$$+ K \sum_{m=n_0}^{n-1} \sum_{k=0}^{m-1} [b(k)p^{-1} + c(m,k)p^{-1}] \cdot p^{-(m-n_0)} r(m),$$

from which, by Gronwell's inequality,

$$|x(n)| \leq K|\varphi(n_0)| \exp(n - n_0)[Kp^{-1}b + \ln p + KLp^{-1}(n - n_0)^{-1}],$$

which implies

$$|x(n)| \leq K \exp(KLp^{-1}|\varphi(n_0)|. \tag{6.4}$$

Let $q > 0$ be such that $M = Kbp^{-1} + \ln p + q < 0$. Then, for all $n \geq n_0$ such that $n - n_0 \geq n_1$, where n_1 is a sufficiently large positive integer, independent of n_0, we have

$$|x(n)| \leq K|\varphi(n_0)| \exp[-M(n - n_0)], \tag{6.5}$$

from which, together with (6.1), we obtain the U.A.S. of the zero solution of (V).

Remark 6.1. If in addition to the hypotheses of Theorem 4.1, we assume that the fundamental matrix Y(n) of (L) is such that $|Y(n)| \to 0$ as $n \to \infty$, then the zero solution of (V) is U.A.S.

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DIFFERENTIAL INEQUALITIES AND VISCOSITY SOLUTIONS IN OPTIMAL CONTROL

BY

STEFAN MIRICA

Dedicated to Professor VIOREL BARBU on the occasion of his 50th birthday anniversary

The aim of the present paper is to prove that the value function of a general optimal control problem satisfies certain differential inequalities that imply other differential properties in the literature, in particular the fact that, in the case of bounded admissible controls, the value function is a strict viscosity of the Bellman-Isaacs equation of Dynamic Programming.

The use of Nonsmooth Analysis has led in the last 10-15 years to a large variety of interesting results ([1],[3],[4],[8]-[13], etc.) "rehabilitating" the Dynamic Programming approach, initially hancapped by the unrealistic assumption according to which, the value function is diferentiable.

Though the real interest of the Dynamic Programming method seems to consist in the sufficient optimality conditions of the type of "verification theorem" ([3],[5],etc.) the necessary optimality conditions of this type are increasingly important and seem to close-in to the point of "overpowering" the Pontryagin's Maximum Principle approach as the recent results in [1].[13],[15] show that the Maximum Principle may easily be derived from necessary optimality conditions of Dynamic Programming type for more and more general type of optimal control problems.

The results to follow significantly improve the corresponding results in [6] and [12] showing that the value function has stronger differential properties than that of being a viscosity solution ([1],[6],[8],etc.) of the Bellman- Isaacs equation of Dynamic Programming.

For notational convenience we consider only Mayer optimal control problems defined by continuously parametrized differential equations, the result to follow may easily be extended to Bolza problems (which may allways be written as Mayer problems ([2],[5])) and to problem defined by differential inclusions using the results cocerning the tangent and contingent directions to their trajectories in [14].

We consider the Mayer optimal control problem of minimizing the functional

$C(\cdot)$ of the form:

$$C(x(\cdot)) = g(t_1, x(t_1)), \quad x(\cdot) \in \mathcal{X}_a(s, y, E_1) \tag{1}$$

over a specified class $\mathcal{X}_a(s, y, E_1)$, of admissible solutions, $x(\cdot) : [s, t_1] \to R^n$, of the parametrized differential inclusion:

$$x' \in f(t, x, U), \quad x(s) = y \tag{2}$$

satisfying the end-point and, respectively, the phase space, constraints of the form:

$$(t_1, x(t_1)) \in E_1 \subset R \times R^n \tag{3}$$

$$(t, x(t)) \in E_0 \subset R \times R^n \quad (\forall) t \in [s, t_1). \tag{4}$$

The set $\mathcal{X}_a(s, y, E_1)$ of admissible trajectories may be generated by a corresponding class, $\mathcal{U}_a(s, y, E_1)$, of measurable (admissible) controls $u(\cdot); [s, t_1] \to U$ for which the problem:

$$x' = f(t, x, u(t)), \quad x(s) = y \tag{5}$$

has a Caratheodory solution, $x(\cdot) \in \mathcal{X}_a(s, y, E_1)$, satisfying (3)-(4).

As usual in Optimal Control Theory, the class $\mathcal{X}_a$ of admissible trajectories may be one of the following sets of solutions of the differential inclusion in (2): the set $\mathcal{X}_{pc}$ of piecewise differentiable solutions (for which the derivatives, $x'(\cdot)$, are piecewise continuous), the set $\mathcal{X}_r$ of regular ones (for which $x'(\cdot)$ is regulated, having a countable number of discontinuities, all of the first kind) and for any $p \in [1, \infty]$, the set $\mathcal{X}_p$ of solutions $x(\cdot)$ for which $x'(\cdot) \in L^p([s, t_1]; R^n)$.

We note that the set $\mathcal{X}_c$ of classical (continuosly diferentiable) solution of (2) is usually avoided since the problem (1) - (4) may not have a solution in this class.

As it is well known, the Dynamic Programming approach uses the value function defined by:

$$W(s, y) = \begin{cases} g(s, y) & , if\ (s, y) \in E_1 \\ \inf\{C(x(\cdot));\ x(\cdot) \in \mathcal{X}_a(s, y, E_1)\} & , if\ (s, y) \in E_0 \end{cases} \tag{6}$$

to obtain necessary and/or sufficient optimality conditions.

In that follows we shall use the following subsets of the set $E = E_0 \cup E_1$ on which the value function is defined:

$$E_0^m = \{(s, y) \in E_0; \tilde{\mathcal{X}}_a(s, y, E_1) = \{x(\cdot) \in \mathcal{X}_a(s, y, E_1); W(s, y) = C(x(\cdot))\} \neq \emptyset\}$$

$$\tilde{E} = \{(s, y) \in E; (\exists)(t_0, x_0) \in E_0^m, \tilde{x}(\cdot) \in \tilde{\mathcal{X}}_a(t_0, x_0, E_1) : t_0 < s, x(s) = y\} \tag{7}$$

$$E_0^i = \{(s, y) \in E_0; W(s, y) \in R\} = E_0 \cap \operatorname{dom} W(.,.)$$

for which the following relations hold : $\tilde{E} \cap E_0 \subset E_0^m \subset E_0^i$.

For every $(s, y) \in E$ we denote by $\mathcal{X}_a(s, y)$ the set of locally admissible trajectories from (s, y) which are the solutions $x(\cdot) : I(x(\cdot)) \subset R \to R^n$ of (2), of the same type as those in $\mathcal{X}_a$ for which the phase space constraints in (4) are satisfied on the interval $I(x(\cdot))$; for every $(s, y) \in E_0$ we define the reachable set of the system (2) at $t > s$ from (s, y) as follows:

$$R_a(t; s, y) = \{x(t); x(\cdot) \in \mathcal{X}_a(s, y), t \in I(x(\cdot))\} \tag{8}$$

At the origin of the Dynamic Programming results in the literature seem to be the first two statements of the following:

Proposition 1. ([2],[5],etc.). *The value function in (6) has the following properties:*

(i) *For any $(s, y) \in E_0, x(\cdot) \in \mathcal{X}_a(s, y)$, the function $w_x(t) = W(t, x(t))$ is nondecreasing on the interval $I(x(\cdot))$;*

(ii) *If $(s, y) \in E_0^m$ than the admissible trajectory $\tilde{x}(\cdot) \in \mathcal{X}_a(s, y, E_1)$ is optimal iff the function $w_{\tilde{x}}(t) = W(t, \tilde{x}(t))$ is constant on the interval $[s, \tilde{t}_1]$;*

(iii) *If $(s, y) \in E_0^m$ and $\tilde{x}(\cdot) \in \mathcal{X}_a(s, y, E_1)$ is optimal for the problem (1)-(4) then one has:*

$$W(t, z) > W(t, \tilde{x}(t))\ (\forall)\ z \in R_a(t; s, y) \subset E_0(t) = \{z \in R^n; (t, z) \in E_0\} \tag{9}$$

for any $t \in (s, t_1)$.

Since, generally, the value function is not differentiable, the monotonicity properties in Proposition 1 may be converted into (more efficient) differential relations only in terms of suitable generalized differentiability concepts.

The already existing results in Nonsmooth Analysis ([3],[4],[6],[8] - [18], etc.) suggest the following possible way of choosing the generalized differentiability concepts that are most suitable for a given problem: for any type of tangent cone to a subset $X \subset R^n$ at a point $x \in X$, say $T_x X$, one may introduce the corresponding T-extreme directional derivatives of a function $g(\cdot) : X \to R$ at a point $x \in \operatorname{dom} g()$ as follows:

$$\overline{D}_T g(x; v) = \sup\{w \in R; (v, w) \in T_{(x, g(x))} H(g)\}, v \in T_x X \tag{10}$$

$$\underline{D}_T g(x; v) = \inf\{w \in R; (v, w) \in T_{(x, g(x))} E(g)\}, u \in T_x X$$

$$\overline{D}_{\overline{T}} g(x; v) = -\underline{D}_T g(x; -v), \underline{D}_{\overline{T}} g(x; v) = -\overline{D}_T g(x; -v), v \in T_{\overline{x}} X = -T_x X$$

where the hypograph and the epigraph are, respectively, defined by:

$$H(g) = \{(x, y); x \in X, y \leq g(x)\}; \quad E(g) = \{(x, y); x \in X, y \geq g(x)\}$$

Some differential properties of nonsmooth functions may be expressed in terms of the associated semidifferentials defined by:

(11) $$\overline{\partial}_T g(x) = \{g \in R^n; < q, v >\geq \overline{D}_T g(x; v)\ (\forall)\ v \in T_x X\}$$

$$\underline{\partial}_T g(x) = \{g \in R^n; < q, v >\geq \underline{D}_T g(x; v)\ (\forall)\ v \in T_x X\}$$

and by other differentiability concepts such as set-valued directional derivatives, prederivatives, etc. .

The results to follow seem to suggest that, from the multitude of tangent cones in the literature (e.g. [16]), the most suitable for our problem are the Bouligand-Severi contingent cones;

$$K_x^+ X = \{v \in R^n; (\exists)(\theta_m, v_m) \rightarrow$$

(12) $$\rightarrow (O+, v) : x + \theta_m v_m \in X\ (\forall)\ m \in N\}$$

$$K_x^- = -K_x^+ X = \{v \in R^n; (\exists)(\theta_m, v_m) \rightarrow$$

$$\rightarrow (O-, v) : x + \theta_m v_m \in X\ (\forall)\ m \in N\}$$

the quasitangent ("intermediate") cones, defined by:

$$Q_X^+ X = \{v \in R^n;\ (\forall)\ \theta_m \rightarrow O+\ (\exists)$$

(13) $$v_m \rightarrow v : x + \theta_m v_m \in X(\forall) m \in N\}$$

$$Q_x^- X = -Q_X^+ X = \{v \in R^n; (\forall)\theta_m \rightarrow O - (\exists)$$

$$v_m \rightarrow v : x + \theta_m v_m \in X(\forall) m \in N\}$$

and Clarke's ("peritangent") cones defined by:

$$C_X^+ X = \{v \in R^n; (\forall)(y_m \theta_m) \rightarrow (x, O+), y_m \in X, (\exists)$$

(14) $$v_m \rightarrow v : y_m + \theta_m v_m \in X\}$$

$$C_x^- X = -C_x^+ X$$

and the corresponding differentiability concepts introduced like in (10) and (11).

From the definitions in (10) and (11) of the extreme directional derivatives and, respectively, semidifferentials, it follows that the obvious inclusions:

$$C_x^+ X \subset Q_x^+ X \subset K_x^+ X, \quad C_x^- X \subset Q_x^- X \subset K_x^- X$$

imply the following relations:

(15) $$\overline{D}_K^+(x; v) \geq \overline{D}_Q^+ g(x; v) \geq \overline{D}_C^+ g(x; v),$$

$$\underline{D}_K^+(x; v) \leq \underline{D}_Q^+ g(x; v) \leq \underline{D}_C^+ g(x; v),$$

(16) $$\overline{\partial}_K g(x) \subset \overline{\partial}_Q g(x) \subset \overline{\partial}_C g(x),$$

$$\underline{\partial}_K g(x) \subset \underline{\partial}_Q g(x) \subset \underline{\partial}_C g(x)$$

The results in [6],[8],[17], etc. show that the contingent semidifferentials $\overline{\partial}_K g(x), \underline{\partial}_K g(x)$ generalize the usual (Fréchet)derivative and coincide with the semidifferentials recently used in the theory of viscosity solutions of Hamilton-Jacobi equations ([8],[9], etc.) while Clarke's semidifferentials $\overline{\partial}_C g(x), \underline{\partial}_C g(x)$ defined as in (11) by Clarke's tangent cone, $C_x^+ X$ in (14), generalize the strict (Fréchet) derivative and coincide with Clarke's generalized gradient, $\partial_C g(x)$, if $g(\cdot)$ is locally Lipschitz at x.

It is interesting to note that if $g(\cdot)$ is locally Lipschitz at $x \in \mathrm{Int}(X)$ then the inequalities in (15) may be completed as follows:

$$\sup\{< p, v >; p \in \partial_C g(x)\} = \limsup_{(y,\theta)\rightarrow(x,\theta+)} (g(y + \theta.v) - g(y))/\theta =$$

(17) $$= \underline{D}_C^+ g(x; v) \geq \underline{D}_Q^+ g(x; v) = \overline{D}_K^+ g(x; v) \geq \underline{D}_K^+ g(x; v) =$$

$$= \overline{D}_Q^+ g(x; v) \geq \overline{D}_C^+ g(x; v) = \inf\{< p, v >; p \in \partial_C g(x)\}$$

$$= \liminf_{(y,\theta)\rightarrow(x,0+)} (g(y + \theta.v) - g(y))/\theta$$

and Clarke's generalized gradient, $\overline{\partial}_C g(x) = \underline{\partial}_C g(x) = \partial_C g(x)$ is a nonempty compact subset of R^n while other semidifferentials may be empty (e.g. if $g(x_1, x_2) = |x_1| - |x_2|$ then $\overline{\partial}_Q g(0,0) = \partial_Q g(0,0) = \emptyset$)

Most of the results to follow will be proved in the very general case the parametrized vector field $f(.,.,.)$ in (2) satisfes the following:

Hypothesis 2. *U is a nonempty set, $D \subset R \times R^n$ is open and $f : D \times U \rightarrow R^n$ is such that $f(.,.,.u) : D \rightarrow R^n$ is continuous for any $u \in U$.*

We prove first the following preliminary result which may be of more general interest:

Lemma 3. *Let Hypothesis 2 be satisfied and $(t_0, x_0) \in D$.*

(i) *For any $u \in U$ there exists $a, r > 0$ such that for any $s \in I_0 = [t_0 - a, t_0 + a]$, $y \in B_r(x_0) = \{y \in R^n; \|y - x_0\| < r\}$ there exists a solution $x(\cdot; s, y) \in \mathcal{X}_c(s, y)$ of the problem:*

(18) $$x' = f(t, x, u), \quad x(s) = y$$

such that

(19) $$\lim_{(s,y,\theta)\to(t_0,x_0,0)}(x(s+\theta;s,y)-y)/\theta = f(t_0,x_0,u)$$

(ii) *If* $f(t_0,x_0,U)\subset \mathbf{R}^n$ *is bounded and its* **cone of unbounded directions** *is defined by:*

(20) $$D^+_\infty f(t_0,x_0,U)=\{v\in\mathbf{R}^n;(\exists)\{v_p\}\subset f(t_0,x_0,U),$$
$$r_p\to o^+ : \|v_p\|\to+\infty, r_p\cdot v_p\to v \text{ as } p\to\infty\}$$

then for any $v\in D^+_\infty f(t_0,x_0,U)$ *and any* $(s_m,y_m,\theta_m)\to(t_0,x_0,0)$ *there exists* $t_m\to o^+$ *and* $m_0\in\mathbf{N}$ *such that for any* $m\geq m_0$ *there exists a solution* $x_m(\cdot)\in\mathcal{X}_c(s_m,y_m)$ *of (2) such that:*

(21) $$\lim_{m\to\infty}(x_m(s_m+\theta_m t_m)-y_m)/\theta_m = v$$

Proof: Since the vector field $f(.,.,u)$ is continuous, from a well known refinement of Peano's existance theorem (e.g. Theorem 10.1.1 in [7]) it follows that there exist $a,r>0$ such that for any $(s,y)\in I_0\times B_r(x_0)$ there exists a solution $x(.;s,y):I_0\to\mathbf{R}^n$ of the problem (18) and moreover, for any such solution one has:

(22) $$\|x(t;s,y)-x_0\|\leq 2r\ (\forall)\ t,s\in I_0, y\in B_r(x_0)$$

Therefore, if $K>1$ is taken such that:

(23) $$K\geq\sup\{\|f(t,x,u)\|;\ (t,x)\in I_0\times B_{2r}(x_0)\subset D\}$$

then one has:

(24) $$\|x(t;s,y)-y\|\leq K|t-s|\ (\forall)\ t,s\in I_0, y\in B_r(x_0)$$

Further on, from the continuity of $f(.,.,u)$ at $(t_0,x_0)\in D$ it follows that for any $\varepsilon>0$ there exists $\delta_\varepsilon>0,\delta_\varepsilon\leq\min(a,r)$ such that $\|f(t,x,u)-f(t_0,x_0,u)\|\leq \leq\varepsilon\ (\forall)\ |t-t_0|<\delta_\varepsilon, \|y-x_0\|<\delta_\varepsilon$.

Hence if we take s,y,θ,t such that : $|s-t_0|<\delta_\varepsilon/3$, $\|y-x_0\|<\delta_\varepsilon/3$, $|\theta|<\delta_\varepsilon/3K$ and $t\in[s,s+\theta]$ then we have: $\|x(t;s,y)-x_0\|\leq\|y-x_0\|+\|x(t;s,y)-y\|\leq \|y-x_0\|+K|t-s|<\delta_\varepsilon$ and $|t-t_0|\leq|t-s|+|s-t_0|<\delta_\varepsilon$ and therefore $\|f(t,x(t;s,y),u)-f(t_0,x_0,u)\|\leq\varepsilon$.

Finally, from the associated integral equation it follows that we may write successively: $\|x(s+\theta;s,y)-y)/\theta-f(t_0,x_0,u)\|\leq(1/|\theta|)\ |\int_s^{s+\theta}\|f(t,x(t;s,y), u)-f(t_0,x_0,u)\|dt|\leq\varepsilon\ (\forall)\ |s-t_0|<\delta_\varepsilon/3$, $\|y-x_0\|<\delta_\varepsilon/3$, $|\theta|<\delta_\varepsilon/3K$ and (19) is proved.

(ii) If $v\in D^+_\infty f(t_0,x_0,U)$ then according to (20) there exist $t_p\to o^+$ and $\{u_p\}\subset U$ such that if $v_p=f(t_0,x_0,u_p)$ then $\|v_p\|\to+\infty$ and $t_pv_p\to v$ as $p\to\infty$.

From the continuity of the mappings $f(.,.,u_p)$ at (t_0,x_0) and from Theorem 10.1.1. in [7] it follows that for any $p\in N$ there exist $a_p,r_p>0$ such that $\|f(t,x,u_p)-f(t_0,x_0,u_p)\|<1/p\cdot t_p$ and such that for any $p\in\mathbf{N}, s\in I_p= [t_0-a_p,t_0+a_p], y\in B_{r_p}(x_0)$ there exists a solution $x_p(.;s,y):I_p\to\mathbf{R}^n$ of (18) that satiefies (22) - (24).

Further on, if $(s_m,y_m,\theta_m)\to(t_0,x_0,0)$ then there exists $m_0\in\mathbf{N}$ such that for any $m\geq m_0$ there exists $p_m\in\mathbf{N}$ such that $|s_m-t_0|\leq a_{p_m}/2$, $\|y_m-x_0\|\leq r_{p_m}/2$, $|\theta_m|\leq\min(a_{p_m}/2, r_{p_m}/2\cdot K_{p_m}), p_m\to\infty$ hence if we denote : $x_m(\cdot)= x_{p_m}(\cdot;s_m,y_m)$,$\tau_m=r_{p_m}\to 0^+$ as $m\to\infty$ then we have: $\|x_m(t)-x_0\|\leq\|x_m(t)-y_m\|+\|y_m-x_0\|\leq K_{p_m}|t-s_m|+\|y_m-x_0\|<r_{p_m}\ (\forall)\ t\in[s_m,s_m+\theta_m\cdot\tau_m], m\geq m_0$ and therefore we may write successively : $\|(x_m(s_m+\theta_m\tau_m)-y_m)/\theta_m-\tau_m\cdot f(t_0,x_0,u_{p_m})\|\leq(1/|\theta_m|)\ |\int_{s_m}^{s_m+\theta_m\tau_m}\|f(t,x_m(t),u_{p_m})-f(t_0,x_0,u_{p_m})dt\|dt\ |\leq 1/p_m\to 0$ as $m\to\infty$ and (21) is proved.

Some of the results to follow will be expressed in terms of the following sets of directions associated to the multifunction $F(s,y)=f(s,y,U),(s,y)\in D$ that defines the differential inclusion in (2) ([14]):

$$\hat{F}(s+,y)=\bigcap_{\theta,r>o}\overline{co}F([s,s+\theta),B_r(y)), \hat{F}(s-,y)=\bigcap_{\theta,r>o}\overline{co}F(s-\theta,s],B_r(y))$$

(25) $$\hat{F}_\infty(s+,y)=\bigcap_{\theta,r,p>o}[o,p)\cdot\overline{co}F([s,s+\theta),B_r(y))$$
$$\hat{F}_\infty(s-,y)=\bigcap_{\theta,r,p>o}[o,p)\cdot\overline{co}F(s-\theta,s],B_r(y))$$

for which one has:

(26) $$\overline{co}F(s,y)\subset\hat{F}(s+,y)\cap\hat{F}(s-;y), D^+_\infty f(s,y)\subset\hat{F}_\infty(s+,y)\cap\hat{F}_\infty(s-,y)$$

Moreover, if $F(.,.)$ is a closed-valued continuous multifunction (in particular, if U is a compact topological space and $f(.,.,.)$ is continuous then one has:

(27) $$\hat{F}(s+,y)=\hat{F}(s-,y)=\overline{co}F(s,y)$$

and if $F(.,.)$ is locally bounded then:

(28) $$\hat{F}_\infty(s+,y)=\hat{F}_\infty(s-,y)=\{0\}=D^+_\infty F(s,y)$$

As a corollary of the mean value theorem for the Lebeague integral one obtains the following result:

Proposition 4 ([14]). *Let* $f:D\times U\to\mathbf{R}^n$ *satisfy Hypothesis 2,* $(s,y)\in D$, *let* $x_m(\cdot):[s-\theta_0,s+\theta_0]\to\mathbf{R}^n, m\in\mathbf{N}$ *be solutions of the problem (2) and let* $\theta_m\to o+$.

(i) *If* $\lim_{m\to\infty}(x_m(s+\theta_m)-y)/\theta_m = v$ (*resp.*, $\lim_{m\to\infty}(x_m(s-\theta_m)-y)/(-\theta_m)=v$) *then* $v\in\hat{F}(s+,y)$ (*respectively* , $v\in\hat{F}(s-,y)$).

(ii) *If* $x_m(s+\theta_m)\to y$ *and* $\|(x_m(s+\theta_m)-y)/(\theta_m)\|\to+\infty$) *then there exist* $t_k, r_k\to o+$ *and* $m_k\to\infty$ *such that* $\lim_{k\to\infty}(x_{m_k}(s+t_kr_k)-y)/t_k = v\in \hat{F}_\infty(s+,y)\backslash\{0\}$ (*respectively* $\lim_{k\to\infty}(x_{m_k}(s-t_kr_k)-y)/-t_k = v\in\hat{F}_\infty(s-,y)\backslash\{0\}$).

The main result of the paper is the following:

Theorem 5. *If Hypothesis 2 is satisfies then the value function defined in (6) satisfies the following inequalities:*

$$\bar{D}^+_C W((s,y);(1,v))\geq 0(\forall)v\in \bar{co}f(s,y,U),(s,y)\in E^i_0\cap \mathrm{Int}(E_0) \tag{29}$$

$$\bar{D}^-_C W((s,y);(1,v))\geq 0(\forall)v\in \bar{co}f(s,y,U),(s,y)\in E^i_0\cap \mathrm{Int}(E_0) \tag{30}$$

$$\bar{D}^+_C W((s,y);(0,v))\geq 0(\forall)v\in \bar{co}D^+_\infty f(s,y,U),(s,y)\in E^i_0\cap \mathrm{Int}(E_0) \tag{31}$$

$$\bar{D}^+_C W((s,y);(0,v))\geq 0(\forall)v\in \bar{co}D^+_\infty f(s,y,U),(s,y)\in E^i_0\cap \mathrm{Int}(E_0) \tag{32}$$

$$\min\{\inf_{v\in\hat{F}(s+,y)}\underline{D}^+_K W((s,y);(1,v)), \inf_{v\in\hat{F}_\infty(s+,y)\backslash\{0\}}\underline{D}^+_K((s,y);(0,v))\}\leq 0(\forall)\quad (s,y)\in E^m_0 \tag{33}$$

$$\min\{\inf_{v\in\hat{F}(s-,y)}\underline{D}^+_K W((s,y);(1,v)), \inf_{v\in\hat{F}_\infty(s-,y)\backslash\{0\}}\underline{D}^+_K((s,y);(0,v))\}\leq 0(\forall)\quad (s,y)\in \tilde{E} \tag{34}$$

Moreover, in the multifunction $F(.,.)=f(.,.,U)$ is compactvalued and Hausdorff continuous (in particular, if U is a compact topological space and $f(.,.,.)$ is continuous), $E_1\subset \mathbf{R}\times\mathbf{R}^n$ is closed and $E_0\cap E_1=\emptyset$ then (33) - (34) may be replaced by:

$$\inf_{v\in\bar{co}f(s,y,U)}\underline{D}^+_K W((s,y);(1,v))\leq 0\ (\forall)\quad (s,y)\in E^i_0 \tag{35}$$

$$\inf_{v\in\bar{co}f(s,y,U)}\underline{D}^-_K W((s,y);(1,v))\leq 0\ (\forall)\quad (s,y)\in \tilde{E} \tag{36}$$

Proof. If $(s,y)\in E^i_0\cap\mathrm{Int}(E_0), u\in U, v=f(s,y,u),\theta_m\to o^+$ and $(s_m,y_m,w_m)\to(s,y,W(s,y))$ is such that $w_m\leq W(s_m,y_m)$ $(\forall)$ $m\in\mathbf{N}$ then from the first statement in Lemma 3 it follows that there exists $m_0\in\mathbf{N}$ such that for any $m\geq m_0$ there exsits a locally admissible solution $x_m(\cdot)\in\mathcal{X}_c(s_m,y_m)$ such that $(x_m(s_m+\theta_m)-y_m)/\theta_m = v_m\to v$ as $m\to\infty$; on the other hand, from Proposition 1 it follows that $W(s_m,y_m)\leq W(s_m+\theta_m,x_m(s_m+\theta_m)) = W(s_m+\theta_m,y_m+\theta_m v_m)$ hence if it take : $r_m=1,\bar{w}_m=0$ then $(s_m+\theta_m r_m, y_m+\theta_m v_m, w_m+\theta_m w_m)\in H(W)$ $(\forall)m\geq m_0$ which proves that $(1,v,o)\in C^+_{(s,y,W(s,y))}H(W)$ and (29) is proved since Clarke's upper directional derivative defined as in (10) by Clarke's tangent cone in (14) is concave and upper semicontinuous.

The inequalities (30) - (32) follow in the same way using the other statements in Lemma 3.

To prove (33) we note that from the definition in (7) of the subset $E^m_0\subset E_0$ it follows that if $(s,y)\in E^m_0$ then there exists an optimal trajectory, $\tilde{x}(\cdot)\in\mathcal{X}_a(s,y,E_1)$, hence according to Proposition 1 one has:

$$W(s+\theta,\tilde{x}(s+\theta)) = W(s,y)\quad (\forall)\quad \theta\in[0,t_1-s] \tag{37}$$

Obviously, only one of the following two possibilities may occur: 1) there exists $\theta_m\to 0+$ such that $(\tilde{x}(s+\theta_m)-y)/\theta_m = v_m\to v\in\hat{F}(s^+,y)$ (Proposition 4); 2) $\lim_{\theta\to 0+}\|(\tilde{x}(s+\theta)-y)/\theta\| = +\infty$.

In the first case , from (37) it follows that if we take $r_m=1,\bar{w}_m=0$ then $(s+\theta_m r_m, y+\theta_m v_m, W(s,y)+\theta_m\bar{w}_m)\in \mathrm{graph}(W)\subset E(W)$ $(\forall)m$ hence $(r_m,v_m,\bar{w}_m)\to(1,v,0)\in K^+_{(s,y,W(s,y))}E(W)$ which proves that $\underline{D}^+_K W((s,y);(1,v))\leq 0$.

In case 2), (which may happen only if $f(.,.,U)$ is not locally bounded and $a\in[1,\infty)$) from Proposition 4 it follow that there exist $t_k,r_k\to 0^+$ such that $(\tilde{x}(s+t_kr_k)-y)/r_k = v_k\to v\in\hat{F}_\infty(s+,y), \|v\|=1$, hence from (37) it follows that $(s+t_kr_k, y+t_kv_k, W(s,y))\in\mathrm{graph}(W)\subset E(W)$ $(\forall)$ $k\in\mathbf{N}$ and therefore $\underline{D}^+_K W((s,y);(0,v))\leq 0$ and (33) is proved.

The inequality (34) follows in the same way using the definition in (7) of the subset $\tilde{E}\subset E$.

If $(s,y)\in E^m_0$ then (35) - (36) follow from (34) - (45) and (27) - (28) ; to prove (35) for $(s,y)\in E^i\backslash E^m_o$ we note that from the definition in (6) of the value function it follows that for any $m\in\mathbf{N}$ there exsits an admissible trajectory $x_m(.)\in\mathcal{X}_a(s,y,E_1)$ such that : $W(s,y)<g(t^m_1,x_m(t^m_1))\leq W(s,y)+1/m$.

Since for any $\theta\in(0,t^m_1-s)$ the restriction map $x_m(.)|[s+\theta,t^m_1]$ is an admissible trajectory with respect to $(s+\theta,x_m(s+\theta))$ we have:

$$W(s,y)\leq W(s+\theta,x_m(s+\theta))\leq g((t^m_1,x_m(t^m_1))\leq \leq W(s,y)+1/m\ (\forall)\ m\in\mathbf{N} \tag{38}$$

On the other hand, if $f(.,.,U)$ is locally bounded then $\mathcal{X}_a(s,y)\subset\mathcal{X}_\infty(s,y)$ $(\forall)$ $(s,y)\in E_0$ and the sequence $\{(x_m(s+\theta_m)-y)/\theta_m\}$ is bounded for any $\theta_m\to o+$; moreover, since $E_1\subset\mathbf{R}\times\mathbf{R}^n$ is closed and $E_0\cap E_1=\emptyset$, there exist $\theta_0>0, m_0\in\mathbf{N}$ such that $t^m_1-s\geq\theta_0$ $(\forall)m\geq m_0$ (otherwise ($t^m_1,x_m(t^m_1))\to(s,y)\in E_0\cap E_1$)

hence there exists $\theta_m \to o+$ such that $1/m, \theta_m \to 0$ as $m \to \infty$ and such that $(x_m(s+\theta_m)-y)/\theta_m = v_m \to v \in \hat{F}(s+,y) = \bar{co}f(s,y,U)$.

As in the previous cases, from (38) we infer that $(s+\theta_m, y+\theta_m v_m, W(s,y)+\theta_m(1/m \cdot \theta_m)) \in E(W)$ $(\forall)$ $m \in \mathbf{N}$ hence $(1, v_m, 1/m \cdot \theta_m) \to (1,v,0) \in K^+_{(s,y,W(s,y))}E(W)$ which proves that $\underline{D}^+_K W((s,y);(1,v)) \leq 0$ and Theorem 5 is proved.

The differential inequalities in (29) - (36) may be expressed in weaker forms in terms of the semidifferentials defined as in (11) by the contingent and Clarke's tangent cones and of the following hamiltonians:

$$H(s,y,p,u) = < p, f(s,y,u) > , \quad \tilde{H}(s,y,p) = \inf\{H(s,y,p,u); u \in U\}$$

$$\tilde{H}_\infty(s,y,p) = \inf\{< p, v >; v \in D^+_\infty f(s,y,U), \|v\| = 1\}$$

$$\hat{H}(s+,y,p) = \inf\{< p, v >; v \in \hat{F}(s+,y)\} \tag{39}$$

$$\hat{H}(s-,y,p) = \inf\{< p, v >; v \in \hat{F}(s-,y)\}$$

$$\hat{H}_\infty(s+,y,p) = \inf\{< p, v >; v \in \hat{F}(s+,y), \|v\| = 1\}$$

$$\hat{H}_\infty(s-,y,p) = \inf\{< p, v >; v \in \hat{F}(s-,y), \|v\| = 1\}$$

Corollary 6. *If Hypothesis 2 holds then Clarke's and contingent semidifferentials of the value function in (6) satisfy the following relations:*

$$p_0 + \tilde{H}(s,y,p) \geq 0 \ (\forall) \ (p_0,p) \in \partial_C W(s,y) = \bar{\partial}_C W(s,y) \cup \underline{\partial}_C W(s,y), \quad (\forall) \ (s,y) \in E^i_0 \cap \mathrm{Int}(E_0) \tag{40}$$

$$\tilde{H}_\infty(s,y,p) \geq 0 \ (\forall) \ (p_0,p) \in \partial_C W(s,y), (s,y) \in E^i_0 \cap \mathrm{Int}(E_0) \tag{41}$$

$$\min\{p_0 + \hat{H}(s+,y,p), \hat{H}_\infty(s+,y,p)\} \leq \leq 0 \ (\forall) \ (p_0,p) \in \underline{\partial}_K W(s,y), (s,y) \in E^m_0 \tag{42}$$

$$\min\{p_0 + \hat{H}(s-,y,p), \hat{H}_\infty(s-,y,p)\} \leq \leq 0 \ (\forall) \ (p_0,p) \in \bar{\partial}_K W(s,y), (s,y) \in \tilde{E} \tag{43}$$

Moreover , if the multifunction $f(.,.,U)$ is compact-valued and Hausdorff continuous then (42)-(43) may be replaced by:

$$p_0 + \tilde{H}(s,y,p) \leq 0 \ (\forall) \ (p_0,p) \in \underline{\partial}_K W(s,y), (s,y) \in E^i_0 \tag{44}$$

$$p_0 + \tilde{H}(s,y,p) \leq 0 \ (\forall) \ (p_0,p) \in \bar{\partial}_K W(s,y), (s,y) \in \tilde{E} \tag{45}$$

and if, in addition, $W(.,.)$ is locally Lipschitz at $(s,y) \in E^i_0 \cap \mathrm{Int}(E_0)$ then one has :

$$\min\{p_0 + \tilde{H}(s,y,p); (p_0,p) \in \partial_C W(s,y)\} = 0 \tag{46}$$

Remark 7. From the inclusions in (16) and from (40)-(41), (44)-(45) it follows that if Hypothesis 2 holds and $f(.,.,U)$ is a compact-valued and Hausdorff continuous multifunction then the value function is a strictviscosity supersolution ([8],[9],etc.) on the subset $E^i_0 \cap \mathrm{Int}(E_0)$, of the Hamilton-Jacobi equation:

$$-\frac{\partial W}{\partial s}(s,y) - \tilde{H}(s,y,\frac{\partial W}{\partial y}(s,y)) = 0 \tag{47}$$

in the sens that, instead of an inequality, satisfies the equality:

$$p_0 + \tilde{H}(s,y,p) = 0 \ (\forall) \ (p_0,p) \in \underline{\partial}_K W(s,y), (s,y) \in E^i_0 \cap \mathrm{Int}(E_0) \tag{48}$$

and a strictviscositysolutionon $\tilde{E} \cap \mathrm{Int}(E_0)$ in the sens that:

$$p_0 + \tilde{H}(s,y,p) = 0 \ (\forall) \ (p_0,p) \in \partial_K W(s,y) = \bar{\partial}_K W(s,y) \cup \underline{\partial}_K W(s,y) \quad (\forall) \ (s,y) \in \tilde{E} \cap \mathrm{Int}(E_0) \tag{49}$$

We note also, that, in view of the relations in (15)-(16) between the contingent and Clarke's generalized derivatives, the inequalities in (29)-(30) are stronger than the corresponding ones in [6] and [12] and the property in (40) is stronger than the fact that the value function is a viscosity subsolution of the equation (47) on $E^i_0 \cap \mathrm{Int}(E_0)$; moreover, the results in Theorem 5 show that in the case of parametrized differential inclusions of the form (2), the lipschitzianity assumption in [6] may be replaced by mere continuity.

Remark 8. In spite of the abundance of uniqueness results for viscosity solutions, simple examples show that the value function may not be only viscosity solution of the equation (47):

Example 9. If we consider the optimal control problem defined by the following data:

$$g(t_1,(t_1,y_2)) = y_2 \ if \ (t_1,(t_1,y_2)) \in E_1 = \{(t_1,(t_1,y_2)); t_1, y_2 \in \mathbf{R}\}$$

$$\begin{cases} x'_1 = u(t) \in [-2,2], & x_1(s) < s, \quad x_1(t_1) = t_1 > s \\ x'_2(u(t))^2, & x_2(s) = y_2 \in \mathbf{R}, \ x_1(t) < t \ (\forall) \ t \in [s,t_1) \end{cases}$$

then standard computations using, for instance, Cauchy's method of characteristics and a verification theorem, show that the value function is given by: $W(s,y) = y_2 + 4(s-y_1)$, $(s,y) \in E = \{(s,y) \in \mathbf{R} \times \mathbf{R}^2; y_1 \leq s\}$ for any $(s,y) \in E_0 = \{(s,y) \in$

$\mathbf{R}\times\mathbf{R}^2; y_1 < s\}$ the corresponding optimal control is given by : $\tilde{u}_{s,y}(t) = 2$ $(\forall)$ $t \in [s, \tilde{t}_1(s,y)]$ and the hamiltonian in (39) is given by:

$$\tilde{H}(s,y,p) = \begin{cases} -(p_1)^2/4p_2 & if \quad p_2 > 0, p_1 \in [-4p_2, 4p_2] \\ -2p_1 + 4p_2 & if \quad p_2 \le 0, p_1 > 0 \quad or \quad p_1 > 4p_2 > 0 \\ 2p_1 + 4p_2 & if \quad p_2 \le 0, p_1 < 0 \quad or \quad p_1 < -4p_2 < 0 \end{cases}$$

It is easy to see that in this case, the function $W_1(s,y) = y_2$ $(\forall)$ $(s,y) \in E = E_0 \cup E_1$ is also a classical (hence a viscosity) solution of the Hamilton-Jacobi equation in (47).

Remark 10. The same proof as that of (33)-(34) shows that the inequalities (35)-(36) are verified at almost any point of an optimal trajectory; however, simple examples show that if $f(.,.,U)$ is not locally bounded and $a \in [1,\infty)$ then (35)-(36) may not be verified at each point $(s,y) \in E_0^m$ hence the value function may not be a viscosity solution of (47) on this set:

Example 11. If we consider the optimal control problem defined by the following data:

$$T \in \mathbf{R}, x_1^1 > 0, g(T,y) = y_2 \quad if \quad (T,y) \in E_1 = \{(T,(x_1^1, y_2)); y_2 \in \mathbf{R}\}$$

$$\begin{cases} x'_1 = u(t) \in \mathbf{R}, x_1(s) = y_1 \ge 0, x_1(T) = x_1^1, x_1(t) \ge 0 \ (\forall)\ t \in [s,T) \\ x'_2 = x_1(u(t))^2, x_2(s) = y_2 \in \mathbf{R} \end{cases}$$

then standard computations show that the value function is given by:

$$W(s,y) = \begin{cases} y_2, & \text{(s,y) = (T,y)} \in E_1 \\ y_2 + 4\left((y_1)^{3/2} - (x_1^1)^{3/2}\right)^2/(T-s), & \text{(s,y)} \in E_0 = \\ & = (-\infty, T) \times \boldsymbol{R}_+ \times \boldsymbol{R} \end{cases}$$

This function is of class C^1 on E_0 but does not verify the inequality (35) at any point $(s,(0,y_2)) \in E_0$ since $DW(s,(o,y_2))\cdot(1,v) = 4(x_1^1)^3/(T-s)^2 + v_2 > 0$ $(\forall)$ $v = (v_1,v_2) \in \hat{F}(s+,(o,y_2)) = \mathbf{R}\times\mathbf{R}_+$; one may see, however, that (33) is verified since $DW(s,(0,y_2))\cdot(0,v) = v_2$ and $\hat{F}_\infty(s+,(0,y_2)) = \mathbf{R}\times\mathbf{R}_+, \mathbf{R}_+ = [0,\infty)$.

Remark 12. The inequality in (29) may be used for an easy proof of the Pontryagin's Minimum Principle for free right end-point problems (for which $E_1 = \{t_1\}\times\mathbf{R}^n$ for which $f(.,.,.)$ is of class C^1 with respect to the second variable as follows ([13]): let $\tilde{x}(\cdot) \in \mathcal{X}_a(t_0, x_0, E_1)$ be an optimal trajectory, let $\tilde{u}(\cdot)$ be the corresponding optimal control such that $(t,x) \mapsto \hat{f}(t,x) = f(t,x,\tilde{u}(t))$ and $(t,x) \mapsto D_2\hat{f}(t,x)$ are of Caratheodory type on some open subset $\tilde{D} \subset D$ and for any $(s,y) \in \tilde{D}$ let $\hat{x}(\cdot; s,y)$ be the maximal (noncontinuable) solution of the problem: $x' = \hat{f}(t,x), x(s) = y$.

From continuity reasons (e.g. [7]) it follows that there exists $r > 0$ such that $\hat{x}(\cdot; s,y)$ is defined on $[s,t_1]$ for any $s \in [t_0,t_1), y \in B_r(\tilde{x}(s))$ hence from (6) it follows:

$$g(\hat{x}(t_1; s,y)) \ge W(s,y) \quad (\forall) \quad s \in [t_0,t_1), y \in B_r(\tilde{x}(s)) \tag{50}$$
$$g(\hat{x}(t_1; s,\tilde{x}(s)) = W(s,\tilde{x}(s)) \quad (\forall) \quad s \in [t_0,t_1]$$

Further on, from the definition in (10), (13) of the extreme quasitangent directional derivatives and from (50) one obtains:

$$\bar{D}_Q^+ g(\tilde{x}(t_1); D_3\hat{x}(t_1; s,\tilde{x}(s))\cdot(v - r\cdot\hat{f}(s,\tilde{x}(s))) \ge \bar{D}_Q^+ W((s,\tilde{x}(s));(r,v))$$

for any $(r,v) \in \mathbf{R}\times\mathbf{R}^n$ and for almost all $s \in [t_0,t_1]$, hence from (50) and (15) it follows:

$$\bar{D}_Q^+ g(\tilde{x}(t_1); D_3\hat{x}(t_1; s,\tilde{x}(s))\cdot(v - \hat{f}(s,\tilde{x}(s))) \ge 0 \ (\forall)\ v \in \bar{co}\hat{f}(s,\tilde{x}(s),U), a.e.$$

Finally, from (11) and from the differential properties of the flow $\hat{x}(.;.,.)$ it follows that if $q \in \bar{\partial}_Q g(\tilde{x}(t_1))$ then the mapping $p(s) = (D_3\hat{x}(t_1; s,\tilde{x}(s)))^*\cdot q, s \in [t_0,t_1]$, is an absolutely continuous solution of the adjoint equation, $p' = -D_2H(t,\tilde{x}(t),p,\tilde{u}(t))$, and satisfies the minimum condition : $< p(s), f(s,\tilde{x}(s),u) - \hat{f}(s,\tilde{x}(s)) > = < q, D_3\hat{x}(t_1; s,\tilde{x}(s))\cdot(f(s,\tilde{x}(s),u) - \hat{f}(s,\tilde{x}(s)) > \ge 0$ $(\forall)$ $u \in U, a.e.([t_0,t_1])$

Using the last statement in Proposition 1, one may obtain a differential inequality along an optimal trajectory that allow the extension of the proof above, of the Minimum Principle, to some constrained right end-point problems ([15]).

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SOME CONDITIONS ON REAL HYPERSURFACES OF QUATERNIONIC PROJECTIVE SPACE

BY

JUAN DE DIOS PEREZ

1.Introduction. Let M be a connected real hypersurface of the quaternionic projective space $QP^m, m \geq 2$, endowed with the metric g of constant quaternionic sectional curvature 4. Let N be a unit local normal vector field on M and $U_k = -J_k N, k = 1,2,3$, where $\{J_k\}_{k=1,2,3}$ is a local basis of the quaternionic structure of QP^m ,[3]. Let us denote by $D^\perp = \text{Span}\{U_1, U_2, U_3\}$ and by $\mathbf{D}$ its orthogonal complement in TM. $\mathbf{D}$ can be considered as a subbundle of TM , [11].

Recently, [1], J.B e r n d t , generalizing previous results of A. M a r t i n e z and the author, [8], has proved that any real hypersurface of $QP^m, m \geq 2$ such that $g(A\mathbf{D}, \mathbf{D}^\perp) = \{0\}$, where A is the Weingarten endomorphism of M is congruent to an open subset of either a tube of radius r, $0 < r < \pi/2$, over the canonically (totally geodesic) embedded quaternonic projective space $QP^k, k \in \{0, .., m-1\}$, or of a tube of radius $r, 0 < r < \pi/4$, over the canonically (totally geodesic) embedded complex projective space CP^m.

In [7], A.M a r t i n e z introduced ruled real hypersurfaces of QP^m, obtaining several examples, as real hypersurfaces for which the distribution $\mathbf{D}$ is integrable. This is equivalent to the fact that $g(A\mathbf{D}, \mathbf{D}) = \{0\}$, [9].

The purpose of the present paper is to prove the following

Theorem 1. *Let M be a areal hypersurface of $QP^m, m > 3$. Then if $\mathbf{D}$ is a curvature inariant subspace of TM and $g(A\mathbf{D}, \mathbf{D}^\perp) \neq \{0\}$, then M is congruent to an open subset of a ruled real hypersurface of QP^m.*

Theorem 2. *Let M be a hypersurface of $QP^m, m \geq 2$, satisfying $L_{U_i}\Phi_i = 0$, $i = 1,2,3$ where L denotes the Lie derivative and Φ_i is the tangent component of J_i Then M is congruent to an open subset of a tube of radius r , $0 < r < \pi/2$, over the canonically (totally geodesic) embedded quaternionic projective space $QP^k, k \in \{0, ..., m-1\}$.*

It is known, [10, Lemma 4.2] that there do not exist parallel real hypersurfaces in quaternionic projective space. Thus we study weaker conditions in the following Theorems.

Theorem 3. *There do not exist real hypersurfaces M of $QP^m, m \geq 2$, satisfying $\sigma(R(X,Y)AZ) = 0$, for any X, Y, Z tangent to M, where σ denotes the*