Definition. *A class which is not groundless is said to be grounded.*

Lemma 1. *Let K be the class of all grounded classes. Then K is groundless and grounded.*

Proof. Let us assume that K is a groundless class. Then there is an infinite progression of classes $A_1, A_2, \ldots$ such that $\ldots \in A_2 \in A_1 \in K$.

Since $A_1 \in K$, A_1 is a grounded class; since $\ldots \in A_3 \in A_2 \in A_1$ A is also a groundless class. But this is impossible.

Therefore K is a grounded class. Hence $K \in K$, and we have $\ldots \in K \in K \in K$. Therefore K is also a groundedless class.

Lemma 2. *If $f : V \to V$ is an onto map, then we can make two disjoint paradoxical classes using f.*

Proof. First $\{x : x \notin f'x\}$ is one. And define $g : V \to V$ by $g'x = (f'x)^c$ for all x. Then g is onto since for given y, there is x such that $f'x = y^c$ and so $g'x = (f'x)^c = y$. Hence $\{x : x \notin g'x\} = \{x : x \notin (f'x)^c\}$ is another paradoxical class and since $\{x : x \notin g'x\} = \{x : x \notin f'x\}^c$, two classes are disjoint.

Theorem. *There are infinitely many paradoxes classes which are not generated by the uniform method.*

Proof. Since $P(V) \subseteq V$, V is a paradoxical class. Suppose V is generated by a map f. Then by Lemma 2, $V^c = \Phi$ would be a paradoxical class, which is absurd.

Now for any nonparadoxical singleton $\{a\}$, consider a class $V \setminus \{a\}$. Let $g_a : P(V \setminus \{a\}) \to V \setminus \{a\}$ be a map defined by

if $a \notin P(V \setminus \{a\})$, $g_a(x) = x$ for all $x \in P(V \setminus \{a\})$,

if $a \in P(V \setminus \{a\})$, $g_a(x) = \begin{cases} x & x = a \\ V & x \neq a. \end{cases}$

Then clearly g_a is $1-1$, and so $V \setminus \{a\}$ is a paradoxical class.

$V \setminus \{a\}$ is not generated by the uniform method, since its complement $\{a\}$ is not paradoxical.

Remark. It remains still open whether there exists a map α such that $\alpha : V \to V$, $\alpha''V = V$ and $C(\alpha, V) = K = \{x \mid \text{is grounded}\}$ as in Lemma 1.

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Department of Mathematics
Yonsei University, Seoul
KOREA

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THE METHOD OF COMPLEX POTENTIALS IN SOLVING TWO-DIMENSIONAL PROBLEMS FOR MEDIA WITH VOIDS

BY

A. MANOLACHI

0.Introduction. The infinitesimal theory of elastic materials possessing a structure of distributed small voids and some questions concerning the behaviour of these materials have been approached by S.C. C o w i n and J.W. N u n z i a t o in [1]. Starting from this theory, S.C. C o w i n defines in [2] the quasi-static problems of plane strain and plane stress applying the appropriate field equations to the study of the problem of stress concentration around a circular hole in an infinite plate subjected to uni and bi-axial tension.

In this paper we give a general method of solving Cowin's equations for arbitrary domains and boundary data. The method which we propose is that of complex potentials. Accordingly, in the first stage one establishes a representation formulae of the general solution of equilibrium equations in terms of three potentials among two are analytic complex valued and, one - real valued - of Helmholtz's type. Then, the obtained representation is used to reduce the boundary value problems of plane strain and plane stress to some problems of complex function theory.

The content of the paper is the the following: the field equations and some of their consequences are presented in Section 1. After the general solution of equilibrium equations is found - Section 2 - we formulate the fundamental boundary value problem in terms of complex functions in Section 3. In Section 4 we deduce the expression of complex potentials in the case of multiply-connected finite and infinite regions. To illustrate the method we have included solutions of some concrete problems in the final part of Sections 3 and 4.

1.Field equations. Consequences.

1.1. *Field equations.* Let $D \subset R^2$ be the domain occupied by an isotropic homogeneous material and C - the frontiere of D. We assume the curve C be of class C^1. The two-dimensional field equations describing the equilibrium state of an elastic solid with voids structure are
- equilibrium equations

$$(1.1) \qquad \sigma_{\gamma\delta,\delta} + \rho b_\gamma = 0, \quad h_{\delta,\delta} + g + \rho\ell = 0, \quad \gamma, \delta = 1, 2.$$

- kinematical relations

$$(1.2) \qquad 2\varepsilon_{\gamma\delta} = u_{\gamma,\delta} + u_{\delta,\gamma}, \quad u_{\gamma,\delta} = \frac{\partial u_\gamma}{\partial x_\delta}, \ldots$$

- linear constitutive equations in the stationary case ($\dot{\varphi} = 0$)

$$(1.3) \qquad \begin{aligned} &\sigma_{\gamma\delta} = (\lambda\theta + \beta\varphi)\delta_{\gamma\delta} + 2\mu\varepsilon_{\gamma\delta}, \\ &h_\delta = \alpha\varphi_{,\delta}, \quad g = -\xi\varphi - \beta\theta, \quad \theta = \varepsilon_{11} + \varepsilon_{22}. \end{aligned}$$

- boundary conditions at C

$$(1.4) \qquad \sigma_{\gamma\delta} n_\delta = X_\gamma(s); \quad h_\delta n_\delta = H(s), \quad at \quad C_1$$

$$(1.5) \qquad u_\gamma = u^*_\gamma(s); \quad \varphi = \varphi^*(s), \quad at \quad C_2$$

where C_1 and C_2 are portions of curve C such that $C_1 \cup C_2 = C$, $C_1 \cap C_2 = \emptyset$, and $X_\gamma(s)$, $H(s)$, $u^*_\gamma(s)$, $\varphi^*(s)$, $\gamma = 1, 2$, are prescribed functions of the arc s.

In the above relations we have used the following notations: ρ - the density in the reference configuration, $u_\gamma = u_\gamma(x_1, x_2)$ - the displacement vector components, $\varphi = \varphi(x_1, x_2)$ - the function of change in volume-fraction, $\varepsilon_{\gamma\delta}$ - the strain tensor components, $\sigma_{\gamma\delta}$ - the stress tensor components, h_δ - the equilibrated stress vector components, b_γ and ℓ - the external body loads, g - the intrinsic equilibrated body force. The coefficients λ, μ, α, β, ξ are the constants of the material, n_γ are the components of unit outward normala at C and comma denotes the operation of differentiation. When $\sigma_{\gamma\delta}$, h_γ and g given by (1.3) are substituted into (1.1) we obtain Lamé's form of equilibrium equations as follows

$$(1.6) \qquad \begin{aligned} &\mu\Delta u_\gamma + (\lambda + \mu)\theta_{,\gamma} + \beta\varphi_{,\gamma} + \rho b_\gamma = 0, \\ &\alpha\Delta\varphi - \beta\theta - \xi\varphi + \rho\ell = 0. \end{aligned}$$

This system of equations characterizes both the plane strain and plane stress with the mention that in the plain stress problems the constants λ, β and ξ should be substituted by

$$(1.7) \qquad \lambda_1 = \frac{2\lambda\mu}{\lambda + 2\mu}, \quad \beta_1 = \frac{2\beta\mu}{\lambda + 2\mu}, \quad \xi_1 = \xi - \frac{\beta^2}{\lambda + 2\mu}$$

The solution of the system (1.6) must satisfy some boundary conditions among (1.4) and (1.5). With reference to these conditions we introduce the following classification:

- when $C_1 = C$, ($C_2 = \emptyset$), (1.6) and (1.4) define *the stress boundary problem* (S – *problem*),
- when $C_2 = C$, ($C_1 = \emptyset$), we refer to (1.6) and (1.7) as *the displacement boundary problem* (D – *problem*) and,
- when $C_1 \neq \emptyset$, $C_2 =\neq \emptyset$, (1.6) and (1.4), (1.5) define *the mixed boundary value problem* (M – *problem*).

We refer to these three problems as the fundamental boundary value problems of two-dimensional theory of media with voids.

In applications one often meets also the so-called *third problem* in which the solution of the system (1.6) must satisfy some combinations among the conditions (1.4) and (1.5) at C, but we do not consider here this problem.

1.2. *Energy considerations.* The density of internal energy for a medium possessing voids read as follows

$$(1.8) \qquad 2\mathcal{V} = \lambda\varepsilon_{\gamma\gamma}\varepsilon_{\delta\delta} + 2\mu\varepsilon_{\gamma\delta}\varepsilon_{\gamma\delta} + 2\beta\varepsilon_{\gamma\gamma}\varphi + \alpha\varphi_{,\delta}\varphi_{,\delta} + \xi\varphi^2$$

In view of [1], the necessary and sufficient conditions for the quadratic form (1.8) be non-negative are

$$(1.9) \qquad \mu \geq 0, \quad \alpha \geq 0, \quad \xi \geq 0, \quad 3\lambda + 2\mu \geq 0, \quad (3\lambda + 2\mu)\xi \geq 3\beta^2.$$

Based on these conditions and geometrical relations (1.2) we can prove that $\mathcal{V} = 0$, if and only if, u_γ and φ one reduces to a rigid displacement field of the form

$$(1.10) \qquad \tilde{u}_\gamma = a_\gamma + \in_{\delta\gamma} \omega x_\delta, \quad \tilde{\varphi} = 0, \quad \gamma, \delta = 1, 2,$$

where a_γ and ω are arbitrary constants and $\in_{\rho\delta}$ - the alternating tensor.

1.3. *Betti's formulae.* Let $u = (u_1, u_2, \varphi)$ and $v = (v_1, v_2, \psi)$ be two vector functions of class $C^2(D) \cap C^1(D + C)$. Multiplying the first two equations among (1.6) by v_γ, the third by ψ, summing up the results and integrating over D, in view of Green-Gauss theorem and Cauchy's formula (1.4), we obtain

$$\int_D [v_\delta \mathcal{L}_\delta(u) + \psi \mathcal{L}_3(u)]\, da = = \int_D \mathcal{V}(u,v) da - \int_C [v_\delta X_\delta(u) + \psi H(u)]\, ds \tag{1.11}$$

where $\mathcal{V}(u,v)$ is the bilinear form associated to (1.8) and $\mathcal{L}_\delta(u)$, $\mathcal{L}_3(u)$, $X_\delta(u)$, $H(u)$ are given by

$$\begin{aligned} &-\mathcal{L}_\delta(u) = \mu\Delta u_\delta + (\lambda+\mu)\theta_{,\delta} + \beta\varphi_{,\delta},\\ &-\mathcal{L}_3(u) = \alpha\Delta\varphi - \beta\theta - \xi\varphi,\\ &X_\delta(u) = \sigma_{\delta\gamma}(u)n_\gamma, \quad H(u) = h_\delta(u)n_\delta. \end{aligned} \tag{1.12}$$

Setting $v = u$ in the formulae (1.11) and noting that $\mathcal{V}(u,u) = 2\mathcal{V}(u)$, we obtain the following Betti's relation

$$\int_D [u_\delta \mathcal{L}_\delta(u) + \varphi \mathcal{L}_3(u)]\, da = = 2\int_D \mathcal{V}(u) da - \int_C [u_\delta X_\delta(u) + \varphi H(u)]\, ds. \tag{1.13}$$

On the base of this formulae and positiveness property of the internal energy we can prove the uniqueness of the solution of fundamental boundary value problems. This is done by the theorem which follows

Uniqueness Theorem. *Let the conditions (1.9) be satisfied in the plane domain D with the frontiere C. Then D and M-problems have at most one solution; the solution of (S)-problem is unique up to a rigid displacement field of the form (1.10).*

1.4. *Necessary conditions of equilibrium.* It is known from the classical elasticity that the stress boundary value problem admits solutions only if the external loads satisfy certain global conditions of equilibrium. In what follows we establish their counterpart in the theory of media with voids.

In this respect let $u = (u_1, u_2, \varphi)$ be a regular solution of the (S)-problem. According to above results this solution verifies relation (1.13). In view of the uniqueness theorem the most general solution of the (S)-problem is obtained by superposing on u a displacement field of the form (1.10). Let $w = (w_1, w_2, \chi)$ be this solution. We have

$$w_\gamma = u_\gamma + \omega \in_{\delta\gamma} x_\delta + a_\gamma, \quad \chi = \varphi.$$

Observing that

$$\mathcal{V}(w) = \mathcal{V}(u), \quad \mathcal{L}_\gamma(w) = \mathcal{L}_\gamma(u) = \rho b_\gamma, \quad \mathcal{L}_3(w) = \mathcal{L}_3(u) = \rho\ell.$$
$$\sigma_{\gamma\delta}(w) = \sigma_{\gamma\delta}(u); \quad h_\gamma(w) = h_\gamma(u)$$

from the formulae (1.13), we deduce

$$a_\gamma\left(\int_D \rho b_\gamma da + \int_C X_\gamma(u) ds\right) + + \omega\left(\int_D \in_{\delta\gamma} x_\delta \rho b_\gamma da + \int_C \in_{\delta\gamma} x_\delta X_\gamma(u) ds\right) = 0$$

As this relation must be satisfied by any system of arbitrary constants a_γ and ω, it follows

$$\int_D \rho b_\gamma da + \int_C X_\gamma ds = 0 \tag{1.14}$$

$$\int_C \in_{\delta\gamma} x_\delta \rho b_\gamma da + \int_C \in_{\delta\gamma} x_\delta X_\gamma ds = 0 \tag{1.15}$$

These relations express just the necessary conditions of global equilibrium for a medium with voids when the external loads are prescribed over the closed domain D. We note that the extrinsic body force g and equilibrated stress vector H does not appear in these relations.

2. Complex representations. In what follows we establish the representation formulae of the general solution of equilibrium equations and give the complex expressions of the stresses in terms of complex functions.

2.1. *Complex formulation of the field equations.* We introduce in D the complex co-ordinate $(z, \bar z)$, $z = x_1 + ix_2$, $\bar z = x_1 - ix_2$, and define the complex displacement $U(z, \bar z)$, complex stress components $\Lambda(z, \bar z)$, $\Gamma(z, \bar z)$ and complex equilibrated stress $\Psi(z, \bar z)$, as follows

$$\begin{aligned} &U = u_1 + iu_2, \ \bar U = u_1 - iu_2, \ i = \sqrt{-1},\\ &\Lambda = \sigma_{11} + \sigma_{22}, \ \Gamma = \sigma_{22} - \sigma_{11} + 2i\sigma_{12}, \ \Psi = h_1 + ih_2 \end{aligned} \tag{2.1}$$

where the bar over an expression denotes its complex conjugate. Using $(2.1)_2$ the equilibrium equations can be written under the form

$$\begin{aligned} &\frac{\partial\Lambda}{\partial\bar z} - \frac{\partial\bar\Gamma}{\partial z} + \rho L = 0, \ f = b_1 + ib_2\\ &\frac{\partial\Psi}{\partial z} + \frac{\partial\bar\Psi}{\partial\bar z} + g + \rho\ell = 0, \ L(z,\bar z) = f\left(\frac{z+\bar z}{2}, \frac{z-\bar z}{2i}\right). \end{aligned} \tag{2.2}$$

Similarly, using $(2.1)_1$ and (1.3), we obtain the complex form of the stress-strain relations as follows

$$\Lambda = 2(\lambda+\mu)(U_{,z}+\bar{U}_{,\bar z})+2\beta\Phi, \qquad \Gamma = -4\mu\bar{U}_{,z}, \qquad \Psi = 2\alpha\Phi_{,\bar z} \tag{2.3}$$

where $U_{,z}=\partial U/\partial z$, $U_{,\bar z}=\partial U/\partial\bar z$ and $\Phi(z,\bar z)=\varphi\left[(z+\bar z)/2,(z-\bar z)/2i\right]$. Substituting (2.3) into (2.2) and making use of the formulae (1.3) which gives the function g, we obtain

$$\mu\Delta U+2(\lambda+\mu)\Theta_{,\bar z}+2\beta\Phi_{,\bar z}+\rho L=0, \qquad \alpha\Delta\Phi-\beta\Theta-\xi\Phi+\rho\ell=0, \qquad \Theta=U_{,z}+\bar U_{,\bar z}, \quad \Delta U=4U_{,z\bar z}. \tag{2.4}$$

Bearing in mind that along the curve C one has $dz=ie^{i\eta}ds$, $d\bar z=-ie^{-i\eta}ds$, where η is the angle of the unit outward normala with $0x_1$ and combining the relations (1.8) we deduce the complex form of the boundary conditions as follows

$$U=U^*(s), \quad \Phi=\Phi^*, \quad at\ C_1 \tag{2.5}$$

$$\Lambda dz+\bar\Gamma d\bar z=2iX(s)ds, \quad \bar\Psi dz-\Psi d\bar z=2iH(s), \quad at\ C_2 \tag{2.6}$$

where

$$U^*(s)=u_1^*(s)+iu_2^*(s), \quad X(s)=X_1(s)+iX_2(s).$$

We write further the well-known relations between the elastic elements in a system of polar and cartesian co-ordinate, namely

$$\Lambda'\equiv\sigma_r+\sigma_\theta=\Lambda, \quad \Gamma'\equiv\sigma_\theta-\sigma_r+2i\tau_{r\theta}=\Gamma e^{2i\theta}, \qquad \Psi'\equiv h_r+ih_\theta=\Psi e^{-i\theta}, \quad U'=u_r+iu_\theta=Ue^{-i\theta} \tag{2.7}$$

where θ is the angle of r-axis with $0x_1$ measured in the anticlock-wise sense.

2.2. *The complex potentials.* Let V be the body force potential. We have $L=-2V_{,\bar z}$. Using the complex form of the two-dimensional Laplace's operator we can rewrite equation $(2.4)_1$ under the form

$$\left[2\mu U_{,z}+(\lambda+\mu)\Theta+\beta\Phi\right]_{,\bar z}=\rho V_{,\bar z}.$$

Integrating this equation with respect to $\bar z$ along an arbitrary path inclosed in D we obtain

$$2\mu U_{,z}+(\lambda+\mu)\Theta+\beta\Phi=\rho V+(1+k)\Omega'(z), \tag{2.8}$$

where $\Omega(z)$ is an arbitrary complex function analytic of variable z in D and $\Omega'=d\Omega/dz$. The constant K has the values

$k=\frac{\lambda+3\mu}{\lambda+\mu}=3-4\sigma$ in the case of plane strain and,

$k=\frac{\lambda_1+3\mu}{\lambda+\mu}=\frac{3-\sigma}{1+\sigma}$ in the case of plane stress.

According to the plane stress hypotheses, the constant λ_1 expresses in terms of Young modulus and Poisson ration, as follows

$$\lambda_1=\frac{2\lambda\mu}{\lambda+2\mu}=\frac{E\sigma}{(1+\sigma)(1-\sigma)}$$

Adding up (2.8) and its conjugate we get the intermediate relation

$$(\lambda+\mu)\Theta+\beta\Phi=\frac{1}{2}(1+k)\left[\Omega'(z)+\bar\Omega'(\bar z)\right]+\rho V \tag{2.9}$$

When the function Θ is eliminated among $(2.4)_2$ and (2.9) we deduce that the function change-in volume-fraction must verify the equation

$$\Delta\Phi-\ell_0^{-2}\Phi=\frac{\beta(1+k)}{2\alpha(\lambda+2\mu)}\left[\Omega'(z)+\bar\Omega'(\bar z)\right]+\Theta_1(z,\bar z) \tag{2.10}$$

where

$$\ell_0^{-2}=\frac{1}{\alpha}\left(\xi-\frac{\beta^2}{\lambda+2\mu}\right), \qquad \Theta_1=-\frac{\rho}{\alpha}\ell(z,\bar z)+\frac{\rho\beta}{\alpha(\lambda+2\mu)}V. \tag{2.11}$$

We note that the constant ℓ_0^2 having the dimension of length square is always positive ([1]).

The general solution of the equation (2.10) is of the form

$$\Phi=\Phi_0-k_0\left[\Omega'(z)+\bar\Omega'(\bar z)\right]+\Phi_1 \tag{2.12}$$

in which $\Phi_0(z,\bar z)$ is the general solution of the equation

$$\Delta\Phi_0-\ell_0^{-2}\Phi_0=0 \tag{2.13}$$

the function $\Phi_1(z,\bar z)$ is given by, ([6]),

$$\Phi_1(z,\bar z)=\int_D K(z,\bar z,\xi,\eta)\Theta_1(\xi,\eta)d\xi d\eta$$

and the constant k_0 has the expression

$$k_0=m\ell_0^2,\quad m=\frac{\beta}{\alpha(\lambda+\mu)} \tag{2.14}$$

Substituting the function Θ given by (2.9) into (2.8) and taking into account (2.12), after some algebraic operations, we obtain

$$2\mu U_{,z}=k_1\Omega'(z)-\eta\bar\Omega'(\bar z)-\frac{1}{2}\beta_1\Phi_0+\frac{\mu}{\lambda+2\mu}(\rho V-\beta\Phi_1) \tag{2.15}$$

where

$$k_1=\frac{k-\beta^2/\xi(\lambda+\mu)}{1-\beta^2/\xi(\lambda+2\mu)},\qquad \eta=\frac{1-\beta^2/\xi(\lambda+\mu)}{1-\beta^2/\xi(\lambda+2\mu)} \tag{2.16}$$

Integrating (2.15) with respect to z by keeping $\bar z$ constant and using the relation $\Phi_0=4\ell_0^2\Phi_{0,z\bar z}$ (see (2.13)), we arrive to the final form of the complex displacement function U namely

$$2\mu U=k_1\Omega(z)-\eta\left[\bar\Omega'(\bar z)+\bar\omega(\bar z)\right]-2\beta_1\ell_0^2\Phi_{0,\bar z}+G(z,\bar z) \tag{2.17}$$

where $\eta\omega(z)$ is a new arbitrary complex valued function analytic of z and ([4]),

$$G(z,\bar z)=-\frac{\mu}{\pi(\lambda+2\mu)}\int_D(\rho V-\beta\Phi_1)\frac{d\xi d\eta}{\bar\varsigma-\bar z};\quad \varsigma=\xi+i\eta.$$

The formulae (2.12) and (2.17) give the complex representation of the general solution of equilibrium equations in terms of two arbitrary complex valued potentials $\Omega(z)$, $\omega(z)$ and one - $\Phi_0(z,\bar z)$ - which is the real solution of the equation (2.13). Knowning these potentials, the stress system and the intrinsic equilibrated body force g are given by (2.3) and $(1.3)_3$.

We note that in the absence of voids there are $K_1=K$, $\eta=1$ and the formulae (2.17) are reduced to the well-known representation of the complex displacement met in the classical two-dimensional elasticity.

2.3. *Complex stresses.* Taking into account the stress-strain relations (2.3), where the expressions for U and Φ are substituted, we obtain

$$\begin{aligned}\Lambda&=2\eta\left[\Omega'(z)+\bar\Omega'(\bar z)\right]+4\beta_1\ell_0^2\Phi_{0,z\bar z}+\Lambda^*,\\ \Gamma&=2\eta\left[\bar z\Omega''(z)+\omega'(z)\right]+4\beta_1\ell_0^2\Phi_{0,zz}+\Gamma^*,\\ \Psi&=2\alpha\left[-k_0\bar\Omega''(\bar z)+\Phi_{0,\bar z}\right]+\Psi^*,\quad \Omega''=\frac{d^2\Omega}{dz^2}.\end{aligned} \tag{2.18}$$

where $\Lambda^*,\Gamma^*,\Psi^*$ are given by

$$\begin{aligned}\Lambda^*&=\frac{\lambda+\mu}{\eta}(G_{,z}+\bar G_{,\bar z})+2\beta\Phi_1,\\ \Gamma^*&=-2\bar G_{,z},\quad \Psi^*=2\alpha\Phi_{1,\bar z}\end{aligned} \tag{2.19}$$

2.4. *Stress resultant and stress moment.* Let $(\sigma_1+\sigma_2)ds$ $(\sigma_\gamma=\sigma_{\gamma\delta}n_\delta)$ be the stress vector exerted on an elementary arc ds of some curve AB drawn in the domain D. In accordance with (2.18), we have

$$\begin{aligned}2i(\sigma_1+i\sigma_2)ds&=\Lambda dz+\bar\Gamma d\bar z=\{2\eta\left[\Omega'(z)+\bar\Omega'(\bar z)\right]+\\ &+4\beta_1\ell_0^2\Phi_{0,z\bar z}+\Lambda^*dz\}+\{2\eta\left[z\bar\Omega''(z)+\bar\omega'(\bar z)\right]+\\ &+4\beta_1\ell_0^2\Phi_{0,\bar z\bar z}+\Gamma^*\}d\bar z=\\ &=2d\{\eta\left[\Omega(z)+z\bar\Omega'(\bar z)+\bar\omega(\bar z)\right]+\\ &+2\beta_1\ell_0^2\Phi_{0,\bar z}\}+\Lambda^*dz+\bar\Gamma^*d\bar z.\end{aligned} \tag{2.20}$$

where $d\{\ldots\}$ denotes the differential of the quantity inside the brackets. Performing the integration with respect to s in the above formulae, we find

$$\begin{aligned}&\{\eta[\Omega(z)+z\bar\Omega'(\bar z)+\bar\omega(\bar z)]+2\beta_1\ell_0^2\Phi_{0,\bar z}\}_A^B+\\ &+\frac{1}{2}\int_{AB}(\Lambda^*dz+\bar\Gamma^*d\bar z)=i\int_{AB}(\sigma_1+i\sigma_2)ds+c\end{aligned}$$

where $\{f\}_A^B=f(B)-f(A)$ and c is an arbitrary complex constant.

Denoting by Hds the equilibrated stress vector which acts on an elementary arc ds of curve AB, in view of (2.18), we deduce

$$\begin{aligned}2iHds&=\bar\Psi dz-\Psi d\bar z=2i\,Im(\bar\Psi dz)=\\ &=4i\alpha\,Im\left\{\left[-k_0\bar\Omega''(\bar z)+\Phi_{0,\bar z}+\Phi_{1,\bar z}\right]\frac{dz}{ds}\right\}ds\end{aligned}$$

where $Re\{\ldots\}$ and $Im\{\ldots\}$ denote respectively the real and imaginary parts of the expression inside the brackets. Consequently, along the arc AB, it holds

$$2\alpha Im\left\{\left[-k_0\bar\Omega''(\bar z)+\Phi_{0,\bar z}+\Phi_{1,\bar z}\right]\frac{dz}{ds}\right\}_A^B=\{H(s)\}_A^B \tag{2.21}$$

If AB is a closed contour C, from (2.20) and (2.21) we deduce the expressions of the stress resultant and equilibrated stress resultant acting on this contour, namely

$$
\begin{aligned}
&\{\eta\left[\Omega(z)+z\bar{\Omega}'(\bar{z})+\bar{\omega}(\bar{z})\right]+2\beta_1\ell_0^2\Phi_{0,\bar{z}}\}_C+\\
&+\frac{1}{2}\int_C(\Lambda^* dz+\bar{\Gamma}^* d\bar{z})=i\int_C(\sigma_1+i\sigma_2)ds
\end{aligned}
\tag{2.22}
$$

$$
2\alpha Im\left\{\left[-k_0\bar{\Omega}''(\bar{z})+\Phi_{0,\bar{z}}+\Phi_{1,\bar{z}}\right]\frac{dz}{ds}\right\}_C=\{H(s)\}_C.
\tag{2.23}
$$

When the interior of curve C is a simply-connected domain and no concentrated loads act inside of C, then $\Omega(z)$, $\omega(z)$ and Φ_0 are uniform functions and, from (2.22) and (2.23), we deduce

$$
\frac{1}{2}\int_C(\Lambda^* dz+\bar{\Gamma}^* d\bar{z})=i\int_C(\sigma_1+i\sigma_2)ds
\tag{2.24}
$$

$$
\{H(s)\}_C=0.
\tag{2.25}
$$

Further, by using the complex Stoke's formulae, ([3]),

$$
\int_C[f(z,\bar{z})dz+g(z,\bar{z})d\bar{z}]=2i\int_{D+}(f_{,\bar{z}}-g_{,z})da
$$

where D^+ is the interior domain of curve C, the left hand side of the relation (2.24) read as follows

$$
\frac{1}{2}\int_C\Lambda^* dz+\bar{\Gamma}^* d\bar{z}=i\int_{D+}(\Lambda^*_{,\bar{z}}-\bar{\Gamma}^*_{,z})da.
$$

In view of equilibrium equations (2.2) and (2.24) the integral appearing in the right hand side of this relation can be transformed to obtain

$$
\int_D\rho L da+\int_C(\sigma_1+i\sigma_2)ds=0.
$$

This formulae express the fact that the stress resultant along the contour C equilibrates the resultant of body forces acting inside of C. Otherwise it represent the complex form of the global equilibrium condition (2.18).

Another cosequence of uniformity properties of potentials Ω,ω and Φ_0 expressed by the relation (2.25) is that requiring $H(s)$ be an uniform function along the contour C.

The moment about the origin of the forces acting along AB is given by

$$
\begin{aligned}
M_{AB}&=\int_{AB}(x_1\sigma_2-x_2\sigma_1)ds=Re\left\{\int_{AB}iz(\sigma_1-i\sigma_2)ds\right\}-\\
&-\frac{1}{2}Re\left\{\int_{AB}z(\Lambda d\bar{z}+\Gamma dz)\right\}=\\
&=Re\left\{\eta\left[\omega(z)-z\omega'(z)-z\bar{z}\bar{\Omega}'(\bar{z})\right]+2\beta_1\ell_0^2\Phi_{0,z}\right\}_A^B-\\
&-2\beta_1\ell_0^2\,Re\left\{\int_{AB}\Phi_{0,z}dz\right\}-\frac{1}{2}Re\left\{\int_{AB}z(\Lambda^* d\bar{z}+\Gamma^* dz)\right\}
\end{aligned}
\tag{2.27}
$$

In this formulae the purely imaginary terms have been dropped since they do not affect M_{AB}. When AB is a closed contour C and the functions Ω,ω and Φ_0 vary uniformly inside of C, the formulae (2.27) gives the resultant moment of stresses along C, as follows

$$
\begin{aligned}
\{M\}_C&=\int_C iz(\sigma_1-i\sigma_2)ds=-2\beta_1\ell_0^2\,Re\left\{\int_C\Phi_{0,z}dz\right\}-\\
&-\frac{1}{2}Re\left\{\int_C z(\Lambda^* d\bar{z}+\Gamma^* dz)\right\}.
\end{aligned}
\tag{2.28}
$$

Using again the Stoke's formulae and equilibrium equations (2.2) we can write

$$
\int_C\Phi_{0,z}dz=2i\int_{D+}\Phi_{0,z\bar{z}}da
\tag{2.29}
$$

$$
\int_C z(\Lambda^* d\bar{z}+\Gamma^* dz)=-2\int_{D+}iz\rho\bar{L}da
\tag{2.30}
$$

According to (2.13), relation (2.29) can be transformed as follows

$$
2\beta_1\ell_0^2\int_C\Phi_{0,z}dz=i\beta_1\int_{D+}\Phi_0 da
$$

Thus the first term appearing in the right hand side of (2.28) vanishes. Finally, when (2.30) is substituted in the second term of (2.28) and (2.27) is taken into account, we deduce

$$
Re\left\{\int_C iz(\sigma_1-i\sigma_2)ds+\int_D iz\rho\bar{L}da\right\}=0
\tag{2.31}
$$

It can be see that this formula represents the complex form of the global equilibrium condition (2.19).

3. Complex formulation of the boundary value problems.

3.1. *Boundary conditions.* The representation formulae of the displacement and stress fields established in the previous Section enable us to formulate the fundamental boundary value problems of two-dimensional elasticity for media with voids in terms of complex function theory, namely

To find analytic potentials $\Omega(z)$, $\omega(z)$ of complex argument z and a solution Φ_0 of the equation (2.13) regular in D which satisfy at C the following boundary conditions:

(i) *The stress boundary value problem (S-problem)*

$$\begin{aligned}&\eta\left[\Omega(t)+\bar{\Omega}'(\bar{t})+\bar{\omega}(\bar{t})\right]+2\beta_1\ell_0^2\Phi_{0,\bar{t}}+\\ &\frac{1}{2}\int_0^s\left(\Lambda^*\frac{dt}{ds}+\bar{\Gamma}\frac{d\bar{t}}{ds}\right)ds=f_1(t)+if_2(t)+constant,\\ &2\alpha\, Im\left\{\left[-k_0\bar{\Omega}''(\bar{z})+\Phi_{0,\bar{t}}+\Phi_{1,\bar{t}}\right]\frac{dt}{ds}\right\}=H(s),\end{aligned}\tag{3.1}$$

where f_1 and f_2 are known functions of t at the contour C and are given by

$$f_1(t)+if_2(t)=i\int_0^s(X_1+iX_2)ds,\quad t\in C$$

(ii) *The displacement boundary value problem (D-problem)*

$$\begin{aligned}&K_1\Omega(t)-\eta\left[t\bar{\Omega}'(\bar{t})+\bar{\omega}(\bar{t})\right]-2\beta_1\ell_0^2\Phi_{0,\bar{t}}+G(t,\bar{t})=2\mu U^*(t,\bar{t}),\\ &\Phi_0(t,\bar{t})-k_0\left[\Omega'(t)+\bar{\Omega}'(\bar{t})\right]+\Phi_1(t,\bar{t})=\Phi^*(t,\bar{t}),U^*=u_1^*+iu_2^*.\end{aligned}\tag{3.2}$$

where u_1^* and u_2^* and Φ^* are given functions of the points of C.

(iii) *The mixed boundary value problem (M-problem), when conditions of the form* (3.1) *are prescribed at C_1 and conditions of the form* (3.2) *are prescribed at C_2, where $C_1\cup C_2=C$ and $C_1\cap C_2=\emptyset$.*

Note. It was shown above that the necessary conditions for equilibrium of a medium within any curve drawn in the domain occupied by the material are given by (2.26) and (2.31). Making use of the notation appearing in (3.1), we can rewrite these conditions under the form

$$\begin{aligned}&\frac{1}{2}\int_C\left(\Lambda^*\frac{dt}{ds}+\bar{\Gamma}^*\frac{d\bar{t}}{ds}\right)ds=\left[f_1(t)+if_2(t)\right]_C,\\ &\frac{1}{2}Re\left\{\int_C it\left(\bar{\Lambda}\frac{d\bar{t}}{ds}+\Gamma\frac{dt}{ds}\right)\right\}ds=Re\left\{\int_C f dt\right\}.\end{aligned}\tag{3.3}$$

Particularly, in the absence of body loads these formulae reduces to

$$\{f_1+if_2\}_C=0,\qquad Re\left\{\int_C f dt\right\}=0.\tag{3.4}$$

3.2. *Arbitrariness degree of potentials.* From the expressions (2.18) it results that the stress components and equilibrated stress are not affected, if one replaces

$$\Omega(z)\ by\ \Omega_1(z)=\Omega(z)+i\ Az+B$$
$$\omega(z)\ by\ \omega_1(z)=\omega(z)+d$$
$$\Phi_0(z,\bar{z})\ by\ \Phi_0(z,\bar{z})$$

where A $(=\bar{A})$, B and d are arbitrary complex constants.

Substituting Ω_1,ω_1 and Φ_0 into the expressions of U and Φ and denoting the result by $\tilde{U}_1$ and $\tilde{\Phi}_1$, we have

$$\tilde{U}_1=U+i(k_1+\eta)Az+k_1d-\eta\bar{B},\quad \tilde{\Phi}_1=\Phi\tag{3.5}$$

The displacement U will be not affected, unless

$$i(k_1+\eta)Az+k_1d-\eta\bar{B}=0$$

which is satisfied only if

$$A=0\qquad and\qquad k_1d-\eta\bar{B}=0.$$

The constants appearing in the linear part of (3.5) can be arbitrarily fixed. For instance, if the origin of co-ordinate lies inside of C we can choose them as follows:

- for a given stress state, by conditions

$$\Omega(0)=0\quad Im[\Omega'(0)]=0,\quad \omega(0)=0\tag{3.6}$$

- for a given displacement field, by conditions

$$\Omega(0)=0\quad or\quad \omega(0)=0.\tag{3.7}$$

The constant from the right hand side of the boundary condition (3.1) can be taken equal to zero whenever D is simply-connected domain.

3.3. *Application. Solid disk in rotation.* Consider a solid circular disk of radius R rotating about its axis with constant angular velocity ω. Using the notation of Section 2, we have

$$
(3.8)\quad
\begin{aligned}
&\rho V=-\frac{\rho\omega^2}{2}z\bar{z},\quad \Phi_1=M(4\ell_0^2+z\bar{z}),\quad M=\frac{\beta\ell_0^2\rho\omega^2}{2\alpha(\lambda+2\mu)},\\
&G=-\frac{\mu}{4(\lambda+2\mu)}\rho\omega^2z^2\bar{z}-\frac{\beta_1M}{2}\left(4\ell_0^2+\frac{z\bar{z}}{2}\right)z,\\
&\Lambda^*=-\frac{\lambda+\mu}{\lambda+2\mu}\rho\omega^2z\bar{z}+\beta_1M\left(4\ell_0^2+z\bar{z}\right),\\
&\Gamma^*=\frac{\mu}{2(\lambda+2\mu)}\rho\omega^2\bar{z}^2+\beta_1M\frac{\bar{z}^2}{2},\quad \Psi^*=2\alpha Mz.
\end{aligned}
$$

With these data, at the circumference $|z|=R$, we find

$$
(3.8)'\quad
\begin{aligned}
&\frac{1}{2}\int_0^s\Lambda^*dz+\bar{\Gamma}^*d\bar{z}=-\frac{2\lambda+3\mu}{4(\lambda+2\mu)}\rho\omega^2R^2t+\beta_1M\left(4\ell_0^2+\frac{R^2}{2}\right)t,\\
&t=Re^{i\theta}.
\end{aligned}
$$

Substituting λ by $\lambda_1=E\sigma/(1-\sigma^2)$ according to the hypothesis of plane stress and replacing (3.8)′ into the boundary conditions (3.1), we obtain

$$
(3.9)\quad
\begin{aligned}
&\eta\left[\Omega(t)+t\bar{\Omega}'(\bar{t})+\bar{\omega}(\bar{t})\right]+2\beta_1\ell_0^2\Phi_{0,\bar{t}}=\\
&\qquad=\frac{1}{8}(3+\sigma)\rho\omega^2R^2t-\frac{1}{2}\beta_1M\left(4\ell_0^2+\frac{R^2}{2}\right)t\\
&Im\left\{\left[-k_0\bar{\Omega}''(\bar{t})+\Phi_{0,t}\right]\frac{dt}{ds}\right\}=-M\ Im\left(t\frac{dt}{ds}\right).
\end{aligned}
$$

In (3.9) we have considered $f_1=f_2=H=0$ since the cylindrical surface of the disk is free of external loads and constant $=0$ because D is simply-connected.

The solution of the boundary value problem (3.9) regular in the origin is sought under the form

$$
(3.10)\qquad \Omega(z)=Pz,\quad \omega(z)=0,\quad \Phi_0=QI_0\left(\frac{r}{\ell_0}\right),
$$

where P and Q are real unknown constants and $I_n(\cdot)$ is the modified Bessel's function of first kind and order n. Inserting (3.10) into (3.9) and effecting some calculations, we get

$$
P=\frac{1}{16\eta}(3+\sigma)\rho\omega^2R^2-\frac{1}{8\eta}\beta_1MR^2;\quad Q=-\frac{2MR\ell_0}{I_1(\frac{R}{\ell_0})}.
$$

Comparing the value of the constant A in the present theory with that found in the classical elasticity we remark the second term which contains the characteristic constants ℓ_0^2 and β_1. The stress components can now be determined from (2.18) and (3.8) as follows

$$
(3.11)\quad
\begin{aligned}
&\Lambda=4\eta P+\beta_1QI_0\left(\frac{r}{\ell_0}\right)+\Lambda^*\\
&\Gamma=\beta_1Q\frac{\bar{z}^2}{r^2}I_2\left(\frac{r}{\ell_0}\right)+\Gamma^*\\
&\Psi=\frac{\alpha}{\ell_0}Q\frac{z}{r}I_1\left(\frac{r}{\ell_0}\right)+\Psi^*
\end{aligned}
$$

By using the formulae (2.7) and (3.11) we find the stresses in polar coordinate under the form

(3.12)

$$
\begin{aligned}
&\sigma_r=\frac{1}{8}[(3+\nu)\rho\omega^2-2\beta_1M](R^2-r^2)+2\beta_1M\ell_0^2\left(1-\frac{R}{r}\frac{I_1\left(\frac{r}{\ell_0}\right)}{I_1\left(\frac{R}{\ell_0}\right)}\right)\\
&\sigma_\theta=\frac{1}{8}[(3+\nu)\rho\omega^2-2\beta_1M]R^2-\frac{1}{8}[(1+3\nu)\rho\omega^2-2\beta_1M]r^2+\\
&\quad+\frac{1}{2}M\beta_1r^2-2M\beta_1\ell_0R\frac{I_1\left(\frac{r}{\ell_0}\right)}{I_1\left(\frac{R}{\ell_0}\right)}+2M\beta_1\ell_0^2\left(1+\frac{R}{r}\frac{I_1\left(\frac{r}{\ell_0}\right)}{I_1\left(\frac{R}{\ell_0}\right)}\right)\\
&h_r=2\alpha Mr\left(1-\frac{R}{r}\frac{I_1\left(\frac{r}{\ell_0}\right)}{I_1\left(\frac{R}{\ell_0}\right)}\right),\qquad h_\theta=0.
\end{aligned}
$$

In particular, the radial and tangential stresses in the center and on the periphery of the disk are given by the formula

$$\sigma_r(0) = \sigma_\theta(0) = \frac{1}{8}\left[(3+\nu)\rho\omega^2 - 2\beta_1 M\right] R^2 + $$
$$+ 2\beta_1 M \ell_0^2 \left(1 - \frac{R}{2\ell_0 I_1\left(\frac{R}{\ell_0}\right)}\right)$$
$$\sigma_r(R) = 0, \quad \sigma_\theta(R) = \frac{1}{4}\left[(1+\nu)\rho\omega^2 + 2\beta_1 M\right] R^2 + $$
$$+ 4M\beta_1\ell_0^2 + \frac{1}{2}M\beta_1 R^2 - 2M\beta_1\ell_0 R \frac{I_0\left(\frac{R}{\ell_0}\right)}{I_1\left(\frac{R}{\ell_0}\right)}$$
$$h_r(0) = h_r(R) = 0, \qquad h_\theta = 0.$$

The radial displacement and change in volume-fraction, at $r = R$, are

$$2\mu u_r(R) = (k_1 - \eta)PR - \beta_1\ell_0 Q I_1\left(\frac{R}{\ell_0}\right), \quad u_\theta(R) = 0$$
$$\varphi(R) = -k_0 P + Q I_0\left(\frac{R}{\ell_0}\right) + M(4\ell_0^2 + R^2)$$

We obtain also from (1.3) and (2.12),(2.17) the expressions of the intrinsic body force g.

4. Potentials in multiply-connected domains. When the domain D bounded by the contour C is simply-connected and no concentrated loads act inside of C or at C, the potentials Ω, ω and Φ_0 are uniform functions inside of C and the stress and displacement fields are single-valued in D.

Consider now the case of multiply-connected domains bounded by $n+1$ simple closed arcs of class C^1 : $C_1, C_2, \ldots, C_n, C_{n+1}$, where C_{n+1} surrounds all the others.

Since D may be finite or no, we will consider separately these cases.

4.1. *The multiply-connected domain D is finite.* Assuming the body loads be single valued in D, the stress components and equilibrated stress vector must satisfy the following uniformity conditions

$$[\Lambda]_{C_k} = 0, \quad [\Gamma]_{C_k} = 0, \quad [\Psi]_{C_k} = 0, \quad k = \overline{1,n} \tag{4.1}$$

where $[\cdot]_{C_k} = 0$ denotes the increasing along the contour C_k of the function inside the brackets.

Conditions (4.1) are satisfied when Φ_0 is an uniform function and $\Omega(z)$, $\omega(z)$ have the form ([4])

$$\begin{aligned}\Omega(z) &= \Omega_0(z) + z\sum_{k=1}^{n} A_k \ln(z - z_k) + \sum_{k=1}^{n}\gamma_k \ln(z - z_k)\\ \omega(z) &= \omega_0(z) + \sum_{k=1}^{n}\gamma_k' \ln(z - z_k), \quad A = \bar{A}\end{aligned} \tag{4.2}$$

where z_k are arbitrary points inside of C_k, $\Omega_0(z)$ and $\omega_0(z)$ are two arbitrary functions holomorphic in D and A_k, γ_k, γ_k', arbitrary constants.

The uniformity conditions for the displacement and change in volume-fraction imply

$$[U]_{C_k} = 0, \quad [\Phi]_{C_k} = 0. \tag{4.3}$$

Substituting (4.2) into (2.17) and (2.12) and taken up the increasing of U and Φ along the contour C_k in the anticlockwise sense, we get

$$\begin{aligned}2\mu[U]_{C_k} &= 2\pi i\left[(k_1+\eta)A_k z + k_1\gamma_k + \eta\bar{\gamma}_k'\right]\\ [\Phi]_{C_k} &= 0.\end{aligned} \tag{4.4}$$

The condition (4.3) will be satisfied if and only if the right hand side of (4.4) vanishes, i.e.

$$A_k = 0, \quad k_1\gamma_k + \eta\bar{\gamma}_k' = 0, \quad k = \overline{1,n}. \tag{4.5}$$

Setting $A_k = 0$ in (4.2) and taking into account the expression of the stress resultant along the contour C_k,

$$\sigma_{1k} + i\sigma_{2k} = -\frac{i}{2}\int_{C_k}\left(\Lambda\frac{dz}{ds} + \bar{\Gamma}\frac{d\bar{z}}{ds}\right) ds,$$

we get

$$\sigma_{1k} + i\sigma_{2k} = -2\pi\eta(\gamma_k - \bar{\gamma}_k') \tag{4.6}$$

When $(4.5)_2$ and (4.6) are solved with regard to γ_k and γ_k', one obtains

$$\gamma_k = -\frac{\sigma_{1k} + i\sigma_{2k}}{2\pi(k_1+\eta)}, \qquad \gamma_k' = -\frac{k_1(\sigma_{1k} - i\sigma_{2k})}{2\pi\eta(k_1+\eta)}.$$

Hence the functions $\Omega(z)$ and $\omega(z)$ are of the form

$$(4.7)\qquad \begin{aligned} \Omega(z) &= -\frac{1}{2\pi\eta(k_1+\eta)}\sum_{k=1}^{n}(\sigma_{1k}+i\sigma_{2k})\ln(z-z_k)+\Omega_0(z),\\ \omega(z) &= \frac{k_1}{2\pi\eta(k_1+\eta)}\sum_{k=1}^{n}(\sigma_{1k}-i\sigma_{2k})\ln(z-z_k)+\omega_0(z), \end{aligned}$$

where the functions $\Omega_0(z)$ and $\omega_0(z)$ are holomorphic inside of D.

As in the classical elasticity many-valued displacements can be interpreted as a dislocation in the multiply-connected domain D. We confine our attention to a problem of this type in the following application.

4.2. ***Volterra's dislocations in a circular annulus.*** Consider the equilibrium problem of a circular annulus $R_1 \le |z| \le R_2$, $(R_1 < R_2)$ in the absence of body loads and stresses at the boundary.

We assume that across a cut connecting the inner and outer circumferences of radii R_1, R_2 the complex displacement has a jump of the form

$$(4.8)\qquad U_+ - U_- = i\varepsilon z + m + in$$

where ε, m, n are given real constants while the volume fraction suffers no any jump, i.e.

$$(4.9)\qquad \Phi_+ - \Phi_- = 0.$$

We add to (4.8) and (4.9) the boundary conditions at the circumferences $|z| = R_\delta$, $\delta = 1,2$,

$$(4.10)\qquad X_1 + iX_2 = 0,\quad h_1n_1 + h_2n_2 = 0,\quad at\ \ |z| = R_\delta.$$

According to (4.2) the functions $\Omega(z)$ and $\omega(z)$ are choosen under the form

$$(4.11)\qquad \begin{aligned} \Omega(z) &= Az\ln z + \gamma\ln z + \Omega_0(z),\\ \Omega(z) &= \gamma'\ln z + \Omega_0(z),\quad A = \bar{A}. \end{aligned}$$

Since the function Ω_0 and ω_0 are holomorphic inside of the domain $R_1 \le |z| \le R_2$, then

$$(4.12)\qquad \Omega_0(z) = \sum_{-\infty}^{+\infty} A_k^* z^k,\quad \omega_0(z) = \sum_{-\infty}^{+\infty} B_k^* z^k.$$

In view of (4.4) and (4.11), condition (4.8) yields

$$(4.13)\qquad \frac{\pi}{\mu}\left[(k_1+\eta)Az + k_1\gamma + \eta\bar{\gamma}'\right] = i\varepsilon z + m + in.$$

Bearing in mind (4.10) and (4.11), in the absence of body loads ($\Lambda^* = 0$, $\Gamma^* = 0$ $\Psi^* = 0$), the boundary condition (3.1) expressed in terms of unknown functions Ω_0 and ω_0 and $\Phi_0(z)$, read as follows

$$(4.14)\qquad \begin{aligned} &\eta\left[\Omega_0(t) + t\bar{\Omega}_0'(\bar{z}) + \bar{\omega}_0(\bar{t})\right] + 2\beta_1\ell_0^2\Phi_{0,\bar{t}} = F(t) + h(t)\\ &Im\left\{\left[-k_0\bar{\Omega}''(\bar{z}) + \Phi_{0,t}\right]\frac{dt}{ds}\right\} = G(t), \end{aligned}$$

where

$$F(t) = -\eta\left[At(2\ln|t|+1) + \gamma\ln t + \bar{\gamma}'\ln\bar{t} + \bar{\gamma}\frac{t^2}{|t|^2}\right]$$

$$G(t) = k_0\ Im\left\{\left(\frac{A}{t} - \frac{\gamma}{t^2}\right)\frac{dt}{ds}\right\}$$

$$h(t) = \begin{cases} 0\ at\ |z| = R_1,\\ c\ at\ |z| = R_2, \end{cases}.$$

the constant c being unknown.

By substituting the function $F(t)$ into (3.4) we find

$$(4.15)\qquad 2\pi i(\gamma - \bar{\gamma}') = 0,\ Re\left(\int_C \bar{F}\,dt\right) = 0,\ C = \{|z| = R_1 \cup |z| = R_2\}$$

From (4.13) and (4.15), we obtain

$$A = \frac{\varepsilon\mu}{\pi(k_1+\eta)};\qquad \gamma = \bar{\gamma}' = \frac{(n-im)\mu}{\pi(k_1+\eta)}.$$

According to (4.12) we choose the functions $\Omega_0(z)$ and $\omega_0(z)$ under the form

$$(4.16)\qquad \begin{aligned} \Omega_0(z) &= A_0^* + A_1^*z + A_2^*z^2,\\ \omega_0(z) &= B_{-1}^*z^{-1} + B_{-2}^*z^{-2}, \end{aligned}$$

where A_1^* and B_{-1}^* are real constants. The solution Φ_0 of the equation (3.13) is sought of the form

$$(4.17)\qquad \begin{aligned} \Phi_0(z,\bar{z}) = {} & M_0I_0\left(\frac{r}{\ell_0}\right) + N_0K_0\left(\frac{r}{\ell_0}\right) +\\ & + (M_1e^{i\theta} + \bar{M}_1e^{-i\theta})I_1\left(\frac{r}{\ell_0}\right) +\\ & + (N_1e^{i\theta} + \bar{N}_1e^{-i\theta})K_1\left(\frac{r}{\ell_0}\right) \end{aligned}$$

When (4.16) and (4.17) are substituted into the boundary condition (4.14) and some properties of modified Bessel's function $I_n(\cdot)$, $K_n(\cdot)$, are used, the following values of the unknown constants are obtained

(4.18)

$$A_1^* = -\frac{A}{2} - \frac{R_2^2 \ln R_2^2 - R_1^2 \ln R_1^2}{R_2^2 - R_1^2}$$

$$B_{-1}^* = \bar{B}_{-1}^* = 2\eta A \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} \ln \frac{R_2}{R_1} - 2\beta_1 \ell_0^2 K_0 A,$$

$$M_0 = \frac{2k_0 \ell_0 A}{\Delta R_1 R_2} \left[R_2 k_1 \left(\frac{R_2}{\ell_0}\right) - R_1 K_1 \left(\frac{R_1}{\ell_0}\right) \right]$$

$$N_0 = \frac{2k_0 \ell_0 A}{\Delta R_1 R_2} \left[R_2 I_1 \left(\frac{R_2}{\ell_0}\right) - R_1 I_1 \left(\frac{R_1}{\ell_0}\right) \right]$$

$$\Delta = I_1 \left(\frac{R_1}{\ell_0}\right) K_1 \left(\frac{R_2}{\ell_0}\right) - I_1 \left(\frac{R_2}{\ell_0}\right) K_1 \left(\frac{R_1}{\ell_0}\right)$$

$$M_1 = M_1^* + A_2 P_1, \; M_1^* = -\frac{k_0 \ell_0 \bar{\gamma}}{\Delta_1 R_1^2 R_2^2} \left[R_2^2 K_1^{(1)}(R_2) - R_1^2 K_1^{(1)}(R_1) \right];$$

$$K_1^{(1)}(R_\delta) = \left(\frac{dK_1}{dr}\right)_{r=R_\delta}$$

$$N_1 = N_1^* + A_2 Q_1, \; N_1^* = -\frac{k_0 \ell_0 \bar{\gamma}}{\Delta_1 R_1^2 R_2^2} \left[R_2^2 I_1^{(1)}(R_2) - R_1^2 I_1^{(1)}(R_1) \right];$$

$$I_1^{(1)}(R_\delta) = \left(\frac{dI_1}{dr}\right)_{r=R_\delta}$$

$$\Delta_1 = I_1^{(1)}(R_1) K_1^{(1)}(R_2) - I_1^{(1)}(R_2) K_1^{(1)}(R_1)$$

$$P_1 = 2\ell_0 k_0 \left[K_1^{(1)}(R_2) - K_1^{(1)}(R_1) \right],$$

$$Q_1 = 2\ell_0 k_0 \left[I_1^{(1)}(R_2) - I_1^{(1)}(R_1) \right],$$

$$A_2^* = -\frac{\bar{\gamma}(R_1^2 - R_2^2) - \beta_1 \ell_0 (M_1^* Q_2 - N_1^* P_2)}{\eta(R_1^4 - R_2^4) + \beta_1 \ell_0 (P_2 Q_1 - P_1 Q_2)},$$

$$P_2 = -R_1^2 K_2 \left(\frac{R_1}{\ell_0}\right) + R_2^2 K_2 \left(\frac{R_2}{\ell_0}\right);$$

$$Q_2 = -R_1^2 I_2 \left(\frac{R_1}{\ell_0}\right) + R_2^2 I_2 \left(\frac{R_2}{\ell_0}\right),$$

$$B_{-2}^* = -\gamma R_1^2 - \frac{\beta_1 \ell_0 R_1^2}{\eta} \left[\bar{M}_1 I_2 \left(\frac{R_1}{\ell_0}\right) - \bar{N}_1 K_2 \left(\frac{R_1}{\ell_0}\right) \right] - R_1^2 \bar{A}_2^*$$

$$A_0^* = -2\gamma \ln R_1 - 2\bar{A}_2^* R_1^2 - \frac{\beta_1 \ell_0}{\eta} \left[\bar{M}_1 I_0 \left(\frac{R_1}{\ell_0}\right) - \bar{N}_1 K_0 \left(\frac{R_1}{\ell_0}\right) \right],$$

$$C = \eta \left(A_0^* + 2\bar{A}_2^* R_2^2\right) + \beta_1 \ell_0 \left[\bar{M}_1 I_0 \left(\frac{R_2}{\ell_0}\right) - \bar{N}_1 K_0 \left(\frac{R_2}{\ell_0}\right) \right] + 2\gamma\eta \ln R_2.$$

4.3. *The multiply-connected domain is infinite.* Consider now the case when the contour C_{n+1} is shifted to infinity.Drawing a circle C_R of sufficient large radius R to surround all the contours $C_1, C_2, \dots, C_n$, the functions $\Omega(z)$ and $\omega(z)$ can be represented outside of C_R as follows

(4.19)
$$\Omega(z) = -\frac{X_1 + iX_2}{2\pi(k_1 + \eta)} \ln z + \Omega^*(z); \quad X_1 = \sum_{k=1}^{n} \sigma_{1k},$$
$$\omega(z) = \frac{X_1 - iX_2}{2\pi\eta(k_1 + \eta)} \ln z + \omega^*(z); \quad X_2 = \sum_{k=1}^{n} \sigma_{2k},$$

where the functions $\Omega^*(z)$ and $\omega^*(z)$ are holomorphic outside of C_R, except possibly the point $z = \infty$.

The function $\Phi_0(z, \bar{z})$ regular outside of C_R is given by

$$\Phi_0(z, \bar{z}) = \sum_{s=1}^{\infty} (M_s e^{is\theta} + M_s e^{-is\theta}) K_s \left(\frac{r}{\ell_0}\right).$$

Proceeding as in [4] we deduce that the stresses and equilibrated stress are bounded for $|z| \to \infty$, unless

(4.20)
$$\Omega(z) = -\frac{X_1 + iX_2}{2\pi(k_1 + \eta)} \ln z + Pz + \Omega_1(z)$$
$$\omega(z) = \frac{X_1 - iX_2}{2\pi\eta(k_1 + \eta)} \ln z + P'z + \omega_1(z)$$

where $\Omega_1(z)$ and $\omega_1(z)$ are of the form

(4.21)
$$\Omega_1(z) = \sum_{j=1}^{\infty} a_j z^{-j}, \quad \omega_1(z) = \sum_{j=0}^{\infty} b_j z^{-j}$$

We put

$$P = B + iC, \quad P' = B' + iC'$$

in which B, C, B', C' are constants to be determined.

Using the arbitrariness degree of functions Ω_1 and ω_1 we can fix the constants a_0 and b_0 by conditions

$$\Omega_1(\infty) = 0, \quad \omega_1(\infty) = 0.$$

In addition, from the formula which gives the rotation

$$\omega = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) = \frac{1}{2i}\left(\frac{\partial U}{\partial z} - \frac{\partial \bar{U}}{\partial \bar{z}}\right)$$

and (3.17), in the absence of body loads, one finds

$$\omega = \frac{k_1 + \eta}{4i\mu}\left[\Omega'(z) - \bar{\Omega}'(\bar{z})\right].$$

In view of (4.20) and (4.21), we have

$$\lim_{|z|\to\infty} \omega = \frac{k_1 + \eta}{2\mu} C \equiv \omega_\infty.$$

When the rotation ω_∞ vanishes, we obtain $C = 0$.

To determine the remained constants B, B' and C' we account the stress components at infinity. By using (4.19), (4.20) and (2.18) we find

$$\begin{aligned} 4\eta B &= \lim_{|z|\to\infty} \Lambda(z,\bar{z}) \equiv \Lambda^{(\infty)} \\ 2\eta P' &= \lim_{|z|\to\infty} \Gamma(z,\bar{z}) \equiv \Gamma^{(\infty)} \\ &\lim_{|z|\to\infty} \Psi(z,\bar{z}) = 0. \end{aligned} \tag{4.22}$$

The last relation shows that the equilibrated stresses has no contribution to the distribution of the stress state to infinity.

In the neighborhood of the point $z = \infty$ the displacement and change-in volume-fraction are given by

$$\begin{aligned} 2\mu U(z,\bar{z}) &= -\frac{k_1(X_1 + iX_2)}{2\pi\eta(K_1 + \eta)}\ln(z\bar{z}) + (k_1 P - \eta\bar{P})z - \bar{P}'\bar{z} + \ldots \\ \Phi_0(z,\bar{z}) &= -k_0(P + \bar{P}) + \ldots \end{aligned} \tag{4.23}$$

where the dots denote functions which are bounded for $|z| \to \infty$. From the formulae (4.23) it results that the displacement and function of change-in volume fraction - will be bounded for $|z| \to \infty$, if and only if there are

$$X_1 + iX_2 = 0, \quad P = 0, \quad P' = 0. \tag{4.24}$$

The first condition from (4.21) implies the vanishing of the stress resultant along the all contours C_k, $k = \overline{1,n}$, whereas the second group express that no stresses and rotation exist at infinite distant points.

We give an application of foregoing formula in the problem of

4.4. *Stress concentration around a circular hole.* This problem was originally considered by C o w i n in [2] without to use the method of complex functions. We return to this problem in the perspective to develop the procedure of conformal mapping for solving the stress concentration around a hole of arbitrary shape.

Consider an infinite plate with free force circular hole the center of which coincides with the origin of co-ordinate axis. We suppose that the stress field at infinity reduces to a tension T acting in the x_1-direction. In view of (4.22), we have

$$\begin{aligned} &\Lambda^{(\infty)} = \sigma_{11}^{(\infty)} = T, \quad \Gamma^{(\infty)} = -\sigma_{11}^{(\infty)} = -T, \quad X + iY = 0 \\ &P = \frac{T}{4\eta}, \quad P' = -\frac{T}{2\eta}, \quad \omega_\infty = 0 \end{aligned} \tag{4.25}$$

The boundary conditions along the hole require

$$\sigma_r + i\tau_{r\theta} = 0, \quad h_r = 0, \quad at \ r = R \tag{4.26}$$

where R is the radius of the hole. A system of functions satisfying conditions (4.25) is given by

$$\begin{aligned} &\Omega(z) = \frac{T}{4\eta}z + \frac{A_{-1}}{z}, \quad \omega(z) = -\frac{T}{4\eta}z + \frac{B_{-1}}{z} + \frac{B_{-3}}{z^3} \\ &\Phi_0(z,\bar{z}) = C(e^{2i\theta} + e^{-2i\theta})K_2\left(\frac{r}{\ell_0}\right), \quad r = \sqrt{z\bar{z}}. \end{aligned} \tag{4.27}$$

Substituting these functions in the formulae (3.18) and then into (2.7), we get

$$\begin{aligned} 2\mu(r_r + iu_\theta) &= \frac{k_1 - \eta}{4\eta}rT - \eta\frac{B_{-1}}{r}\left[\frac{k_1 A_1}{r} + \frac{rT}{2} + \right. \\ &\left. + \beta_1\ell_0 C K_1\left(\frac{r}{\ell_0}\right)\right]e^{-2i\theta} + \\ &+ \left[\frac{\eta A_{-1}}{r} - \frac{B_{-3}}{r^3} + \beta_1\ell_0 C K_3\left(\frac{r}{\ell_0}\right)\right]e^{2i\theta} \\ \Phi &= -n\ell_0^2\frac{T}{2\eta} + \left[\frac{A_{-1}}{r^2} + CK_2\left(\frac{r}{\ell_0}\right)\right](e^{2i\theta} + e^{-2i\theta}) \end{aligned} \tag{4.28}$$

$$\sigma_r+\sigma_\theta=T-\left[\frac{-2\eta A_{-1}}{r^2}+\beta_1 C K_2\left(\frac{r}{\ell_0}\right)\right]\left(e^{2i\theta}+e^{-2i\theta}\right)$$

$$\sigma_\theta-\sigma_r+2i\tau_{r\theta}=-\eta\frac{B_{-1}}{r^2}+\left[-T+\beta_1 C K_0\left(\frac{r}{\ell_0}\right)\right]e^{2i\theta}+ \tag{4.29}$$

$$+\left[2\eta\left(\frac{2A_{-1}}{r^2}-\frac{3B_{-3}}{r^4}\right)+\beta_1 C K_4\left(\frac{r}{\ell_0}\right)\right]e^{-2i\theta},$$

$$h_r+ih_\theta=-2\alpha\left[\frac{2m\ell_0^2 A_{-1}}{r^3}+\frac{C}{2\ell_0}K_3\left(\frac{r}{\ell_0}\right)\right]e^{2i\theta}- \tag{4.30}$$

$$-\frac{\alpha}{\ell_0}K_1\left(\frac{r}{\ell_0}\right)e^{-2i\theta},$$

$$g=-\frac{\alpha C}{\ell_0^2}\left(e^{2i\theta}+e^{-2i\theta}\right)K_2\left(\frac{r}{\ell_0}\right). \tag{4.31}$$

In the formula (4.28)-(4.30) we are used recurrence properties of Bessel functions of imaginary arguments, [5].

Expressing the boundary conditions (4.26) with the aid of (4.29) and solving the corresponding system of algebraic equations, we get

$$A_{-1}=\frac{TR^2}{2\eta\Delta}\left[K_1\left(\frac{R}{\ell_0}\right)+K_3\left(\frac{R}{\ell_0}\right)\right],$$

$$B_{-1}=-\frac{TR^2}{2\eta},$$

$$C=-\frac{2mTR^2}{\eta\Delta}\left(\frac{\ell_0}{R}\right)^3\equiv-\frac{TM}{\beta_1\Delta}$$

$$B_{-3}=\frac{TR^4}{2\eta}+\frac{\beta_1}{\eta}\ell_0R^3C\left[K_1\left(\frac{R}{\ell_0}\right)+K_3\left(\frac{R}{\ell_0}\right)\right] \tag{$*$}$$

$$M=\frac{2mR^2\beta_1}{\eta}\left(\frac{\ell_0}{R}\right)^3,$$

$$\Delta=M\left[K_2\left(\frac{R}{\ell_0}\right)-K_0\left(\frac{R}{\ell_0}\right)\right]+K_1\left(\frac{R}{\ell_0}\right)+K_3\left(\frac{R}{\ell_0}\right)$$

In view of (2.10) we can show that $M>0$. Moreover, using relations (3.11) and (3.14) we can write

$$M=N\cdot\left(\frac{\ell_0}{R}\right)$$

where

$$N=\frac{4\mu\beta^2}{(\lambda+2\mu)[\xi(\lambda+\mu)-\beta^2]}\quad(>0)$$

Analysing the stresses at the periphery of the hole, we find

$$(\sigma_\theta)_{r=R}=T\left(1-2\cos2\theta-2\frac{MK_0\left(\frac{R}{\ell_0}\right)}{\Delta}\cos2\theta\right) \tag{4.32}$$

$$(h_\theta)_{r=R}=-\frac{2\alpha}{\ell_0}\frac{TM}{\beta_1\Delta}K_1\left(\frac{R}{\ell_0}\right)\sin2\theta. \tag{4.33}$$

From (4.32) it results that the maximum value of the contour stress σ_θ occurs at $\theta=\pm\frac{\pi}{2}$ and it is given by

$$(\sigma_\theta)_{\max}=T\left(3+\frac{2MK_0\left(\frac{R}{\ell_0}\right)}{\Delta}\right).$$

Also, the contour equilibrated stress h_θ received extreme values at $\theta=\pm\frac{\pi}{4}$, $\theta=\pm\frac{3\pi}{4}$ and these are

$$(h_\theta)_e=\frac{T_\alpha}{\ell_0\beta_1}(3-S_c)\frac{K_1\left(\frac{R}{\ell_0}\right)}{K_0\left(\frac{R}{\ell_0}\right)}sgn\ \theta,$$

where

$$S_c=\frac{(\sigma_e)_{\max}}{T}$$

denotes the stress concentration factor.

From (4.31) we deduce that $\theta=\pm\frac{\pi}{2}$ are extremal directions of the intrinsic body force, too. Along these directions hold

$$(g)_{\theta=\pm\frac{\pi}{2}}=-\frac{2\alpha}{\ell_0}\frac{TN}{\beta_1\Delta}K_2\left(\frac{r}{\ell_0}\right).$$

Using the expressions of M and Δ given by $(*)$ we can rewrite S_c under the form

$$S_c=3+\frac{2N\frac{\ell_0}{R}K_0\left(\frac{R}{\ell_0}\right)}{K_1\left(\frac{R}{\ell_0}\right)\left[1+N\left(\frac{\ell_0}{R}\right)^2\right]+K_3\left(\frac{R}{\ell_0}\right)}.$$

This formulae is similar to that obtained by C o w i n in [2]. Further discussions concerning the variation of the stres concentration factor can be found in the quoted article.

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Department of Mathematics
"Al.I.Cuza" University
B-dul Copou 11
6600 Iaşi
ROMANIA

ANALELE STIINTIFICE ALE UNIVERSITATII "AL.I.CUZA" IASI
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SOLVING A SINGULAR EQUATION BY USE OF FRESNEL INTEGRALS

BY

ADRIAN CORDUNEANU

We consider the singular integral equation

$$y(x) + \lambda \int_{-\infty}^{\infty} k(x-t)y(t)dt = f(x), \quad x \in R \tag{1}$$

where $y = y(x)$ is the unknown function, λ is a real parameter and R stands for the set of all real numbers. We assume that the functions $k = k(x)$ and $f = f(x)$, defined on R, are given by

$$k(x) = \sum_{n\geq 1}(k_n' \cos nx^2 + k_n'' \sin nx^2), \quad \sum_{n\geq 1}(|k_n'| + |k_n''|) < \infty \tag{2}$$

and respectively

$$f(x) = \sum_{n\geq 1}(f_n' \cos nx + f_n'' \sin nx), \quad \sum_{n\geq 1}(|f_n'| + |f_n''|) < \infty. \tag{3}$$

These conditions imply that $k = k(x)$ is continuous and $f = f(x)$ is continuous and 2π-periodic on R. In what follows, we shall search for equation (1) a continuous and 2π-periodic solution $y = y(x)$.

If the function $y = y(x)$, not necessarily a solution of (1), is given by

$$y(x) = \sum_{n\geq 1}(y_n' \cos nx + y_n'' \sin nx), \quad \sum_{n\geq 1}(|y_n'| + |y_n''|) < \infty \tag{4}$$

we shall prove that the integral involved in equation (1) is convergent, writting it as

$$\int_{-\infty}^{\infty} y(x-t)k(t)dt, \quad x \in R. \tag{5}$$