\varepsilon = \nabla g^\varepsilon(t, y^\varepsilon) \ a.e. \ t \in]0,T[$$

$$p^\varepsilon(T) + \nabla\varphi_0^\varepsilon(y^\varepsilon(T)) = 0 \tag{13}$$

whose existence is established in [1], p.76.

Now, we consider the system

$$\frac{d\bar{y}^\varepsilon}{dt}(t) + (\nabla_y F^\varepsilon(y^\varepsilon, u^\varepsilon) + \nabla\beta^\varepsilon(y^\varepsilon))\bar{y}^\varepsilon = -\nabla_u F^\varepsilon(y^\varepsilon, u^\varepsilon)\bar{u}^\varepsilon, \tag{14}$$

$$\frac{dp^\varepsilon}{dt}(t) - (\nabla_y F^\varepsilon(y^\varepsilon, u^\varepsilon) + \nabla\beta^\varepsilon(y^\varepsilon))^* p^\varepsilon = \nabla g^\varepsilon(t, y^\varepsilon). \tag{15}$$

Taking the scalar product of the first equation with p^ε and the scalar product of the second equation with $\bar{y}^\varepsilon$ and adding we get

$$\frac{d}{dt} < p^\varepsilon, \bar{y}^\varepsilon > = < \nabla g^\varepsilon(t, y^\varepsilon), \bar{y}^\varepsilon > - < (\nabla_u F^\varepsilon(y^\varepsilon, u^\varepsilon))^* p^\varepsilon, \bar{u}^\varepsilon > . \tag{16}$$

Integrating relation (16) and using the initial condition

$$\bar{y}^\varepsilon(0) = 0$$

and the final condition

$$p^\varepsilon(T) = -\nabla\varphi_0^\varepsilon(y^\varepsilon(T))$$

we get

$$< p^\varepsilon(T), \bar{y}^\varepsilon(T) > - < p^\varepsilon(0), \bar{y}^\varepsilon(0) > = \tag{16$'$}$$

$$\int_0^T < \nabla g^\varepsilon(t, y^\varepsilon), \bar{y}^\varepsilon > \, dt - \int_0^T < (\nabla_u F^\varepsilon(y^\varepsilon, u^\varepsilon))^* p^\varepsilon, \bar{u}^\varepsilon > \, dt$$

or

$$- < \varphi_0(y^\varepsilon(T)), \bar{y}^\varepsilon(T) > = \int_0^T < \nabla g^\varepsilon(t, y^\varepsilon), \bar{y}^\varepsilon > \, dt - \tag{16"}$$

$$- \int_0^T < (\nabla_u F^\varepsilon(y^\varepsilon, u^\varepsilon))^* p^\varepsilon, \bar{u}^\varepsilon > \, dt$$

Therefore we have

$$\int_0^T < \nabla g^\varepsilon(t, y^\varepsilon), \bar{y}^\varepsilon > \, dt + < \nabla\varphi_0^\varepsilon(y^\varepsilon(T)), \bar{y}^\varepsilon(T) > = \tag{16$'''$}$$

$$= \int_0^T < (\nabla_u F^\varepsilon(y^\varepsilon, u^\varepsilon))^* p^\varepsilon, \bar{u}^\varepsilon > \, dt$$

The relation (16)"' replaced in the (11)' leads to

$$\int_0^T (< (\nabla_u F^\varepsilon(y^\varepsilon, u^\varepsilon))^* p^\varepsilon, \bar{u}^\varepsilon > + h'(v^\varepsilon, \bar{v}) + \tag{17}$$

$$+ < v^\varepsilon - v^*, \bar{v} >)dt \geq 0 \; for \; every \; \bar{v} \in L^2(0, T; \mathbf{R}^m).$$

The Frechet differential $\bar{u}^\varepsilon$ of the application $v^\varepsilon \to u^\varepsilon(v^\varepsilon)$ satisfies the relations

$$\begin{cases} \bar{u}^\varepsilon(t) + \nabla G^\varepsilon(u^\varepsilon(t))\bar{u}^\varepsilon(t) = \bar{v}(t) & , a.e. \; t \in]0, T[\\ \bar{u}^\varepsilon(0) = 0 \end{cases} \tag{18}$$

so that

$$\bar{u}^\varepsilon(t) = U_v^\varepsilon(t) \int_0^t (U_v^\varepsilon(s))^{-1} \bar{v}(s) ds; U_v^\varepsilon(t) = \exp\,(-t\nabla G^\varepsilon(u^\varepsilon)) \tag{19}$$

where

$$\begin{cases} (U_v^\varepsilon(t))' + \nabla G^\varepsilon(u^\varepsilon) U_v^\varepsilon(t) = O_m & , a.e. \; t \in]0, T[\\ U_v^\varepsilon(0) = I_m. \end{cases}$$

Replacing $\bar{u}^\varepsilon$ in (17) with its value given by (19) we obtain

$$\int_0^T < (\nabla_u F^\varepsilon(y^\varepsilon, u^\varepsilon))^* p^\varepsilon, \bar{u}^\varepsilon > \, dt = \tag{19$'$}$$

$$= \int_0^T < (\nabla_u F^\varepsilon(y^\varepsilon, u^\varepsilon))^* p^\varepsilon, \int_0^t (U_v^\varepsilon(t))(U_v^\varepsilon(s))^{-1} \bar{v}(s) ds > dt =$$

$$\int_0^T ds \int_s^T < (\nabla_u F^\varepsilon(y^\varepsilon, u^\varepsilon))^* p^\varepsilon, \; U_v^\varepsilon(t)(U_v^\varepsilon(s))^{-1} \bar{v}(s) > dt$$

In conclusion, the relation (17) becomes

$$\int_0^T ds \int_s^T < ((U_v^\varepsilon(s))^{-1})^* (U_v^\varepsilon(t))^* (\nabla_u F^\varepsilon(y^\varepsilon, u^\varepsilon))^* p^\varepsilon, \bar{v} > dt + \tag{17$'$}$$

$$+ \int_0^T h'(v^\varepsilon, \bar{v}) dt + \int_0^T < v^\varepsilon - v^*, \bar{v} > dt \geq 0, \quad \forall \, \bar{v} \in L^2(0, T; \mathbf{R}^m).$$

Replacing $\bar{v}$ by $-\bar{v}$ we obtain the reverse inequality, which seems that

$$\int_s^T ((U_v^\varepsilon(s))^{-1})^* (U_v^\varepsilon(t))^* (\nabla_u F^\varepsilon(y^\varepsilon, u^\varepsilon))^* p^\varepsilon dt \in \tag{20}$$

$$\in \partial h(v^\varepsilon(s)) + v^\varepsilon - v^* \quad a.e.\ s \in]0,T[.$$

Relations (12), (13) and (20) represent the Euler-Lagrange optimality equations for the approximate control problem (P^ε). Passing to the limit in these equations for $\varepsilon \to 0$, we obtain (8),(9), (10).

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Department of Mathematics
Polytechnic Institute
R - 6600, Iasi
ROMANIA

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ON STABILITY AND BIFURCATION OF IMMUNE SYSTEM

BY

M.M.A. El-SHEIKH and A.A.SOLIMAN

1.Introduction. The immunity process of an organism to infection where devised in mathematical models as a two-component system of ordinary differential equations by B e l l [2],[3]. Recently P r i m b l e y [11], [12], improved Bell's model and discussed oscillation and stability of steady states using L e f s c h e t z approach [8]. Moreover he introduced in [11],[12],[13],[14], a 3-component model describing the immunity process of an organism to infection, the interaction between the produce of antibodies when stimulated by antigen. During his papers, P i m b l e y has been studied oscillation of steady states solutions using an artificial parameter on both the two and three-dimension systems. In this paper, we consider both the two and three-component immune systems introduced by [11],[12]. We discuss the stability and bifurcation of periodic solutions and then we use the technique of [7],[9] to apply the centre manifold theory.

In section 2 we use a very simple technique introduced by P o o r e [15] depends on monotonicity of one of the natural parameters given in the immune 2-component system to discuss existence, index and stability of steady states. In section 3 we study the Pimbley 3-component immune system, Hopf's bifurcation and center manifold theories are considered our approach depends on the improved technique of [6],[9]. Finally in section 4, we use Poincare-Andronov substitution ([1],[16]) to discuss the global stability by discussing the behaviour of path at infinity.

2.Bell's immune 2-component system. In this section we discuss a 2-component immune system recently introduced by [2],[3]. Using a very simple monotonicity technique of [15], we study in details the existence, index and stability of steady states.

Consider the 2-component system

$$u' = uf(u,v) \tag{2.1}$$

$$v' = vg(u,v)$$

where

$$f(u,v) = -\lambda_2 - k\lambda_2 u + k(\alpha_2 - \lambda_2)\bar{v} - k\frac{\alpha_2}{\theta}uv$$