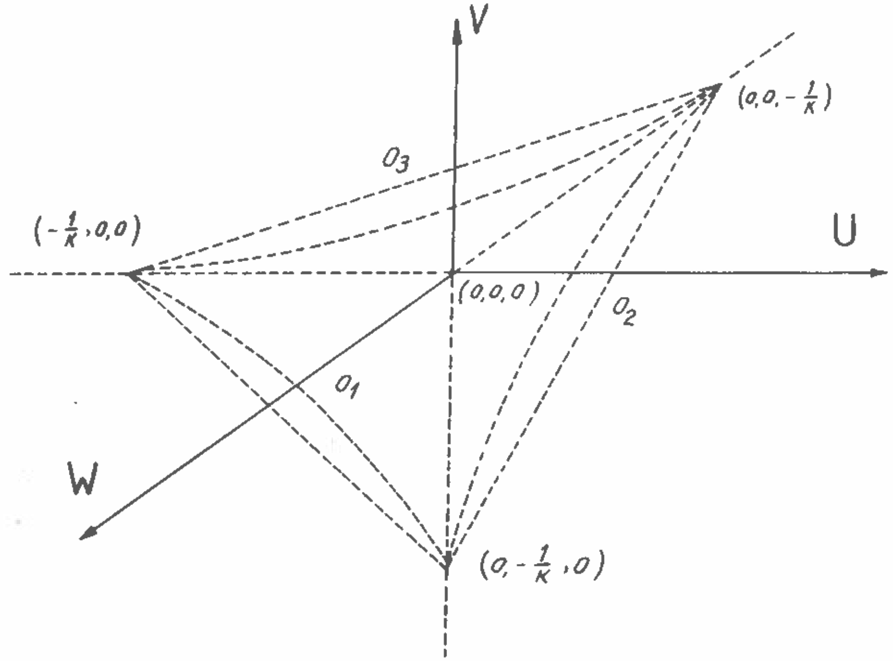

FIG. (3)

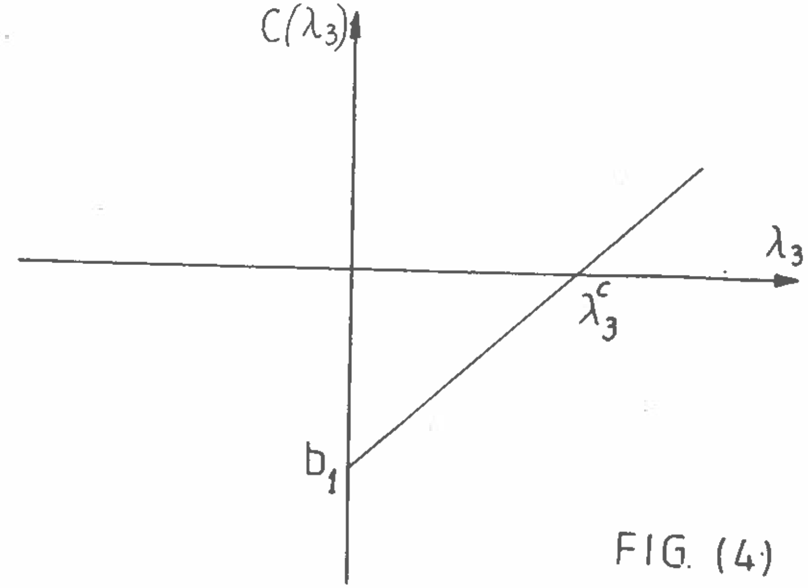

FIG. (4)

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STABILITY CRITERIA FOR NONLINEAR VOLTERRA TYPE DIFFERENCE EQUATIONS

BY

PAVEL TALPALARU

1.Introduction. Difference equations occur in many branches of applied mathematics: numerical analysis, physics, control theory and optimization. In recent years, considerable attention has been paid to the development of the qualitative theory for difference equations.

In order to discuss the asymptotic problems for difference equations, generally speaking, the analogous methods to those for differential and functional-differential equations are aplicable.

The purpose of this pair is to extend a part of the results of T. H a r a, T.Y o n e y a m a and T.I t o h [3] concerning the asymptotic properties of an integro-differential system to the case of corresponding discrete system of the form

$$(V) \qquad x(n+1) = A(n)\,x(n) + \sum_{m=0}^{n-1} B(n,m,x(m))$$

where $A(n)$ is a $q \times q$ matrix on N and $B(n,m,x)$ is a q vector for $n, m \in N, m < n$ and $x \in \mathbf{R}^q$.

The discrete analogue of an integro-differential system has been studied in a particular case , by Ja.V.B y k o v and V.G. L i n e n k o [1] who have obtained interesting results concerning the boundedness and stability of the solutions of such systems, N.L u c a and P.T a l p a l a r u [4] who have considered the discrete analogue of an integro-differential system of the convolution type of the form

$$(1.1) \qquad x(n+1) = Ax(n) + \sum_{m=0}^{n-1} C(n-m-1)x(m) + h(n, x(\cdot)),$$

where A and C are constant $q \times q$ matrices and $h(n, x(\cdot))$ is an operator, $n \in N$. In the study of equation (1.1) were used a technique based on the discrete analogue of the variation of constants formula, the method of the difference resolvent associated with the linear system

$$(1.2) \qquad y(n+1) = Ay(n) + \sum_{m=0}^{n-1} C(n-m-1)y(m)$$

the idea of admissibility and the theory of discrete Laplace transformation.

In this paper we do not assume that $A(n)$ is constant and that $B(n, m, x(m))$ is of the convolution type, and hence it is difficult to apply the above methods to (1.1).

We regard (1.1) as a perturbed system of

$$(L) \qquad y(n+1) = A(n)\, y(n)$$

to obtain some new results on the asymptotic behaviour of solutions of (V) using a fundamental matrix $Y(n)$ for (L). The main tool in our analysis is the variatin of parametres formula in accordance with which, the solution of (V) with the initial function φ on $\{0, 1, ..., n_0\}$ is given by

$$(1.3) \qquad x(n; n_0, \varphi) = Y(n)Y^{-1}(n_0)\varphi(n_0) + \sum_{m=n}^{n-1} Y(n)Y^{-1}(m+1)\sum_{k=0}^{m-1} B(m, k, x(k)).$$

2.Notations,definitions and preliminary results. We adopt the following notations in this paper: $\mathbf{N}$ is the set of non-negative integers ; $N(n_0) = \{n_0, n_0 + 1, ...\}$, where n_0 is a natural number or zero; $N_m(n_0) = \{n_0, n_0 + 1, ..., n_0 + m\}$, $m = 0, 1, 2, ...$; $\mathbf{R}^q$ the q-dimensional real Euclidean space with norm $|x| = \sum_{i=1}^{q} |x_i|$, $x = (x_1, x_2, ..., x_q)$. For a $q \times q$ matrix $A = (a_{ij})$, define its norm $|A|$ by $|A| = \max_j \sum_{i=1}^{q} |a_{ij}|$. The identity matrix is denoted by E. Note that if $Y(n)$ is the fundamental matrix of (L), then it is unique solution of the following matrix difference equation

$$Y(n+1) = A(n)\, Y(n), \text{ with } Y(n_0) = E,$$

and also that $Y(n) = A(n-1)A(n-2)...A(n_0)E$, from which, since $A(n)$ is nonsingular, follows that $Y^{-1}(n)$ exists for any $n \in N(n_0)$. For a vector function φ given on $N_n(0)$ define $||\varphi||_n = \max\{|\varphi(s)|; s \in N_n(0)\}$. Consider the non-linear Volterra type difference equations (V) where $A(n)$ is a non- singular $q \times q$ matrix on $\mathbf{N}$ and $B(n, m, x)$ is a q-vector defined for $m \in N_n(0)$ and $x \in \mathbf{R}^q$. The solution of (V) with initial values (n_0, φ) will be denoted by $x(n; n_0, \varphi)$, where $n_0 \in \mathbf{N}$ and $\varphi : \{0, 1, ..., n_0\} \to \mathbf{R}^q$ is a given functin, and it is given by (1.3). As it is customary, here and in the following, we use the convention $\sum_{i=s}^{k} u(i) = 0$, $\Pi_{i=s}^{k} v(i) = 1$ when $s > k$.

We give some definitions of stability of the zero solution of (V). To this end we assume $B(n, m, 0) = 0$, $m \in N_n(n_0)$, i.e. $x(n) = 0, n \in N$, is a solution of (V).

Definition 2.1. *The zero solution of (V) is* **stable** *(S), if for every $\varepsilon > 0$ and any $n_0 \in N \equiv N(0)$ there exists a $\delta(\varepsilon, n_0) > 0$ such that, $||\varphi||_{n_0} < \delta$ and $n \in N(n_0)$ imply $|x(n; n_0, \varphi)| < \varepsilon$.*

Definition 2.2 *The zero solution of (V) is* **uniformly stable** *(U.S.), if it is stable and the above δ is independent of n_0.*

Definition 2.3. *The zero solution of (V) is* **asymptotically stable** *(A.S.) if it is stable and, for any $n \in \mathbf{N}$ there exists $\delta_0(n_0) > 0$ such that $||\varphi||_{n_0} < \delta_0$ implies $|x(n; n_0, \varphi)| \to 0$ as $n \to \infty$ (i.e., it is* **attractive** *).*

Definition 2.4. *The zero solution of (V) is* **uniformly asymptotically stable** *(U.A.S.), if it is U.S., the above δ_0 in Definition 2.3. is independent of n_0 and for every $\varepsilon > 0$ there exists $T(\varepsilon) \in N(1)$ such that $||\varphi||_{n_0} < \delta_0$ and $n \in N(n_0 + T)$ imply $|x(n; n_0, \varphi)| < \varepsilon$.*

Definition 2.5. *The zero solution of (V) is* **exponentially asymptotically stable** *(Ex.A.S.), if there exists $0 < p < 1$ and for every $\varepsilon > 0$ there exists $\delta(n_0, \varepsilon) > 0$ such that $n_0 \in N, ||\varphi||_{n_0} < \delta$ and $n \in N(n_0)$ imply $|x(n; n_0, \varphi)| < \varepsilon p^{n-n_0}$.*

To establish our results on stability of the zero solution of (V) we need to recall two Lemmas which will be used in the following sections. It is known that various stability properties for (L) are characterized by those of $Y(n)$. We start with a Lemma which connects the stability properties of the equation (L) to the fundamental matrix $Y(n)$.

Lemma 2.1 *([2], Lemma 3). The zero solution of (L) is:*

i). stable if and only if $|Y(n)| \le L, L \ge 1, n \in N$;

ii). uniformly stable if and only if $|Y(n)Y^{-1}(m)| \le L, L \ge 1, m \in N, n \in N(m)$;

iii). asymptotically stable if and only if $|Y(n)| \to 0$ as $n \to \infty$;

iv). uniformly asymptotically stable if and only if, for any $n \in N$, we have $|Y(n)Y^{-1}(m)| \le Lp^{n-m}$, $n \in N(m)$, where $L \ge 1, 0 < p < 1$.

A more general result including ii) from Lemma 2.1 is given by

Lemma 2.2 *([5]). Let $Y(n)$ be a non-singular matrix defined for $n \in N(n_0)$. Suppose there exists a projection P and a constant $L > 1$ such that for $n \in N(n_0)$ the inequality*

$$(2.1) \qquad \sum_{m=n_0}^{n} |Y(n)\; P\; Y^{-1}(m)|^q \le L^q, q > 1,$$

holds. Then, there exists a constant $K > 0$ such that the following inequality is valid for $n \in N(n_0)$:

$$(2.2) \qquad |Y(n)P| \le K(n+1-n_0)^{1-1/q}.\exp[-L^{-1}(n+1-n_0)^{1/q}].$$

We note that from (2.2) it follows $\lim_{n\to\infty}|Y(n)P| = 0$. An analogous result may be obtained for $q = 1$, when the inequality (2.2) becomes

$$(2.3) \qquad |Y(n)P| \le \bar{K}\exp[-\bar{K}^{-1}(n-n_0)], \bar{K} \ge L|Y(n_0)|P.$$

In particular , $P = E$ may be accepted. Sometimes, in the sequel we shall assume that there exists $L > 1$ such that

$$(2.4) \qquad \sum_{m=0}^{n} |Y(n)Y^{-1}(m)| \le L, \text{ for } n \in N.$$

We have noted that under (2.4) it follows

$$|Y(n)| \to 0 \text{ as } n \to \infty, \tag{2.5}$$

and according to Lemma 2.1 that is equivalent to the asymptotic stability of the zero solutions of (L).

Our aim is to find criteria which guarantee that a certain kind of stability of the zero solution of the linear equation (L) is perserved, possible in a weaker sense, by the zero solution of the perturbed equation (V).

3.Stability. In this section, we assume the zero solution of the linear equation (L) is stable, so that there exists $K \geq 1$ such that (Lemma 2.1)

$$|Y(n)| \leq K \ , \ n \geq n_0. \tag{3.1}$$

We make the following assumption

$$(H_1) : |B(n,m,x)| \leq c(n,m)|x|, n > m \geq 0, |x| < H \text{ for some } H > 0.$$

Theorem 3.1. *Suppose that the assumption (3.1) and (H_1) hold and the fundamental matrix $Y(n)$ of (L) satisfies*

$$|Y(n)Y^{-1}(m+1)| \leq L\, b(m) \ , \ n > m \geq n_0, \tag{3.2}$$

where $L \geq 1$ and $b(m) \geq 1$, and in addition ,

$$\sum_{v=n_0}^{\infty} \sum_{k=0}^{v-1} b(v)c(v,k) < \infty. \tag{3.3}$$

Then the zero solution of (V) is stable.

Proof. Let $x(n)$ be a solution of (V) with $x(n_0) = \varphi(n_0)$. By the variation of parametres formula, (3.1) and (3.2) we have

$$|x(n)| \leq K|Y^{-1}(n_0)|\ |\varphi(n_0)| + L \sum_{m=n_0}^{n-1} \sum_{k=0}^{m-1} b(m)c(m,k)|x(k)|, n \geq n_0.$$

If we define $r(n) = \max\{|x(s)|, s = 0,1,...,n\}$, then we have

$$r(n) \leq K|Y^{-1}(n_0)|\ |\varphi(n_0)| + L \sum_{m=n_0}^{n-1} [\sum_{k=0}^{m-1} b(m)c(m,k)]r(m).$$

By Gronwall's inequality

$$|x(n)| \leq r(n) \leq M(n_0)|\varphi(n_0)|,$$

where $M(n_0) = K|Y^{-1}(n_0)|\exp L \sum_{v=n_0}^{\infty} \sum_{k=0}^{v-1} b(v)c(v,k) < \infty$.

This shows that the zero solutions of (V) is stable.

4.Uniformly stability. In this section, we assume the zero solution of the homogeneous linear system (L) is U.S., so that there exists $K \geq 1$ such that (Lemma 2.1)

$$|Y(n)Y^{-1}(m)| \leq K, n \geq m \geq 0, \tag{4.1}$$

where $Y(n)$ is a fundamental matrix of (L). We shall be concerned with the problem when the property of U.S. is preserved for (V) under the effect of a "small" nonlinear perturbation of the forcing term $B(n,m,x)$. With regard to $B(n,m,x)$ we make the same assumption (H_1).

Theorem 4.1. *Suppose that the assumption (4.1) and (H_1) hold and there exists a positive constant $M > 0$ such that*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} c(n,m) \leq M. \tag{4.2}$$

Then the zero solution of (V) is uniformly stable.

Proof. For any $0 < \varepsilon < H$ let $\delta(\varepsilon) < \varepsilon/K.\exp(KM)$ and $\|\varphi\|_{n_0} < \delta(\varepsilon)$. Assume that there exists $n_1 \geq n_0$ such that $|x(n_1)| = \varepsilon$ and $|x(n)| < \varepsilon$ for $n = n_0, n_0+1, ..., n_1 - 1$. By the variation of parametres formula, we have

$$|x(n)| \leq |Y(n)Y^{-1}(n_0)||\varphi(n_0)| + \sum_{m=n_0}^{n-1} |Y(n)Y^{-1}(m+1)| \sum_{k=0}^{m-1} c(m,k)|x(k)| \leq$$

$$\leq K\delta(\varepsilon) + K \sum_{m=n_0}^{n-1} \sum_{k=0}^{m-1} c(m,k)|x(k)|, \ for\ n = n_0, n_0+1, ..., n_1.$$

Define $r(n) = \max|\{x(m)|; m = 0,1,...,n\}$. Then we have

$$r(n) \leq K\delta(\varepsilon) + K \sum_{m=n_0}^{n-1} [\sum_{k=0}^{m-1} c(m,k)]r(m).$$

By Gronwall's inequality, we have

$$|x(n)| \leq r(n) \leq K\delta(\varepsilon)\exp\left[K \sum_{m=n_0}^{n-1} \sum_{k=0}^{m-1} c(m,k)\right] \leq K\exp(KM)\delta(\varepsilon) < \varepsilon$$

on $\{n_0, n_0+1, ..., n_1\}$. Therefore $|x(n_1)| < \varepsilon$, which is a contradiction. Thus the zero solution of (V) is U.S. We now give an example of equation (V) which satisfies the conditions of Theorem 4.1.

Example 4.1. Let $A(n)$ be the 2×2 matrix $A(n) = \begin{bmatrix} 1 & 2^{-1} \\ 0 & 2^{-1} + 2-n \end{bmatrix}$.

We have $|Y(n)Y^{-1}(m)| \le \Pi_{v=m}^{n-1}|A(v)| \le \Pi_{v=1}^{\infty}(1+2^{-v}) \le \exp(\sum_{v=1}^{\infty} 2^{-v}) =$ e and so (4.1) is satisfied with $K =$ e > 1 . Let now $B(n,m,x)$ be given by $B(n,m,x) = \begin{bmatrix} -r^{2n-m} & -nr^n \\ nr^n & -r^{2n-m} \end{bmatrix} \begin{bmatrix} x_1^3 \\ x_2^2 \end{bmatrix}$, $0 < r < 1$, and $|x| = |x_1| + |x_2|$, $|A| = \max\{|a_{1j}| + |a_{2j}|; j = 1,2\}$.

Then we have $|B(n,m,x)| = (nr^n + r^{2n-m})\,(|x_1|^3 + |x_2|^2 \le (nr^n + r^{2n-m}) \cdot$ $\cdot(|x_1| + |x_2|) \equiv c(n,m)|x|$, if we choose $H = 1$.

Condition (4.2) holds because

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} c(n,m) = \sum_{n=0}^{\infty}[n(n+1)r^n + (r^{2n+1} - r^n)/(r-1)] < \infty.$$

5. Asymptotic stability. In this section we study the asymptotic stability of the zero solution of (V). We first assume that the condition (2.4) is satisfied which implies, as we have noted, $|Y(n)| \to 0$, as $n \to \infty$. With regard to $B(n,m,x)$ we shall assume the same hypothesis (H_1) from section 3 is satisfied.

Theorem 5.1. *Suppose that the assumptions (H_1) and (2.4) hold and, in addition ,*

$$\sup_{n \ge 0} \sum_{m=0}^{n} c(n,m) < L^{-1}, \tag{5.1}$$

$$\lim_{s \to \infty} \sum_{m=0}^{n} c(s,m) = 0 \ \textit{for all} \ n \ge 0. \tag{5.2}$$

Then the zero solution of (V) is asymptotically stable.

Proof. We first show the stability of the zero solution. From (5.1) there exists a positive constant γ such that

$$0 < \gamma \le L^{-1} \ \textit{and} \ \sup_{n \ge 1} \sum_{m=0}^{n} c(n,m) \le \gamma. \tag{5.3}$$

From (2.4) there exists a positive constant T such that

$$|Y(n) \le T \ \textit{for all} \ n \in N. \tag{5.4}$$

For any $0 < \varepsilon < H$ and $n_0 \in N$, we choose $\delta = \delta(n_0, \varepsilon) < \min\{(1-\gamma L)\varepsilon/(T|Y(n_0)|), \varepsilon\}$. Consider the solution of (V) such that $\|\varphi\|_{n_0} \le \delta$. Assume that there exists $n_1 \ge n_0$ such that $|x(n_1)| = \varepsilon$ and $|x(n)| < \varepsilon$ on $n \in \{n_0, n_0+1, ..., n_1 - 1\}$. For all $n \in \{n_0, n_0+1, ..., n_1\}$ we have

$$|x(n)| \le |Y(n)Y^{-1}(n_0)||\varphi(n_0)| + \sum_{m=n_0}^{n-1} |Y(n)Y^{-1}(m+1)| \sum_{k=0}^{m-1} c(m,k)|x(k)| <$$

$$< T|Y^{-1}(n_0)|\delta + \varepsilon \sum_{m=n}^{n-1} |Y(n)Y^{-1}(m+1)| \sum_{k=0}^{m-1} c(m,k) < (1-\gamma L)\varepsilon + \gamma\varepsilon L = \varepsilon,$$

therefore $|x(n_1)| < \varepsilon$, which is a contradiction. Thus, the zero solution of (V) is stable. Next we will show that the zero solution of (V) is attractive. From the stability, taking $\varepsilon = 1$, it follows the existence of $\delta_0 = \delta(1, n_0) < 1$ such that $n_0 \ge 1$ and $\|\varphi\|_{n_0} < \delta_0$ imply

$$|x(n; n_0, \varphi)| < \min(H, 1) \ \textit{for all} \ n \ge 1. \tag{5.5}$$

In the sequel we consider the solution such that $\|\varphi\|_{n_0} < \delta_0$ and assume that there exist $\varphi(n)$ and $x(n) = x(n; n_0, \varphi)$ such that

$$\limsup_{n \to \infty} |x(n)| = \mu > 0. \tag{5.6}$$

Since $\gamma L < 1$, by (5.3), there exists a constant θ such that $\gamma L < \theta < 1$. By (5.6) there exists $n_1 \ge n_0$ such that

$$|x(k)| \le \mu/\theta \ \textit{for all} \ n \ge n_1. \tag{5.7}$$

From (5.2) there exists $M > n_1$ such that

$$\sum_{m=0}^{n_1} c(s,m) < (\theta - \gamma L)\mu/2\theta L \ \textit{for} \ s \ge M. \tag{5.8}$$

Then we have

$$|x(n)| \le |Y(n)||Y^{-1}(n_0)|\delta_0 + \sum_{m=n_0}^{n-1} |Y(n)Y^{-1}(m+1)| \sum_{k=0}^{m-1} c(m,k)|x(k)| \le$$

$$\le |Y(n)||Y^{-1}(n_0)|\delta_0 + |Y(n)| \sum_{k=0}^{M-1} |Y^{-1}(m+1)| \sum_{k=0}^{m-1} c(m,k)|x(k)| +$$

$$+ \sum_{m=M}^{n-1} |Y(n)Y^{-1}(m+1)| \sum_{k=0}^{n_1-1} c(m,k)|x(k)| +$$

$$+\sum_{m=M}^{n-1}|Y(n)Y^{-1}(m+1)|\sum_{k=N_1}^{m-1}c(m,k)|x(k)|.$$

From (2.4) , (5.3) and (5.7) we have

$$\sum_{m=M}^{n-1}|Y(n)Y^{-1}(m+1)|\sum_{k=n_1}^{m-1}c(m,k)|x(k)|\le L\gamma\mu/\theta,$$

and from (2.4), (5.5) and (5.8) we have

$$\sum_{m=M}^{n-1}|Y(n)Y^{-1}(m+1)|\sum_{k=0}^{m-1}c(m,k)|x(k)|\le L(\theta-\gamma L)\mu/2\theta L=(\theta-\gamma L)\mu/2\theta.$$

Thus, we have

$$|x(n)|\le|Y(n)|Y^{-1}(n_0)|\delta_0+|Y(n)|\sum_{m=n_0}^{M-1}|Y^{-1}(m+1)| \tag{5.9}$$

$$\sum_{k=0}^{m-1}c(m,k)|x(k)|+(\theta+\gamma L)\mu/2\theta.$$

Since $|Y(n)|\to 0$ as $n\to\infty$, by (2.5), we have $\mu\le(\theta+\gamma L)\mu/2\theta$ and thus $\mu<\mu$, which is impossible. Therefore, the zero solution of (V) is attractive, and the proof is now completed.

Corallary 5.1. *Suppose that (2.4) holds and $|B(n,m,x)|\le d(n-m)|x|$ for $n>m\ge 0$ and $|x|<H$, where $\sum_{k=0}^{\infty}d(k)<L^{-1}$. Then the zero solution of (V) is asymptotically stable.*

Proof. Since $\sup_{n\ge 0}\sum_{m=0}^{n}d(n-m)=\sup_{n\ge 0}\sum_{k=0}^{n}d(k)=\sum_{k=0}^{\infty}d(k)<L^{-1}$, it follows that (5.1) holds. Then, for any $n\ge 0$ we have

$$\lim_{s\to\infty}\sum_{m=0}^{n}d(s-m)=\lim_{s\to\infty}\sum_{k=s-n}^{s}d(k)=0,$$

i.e., (5.2) holds. Thus, by Thorem 5.1, the zero solution of (V) is A.S.

Corollary 5.2. *Suppose that (2.4) holds and $|B(n,m,x)\le f(n)g(m)|x|$ for $n\ge m\ge 0$ and $|x|<H$, where $f(n)\ge 0, g(m)\ge 0$, $\sup_{n\ge 0}f(n)\sum_{m=0}^{n}g(m)<L^{-1}$ and $\sum_{m=0}^{\infty}g(m)=\infty$. Then the zero solution of (V) is A.S.*

Proof. Since $\sum_{m=0}^{n}g(m)$ is an increasing function which tends to ∞ and $\sup_{n\ge 0}f(n)\sum_{m=0}^{n}g(m)<\infty$, then $f(n)\to 0$ as $n\to\infty$. Therefore for any $n\ge 0$ we have $\lim_{s\to\infty}f(s)\sum_{m=0}^{n}g(m)=0$, i.e., (5.2) holds and thus, by Theorem 5.1, the zero soltuion of (V) is A.S.

We now present an example which satisfies the conditions of Theorem 5.1.

Example 5.1 Consider the system of difference equations

$$x(n+1)=A(n)x(n)+\sum_{m=0}^{n-1}B(n,m,x(m)),\ \textit{for}\ n\in N(1), \tag{5.10}$$

where $A(n)=\begin{bmatrix}0 & (n+1)^{-1}\\ (3n^2+1)^{-1} & (n^2+1)^{-1}\end{bmatrix}$ and $B(n,m,x)=4^{-1}\begin{bmatrix}-r^{2n-m} & -nr^n\\ nr^n & -r^{2n-m}\end{bmatrix}\cdot\begin{bmatrix}x_1^3\\ x_2^2\end{bmatrix}$, $0<r<1$. Since $|Y(n)Y^{-1}(m)|\le\Pi_{i=m}^{n-1}|A(i)|<2(n+1)^{-1}$, we have $\sum_{m=0}^{n}|Y(n)Y^{-1}(m)|<2=L$, i.e., (2.4) is satisfied. Then, $|B(n,m,x)|=4^{-1}(nr^n+r^{2n-m})\left(|x_1|^3+|x_2|^2\right)\le 4^{-1}(nr^n+r^{2n-m})|x|$, for $n>m\ge 0$ and $|x|<H=1$. To verify (5.1), we consider

$$\sup_{n\ge 1}4^{-1}\sum_{m=0}^{n}(nr^n+r^{2n-m})=n(n+1)r^n/4+(r^{2n+1}-r^n)/4(r-1)<2^{-1},$$

if we choose $r=e^{-1}$.

On the other hand,

$$\sum_{m=0}^{n}c(s,m)=\sum_{m=0}^{n}(se^{-s}+e^{-2s+m})=s(n+1)e^{-s}+e^{-2s}(1-e^{n+1})/(1-e)\to 0,$$

as $s\to\infty$, for every fixed $n\in N(1)$, and therefore the zero solution of the considered solution is A.S.

Remark 5.1. We note that the condition (5.2) in Theorem 5.1 assures the attractivity of the zero solution of (V). If it is omitted, the zero solution cannot be attractive as we see by the following

Exemple 5.2 Consider the scalar difference equation

$$x(n+1)=\alpha x(n)+\sum_{m=0}^{n-1}b(m)x(m), \tag{5.11}$$

where $0<\alpha<1$, $b(m)=\begin{cases}\gamma, m=0,1,...,k-1,\\ 0, m=k,k+1,...\end{cases}$ k a fixed natural number and $0<\gamma<(1-\alpha)/k$.

It is easily to see that the conditions (2.4) and (5.1) are satisfied, but (5.2) does not. Indeed, for $y(n+1)=\alpha y(n)$ the fundamental matrix is given by

$$Y(n)=A(n-1)A(n-2)...A(n_0)Y(n_0)=\alpha^{n-n_0},$$

and (2.4) becomes

$$\sum_{m=0}^{n} \alpha^{n-n_0}\alpha^{-s+n_0} = (1-\alpha^{n+1})/(1-\alpha) < (1-\alpha)^{-1} = L, L > 1.$$

Then,

$$\sum_{m=0}^{n} c(n,m) = \sum_{m=0}^{k-1} \gamma = k\gamma < 1-\alpha = L^{-1},$$

and therefore (5.1) holds.

On the other hand we have

$$\sum_{m=0}^{n} c(n,m) = \sum_{m=0}^{k-1} \gamma = \gamma k,$$

from which $\lim_{s\to\infty} \sum_{m=0}^{n} c(s,m) = \gamma k \neq 0$,i.e., (5.2) is not satisfied. The solution $x(n)$ satisfying $x(n) = \varphi(n) \equiv x_0$ for $n = 0, 1, 2, ..., n_0$, may be obtained from

$$x(n+1) = \alpha x(n) + \sum_{m=0}^{k-1} b(m)x(m) + \sum_{m=k}^{n-1} b(m)x(m) =$$

$$= \alpha x(n) + \sum_{m=0}^{k-1} b(m)x(m) = \alpha x(n) + k\gamma x_0.$$

Taking $n := n_0, n_0+1, ..., n$, we get $x(n) = \alpha^{n-n_0}x_0 + k\gamma x_0(1-\alpha^{n-n_0})/(1-\alpha)$. Because $x(n) \to k\gamma x_0/(1-\alpha) \neq 0$, as $n \to \infty$, it follows that the zero solution of (5.11) is not attractive.

Now we shall give a result pointing out the connection with the stability property.

Theorem 5.2 *Suppose that the assumptions (3.2) and (3.3) from Theorem 3.1 hold and, in addition, that the fundamental matrix $Y(n)$ of (L) is such that $|Y(n)| \to 0$ as $n \to \infty$ (guaranteed by (2.4), for example). Then the zero solution of (V) is asymptotically stable.*

Proof. Let $n_0 \in N$ and $\varepsilon > 0$ be given. Since condition (3.1) is also satisfied the zero solution of (V) is stable. This assure the existances of $\delta(n_0) > 0$ such that

$$\|\varphi\|_{n_0} \leq \delta(n_0) \textit{ implies } |x(n)| < 1, n \geq n_0. \tag{5.12}$$

We have

$$|x(n)| \leq |Y(n)||Y^{-1}(n_0)x(n_0)| + |Y(n)| \sum_{m=n_0}^{n_1-1} |Y^{-1}(m+1)| \tag{5.13}$$

$$\sum_{k=0}^{m-1} |B(m,k,x(k))| + \sum_{m=n_1}^{n-1} |Y(n)Y^{-1}(m+1)| \sum_{k=0}^{m-1} |B(m,k,x(k))|,$$

where n_1 is an integer, $n_0 < n_1 < n$. Take $x(n_0) = \varphi(n_0)$ satisfying (5.12). Then, for a fixed and sufficiently large n_1, we have

$$\sum_{m=n_1}^{n-1} |Y(n)Y^{-1}(m+1)||B(m,k,x(k))| \leq L \sum_{m=n_1}^{n-1}\sum_{k=0}^{m-1} b(m)c(m,k)|x(k)| \leq$$

$$\leq L \sum_{m=n_1}^{\infty}\sum_{k=0}^{m-1} b(m)c(m,k) < \varepsilon.$$

From the last inequality and (5.13) we get

$$\lim_{n\to\infty}|x(n)| = 0 \; if \; \|\varphi\|_{n_0} < \delta(n_0),$$

because $|Y(n)| \to 0$ as $n \to \infty$ and $\varepsilon > 0$ was arbitrary . This completes the proof.

The last part of the proof of Theorem 5.2 may be simplified as we shall see in the sequel.

We make the following assumption

$$(H_2) : |B(n,m,x) \leq [b(m) + c(n,m)]|x|, n > m \geq 0, |x| < H,$$

where $\sum_{m=0}^{\infty} b(m) = b, 0 < b < L^{-1}$.

Theorem 5.3. *Suppose that the assumptions (H_2) and (2.4) hold and, in addition,*

$$\sum_{m=n_0}^{\infty}\sum_{k=0}^{m-1} c(m,k) < \infty. \tag{5.14}$$

Then, the zero solution of (V) is asmptotically stable.

Proof. First, we prove that the zero solution of (V) is stable. From (2.5), there exists a positive constant L_1 such that $|Y(n)| \leq L_1, n \in N$. Let $n_0 \in N$ and let $x(n)$ be the solution of (V) with $x(n_0) = \varphi(n_0)(x(n) = \varphi(n), n = 0, 1, ..., n_0)$. From the variation of parameters formula we have, for any $n \geq n_0$,

$$|x(n)| \leq L_1|Y^{-1}(n_0)||\varphi(n_0)| + \sum_{m=n_0}^{n-1} |Y(n)Y^{-1}(m+1)| \sum_{k=0}^{m-1} |x(k)|b(k)+$$

$$+ \sum_{m=n_0}^{n-1} |Y(n)Y^{-1}(m+1)| \sum_{k=0}^{m-1} c(m,k)|x(k)|.$$

Define $r(n) = \max\{|x(s)|; s = 0, 1, ..., n\}$. Then, for a certain integer $n_1, n_0 \le n_1 \le n$, we have

$$r(n) = |x(n_1)| \le L_1|Y^{-1}(n_0)||\varphi(n_0)| + br(n)L + L\sum_{m=0}^{n-1}(\sum_{k=0}^{m-1} c(m,k)r(m),$$

from which, since $bL < 1$, we have

$$r(n) = L_1|Y^{-1}(n_0)||\varphi(n_0)|/(1-bL) + \sum_{m=n_0}^{n-1}(\sum_{k=0}^{m-1} L(1-bL)^{-1}c(m,k))r(m).$$

From this, by Gronwall's inequality,

$$|x(n)| \le r(n) \le L_1|Y^{-1}(n_0)|(1-bL)^{-1}\exp L(1-bL)^{-1}\sum_{m=n_0}^{n-1}\sum_{k=0}^{m-1} c(m,k)|\varphi(n_0)|.$$

Therefore, for any $n_0 \ge 0$ there exists a $\delta(n_0) > 0$ such that any solution $x(n)$ of (V), with $\|\varphi\|_{n_0} < \delta$, satisfies $|x(n : n_0, \varphi)| < 1$ for all $n \ge n_0$.

To complete the proof we shall show that, if $\|\varphi\|_{n_0} < \delta(n_0)$, we have $x(n) \to 0$ as $n \to \infty$. To this end define $M = \limsup_{n\to\infty}|x(n)|$. Obviously, $0 \le M \le 1$. Choose d such that $0 < Lb < d < 1$. If $M > 0$, there exists $n_1 \ge n_0$ such that

$$|x(n)| < M/d \textit{ for all } n \ge n_1. \tag{5.15}$$

From the variation of parametres formula we have,

$$|x(n)| \le |Y(n)|[|Y(n_0)||\varphi(n_0)| + \sum_{m=n_0}^{n-1}\sum_{k=0}^{m-1}|Y^{-1}(m+1)||B(m,k,x(k))|+$$

$$+\sum_{m=n_1}^{n-1}|Y(n)Y^{-1}(m+1)|\sum_{k=0}^{n_1-1}|B(m,k,x(k))|,$$

and by virtue of (5.15) and the hypothesis of our theorem, we get

$$|x(n)| \le |Y(n)|\left[|Y^{-1}(n_0)||\varphi(n_0)| + \sum_{m=n_0}^{n_1-1}\sum_{k=0}^{m-1}|Y^{-1}(m+1)||B(m,k,x(k))|\right]+$$

$$+ LbMd^{-1} + LMd^{-1}\sum_{m=n_1}^{n-1}\sum_{k=0}^{m-1}c(m,k).$$

Since $\sum_{m=n_0}^{\infty}\sum_{k=0}^{m-1} c(m,k) < \infty$, taking if necassary a larger n_1 and keeping it fixed, we have

$$|x(n)| \le |Y(n)|\left[|Y^{-1}(n_0)||\varphi(n_0)| + \sum_{m=n_0}^{n_1-1}\sum_{k=0}^{m-1}|Y^{-1}(m+1)||B(m,k,x(k))|\right]+$$

$$+ LbMd^{-1} + \varepsilon.$$

By Lemma 2.2 $|Y(n)| \to 0$ as $n \to \infty$ and hence, $M = \limsup_{n\to\infty}|x(n)| \le LbMd^{-1} + \varepsilon$.

Since $\varepsilon > 0$ is arbitrary and $M > 0$, this implies $Lb \ge d$ in contradiction with our assumption. Therefore $M = 0$ and the Theorem is proved.

6.Uniform asymptotic stability. In this section, we assume that the zero solution of (L) is U.A.S., so that there exists $K \ge 1$ and $0 < p < 1$, such that (Lemma 2.3)

$$|Y(n)Y^{-1}(m)| \le Kp^{n-m}, n \ge m \ge 0. \tag{6.1}$$

Moreover, we assume the kernel $B(n,m,x)$ of (V) satisfies (H_3) : $|B(n,m,x)| \le (b(m) + c(n,m))|x|, n \ge m > 0, |x| \le H$, for some $H > 0$, where $b(m) \ge 0$ is such that $\sum_{m=0}^{\infty} b(m) = b < \infty$.

Thorem 6.1. *Suppose that the assumption (6.1) and (H_3) hold and, in addition,*

$$\sum_{m=n_0}^{\infty}\sum_{k=0}^{m-1} c(m,k) = L < \infty, \tag{6.2}$$

$$\mathrm{K\ b\ p^{-1}} + \ln p < 0. \tag{6.3}$$

Then, the zero solution of (V) is U.A.S.

Proof. Let $n_0 \ge 0$. From the variation parameters formula we get

$$|x(n) \le Kp^{n-n_0}|\varphi(n_0)| + \sum_{m=n_0}^{n-1}\sum_{k=0}^{m-1} K[b(k) + c(m,k)]p^{n-m-1}|x(k)|.$$

Setting $r(n) = \max\{|x(s)|; s = 0, 1, ..., n\}$,we have

$$|x(n) \le r(n) \le Kp^{n-n_0}|\varphi(n_0)| + K\sum_{m=n_0}^{n-1}\sum_{k=0}^{m-1}[b(k) + c(m,k)]p^{n-m-1}r(m).$$

From this, multiplying both sides by $p^{-(n-n_0)}$ we get

$$|x(n)|p^{-(n-n_0)} \le r(n)p^{-(n-n_0)} \le K|\varphi(n_0)|+$$

$$+ K \sum_{m=n_0}^{n-1} \sum_{k=0}^{m-1} [b(k)p^{-1} + c(m,k)p^{-1}] \cdot p^{-(m-n_0)} r(m),$$

from which, by Gronwell's inequality,

$$|x(n)| \le K|\varphi(n_0)| \exp(n - n_0)[Kp^{-1}b + \ln p + KLp^{-1}(n-n_0)^{-1}],$$

which implies

$$|x(n)| \le K \exp(KLp^{-1}|\varphi(n_0)|. \tag{6.4}$$

Let $q > 0$ be such that $M = Kbp^{-1} + \ln p + q < 0$. Then, for all $n \ge n_0$ such that $n - n_0 \ge n_1$, where n_1 is a sufficiently large positive integer, independent of n_0, we have

$$|x(n)| \le K|\varphi(n_0)| \exp[-M(n - n_0)], \tag{6.5}$$

from which, together with (6.1), we obtain the U.A.S. of the zero solution of (V).

Remark 6.1. If in addition to the hypotheses of Theorem 4.1, we assume that the fundamental matrix Y(n) of (L) is such that $|Y(n)| \to 0$ as $n \to \infty$, then the zero solution of (V) is U.A.S.

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Department of Mathematics
Polytechnic Institute
R- 6600, Iasi
ROMANIA

ANALELE STIINTIFICE ALE UNIVERSITATII "AL.I.CUZA" IASI
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DIFFERENTIAL INEQUALITIES AND VISCOSITY SOLUTIONS IN OPTIMAL CONTROL

BY

STEFAN MIRICA

Dedicated to Professor VIOREL BARBU on the occasion of his 50th birthday anniversary

The aim of the present paper is to prove that the value function of a general optimal control problem satisfies certain differential inequalities that imply other differential properties in the literature, in particular the fact that, in the case of bounded admissible controls, the value function is a strict viscosity of the Bellman-Isaacs equation of Dynamic Programming.

The use of Nonsmooth Analysis has led in the last 10-15 years to a large variety of interesting results ([1],[3],[4],[8]-[13], etc.) "rehabilitating" the Dynamic Programming approach, initially hancapped by the unrealistic assumption according to which, the value function is diferentiable.

Though the real interest of the Dynamic Programming method seems to consist in the sufficient optimality conditions of the type of "verification theorem" ([3],[5],etc.) the necessary optimality conditions of this type are increasingly important and seem to close-in to the point of "overpowering" the Pontryagin's Maximum Principle approach as the recent results in [1].[13],[15] show that the Maximum Principle may easily be derived from necessary optimality conditions of Dynamic Programming type for more and more general type of optimal control problems.

The results to follow significantly improve the corresponding results in [6] and [12] showing that the value function has stronger differential properties than that of being a viscosity solution ([1],[6],[8],etc.) of the Bellman- Isaacs equation of Dynamic Programming.

For notational convenience we consider only Mayer optimal control problems defined by continuously parametrized differential equations, the result to follow may easily be extended to Bolza problems (which may allways be written as Mayer problems ([2],[5])) and to problem defined by differential inclusions using the results cocerning the tangent and contingent directions to their trajectories in [14].

We consider the Mayer optimal control problem of minimizing the functional