This formulae is similar to that obtained by C o w i n in [2]. Further discussions concerning the variation of the stres concentration factor can be found in the quoted article.

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Department of Mathematics
"Al.I.Cuza" University
B-dul Copou 11
6600 Iaşi
ROMANIA

ANALELE STIINTIFICE ALE UNIVERSITATII "AL.I.CUZA" IASI
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SOLVING A SINGULAR EQUATION BY USE OF FRESNEL INTEGRALS

BY

ADRIAN CORDUNEANU

We consider the singular integral equation

$$y(x)+\lambda\int_{-\infty}^{\infty}k(x-t)y(t)dt=f(x),\quad x\in R \tag{1}$$

where $y=y(x)$ is the unknown function, λ is a real parameter and R stands for the set of all real numbers. We assume that the functions $k=k(x)$ and $f=f(x)$, defined on R, are given by

$$k(x)=\sum_{n\geq 1}(k_n'\cos nx^2+k_n''\sin nx^2),\quad \sum_{n\geq 1}(|k_n'|+|k_n''|)<\infty \tag{2}$$

and respectively

$$f(x)=\sum_{n\geq 1}(f_n'\cos nx+f_n''\sin nx),\quad \sum_{n\geq 1}(|f_n'|+|f_n''|)<\infty. \tag{3}$$

These conditions imply that $k=k(x)$ is continuous and $f=f(x)$ is continuous and 2π-periodic on R. In what follows, we shall search for equation (1) a continuous and 2π-periodic solution $y=y(x)$.

If the function $y=y(x)$, not necessarily a solution of (1), is given by

$$y(x)=\sum_{n\geq 1}(y_n'\cos nx+y_n''\sin nx),\quad \sum_{n\geq 1}(|y_n'|+|y_n''|)<\infty \tag{4}$$

we shall prove that the integral involved in equation (1) is convergent, writing it as

$$\int_{-\infty}^{\infty}y(x-t)k(t)dt,\quad x\in R. \tag{5}$$

To this end, we need the following

Lemma 1. *Under the condition (2), for every fixed $n \geq 1$, we have*

$$\int_{-\infty}^{\infty} k(t)\cos nt\,dt = \sum_{m\geq 1} \frac{\alpha}{\sqrt{m}}(k'_m a_{n,m} + k''_m b_{n,m}) \tag{6}$$

where $\alpha = (\pi/2)^{\frac{1}{2}}$ is a constant and

$$a_{n,m} = \cos\frac{n^2}{4m} + \sin\frac{n^2}{4m}, \quad b_{n,m} = \cos\frac{n^2}{4m} - \sin\frac{n^2}{4m}, \tag{7}$$

for every $n, m \geq 1$.

Proof. Let a and b be finite, $a < 0 < b$ and let $n \geq 1$ be fixed. Obviously, we have

$$\int_a^b \left(\sum_{m\geq 1} k'_m \cos mt^2\right)\cos nt\,dt = \sum_{m\geq 1} k'_m \int_a^b \cos mt^2 \cos nt\,dt \tag{8}$$

On the other hand, using the Fresnel integrals

$$\int_{-\infty}^{\infty} \cos t^2\,dt = \int_{-\infty}^{\infty} \sin t^2\,dt = \sqrt{\frac{\pi}{2}} \tag{9}$$

we easily deduce

$$\int_{-\infty}^{\infty} \cos mt^2 \cos nt\,dt = \frac{\alpha}{\sqrt{m}} a_{n,m}, \quad n, m \geq 1 \tag{10}$$

Then, using (8) and (10), we can write

$$\begin{aligned} &\left|\int_a^b \left(\sum_{m\geq 1} k'_n \cos mt^2\right)\cos nt\,dt - \sum_{m\geq 1} k'_m \frac{\alpha}{\sqrt{m}} a_{n,m}\right| \leq \\ &\leq \sum_{m\geq 1} |k'_m| \left(\left|\int_b^{\infty} \cos mt^2 \cos nt\,dt\right| + \left|\int_{-\infty}^{a} \cos mt^2 \cos nt\,dt\right|\right) \end{aligned} \tag{11}$$

Now, we will show that for every $\varepsilon > 0$ and fixed n, we can find $B(\varepsilon) > 0$, such that

$$\left|\int_b^{\infty} \cos mt^2 \cos nt\,dt\right| < \varepsilon \tag{12}$$

for every $b \geq B(\varepsilon)$ and for every $m \geq 1$. This easily follows from the equality

$$\begin{aligned} 2\int_b^{\infty} \cos mt^2 \cos nt\,dt = &\frac{1}{\sqrt{m}} \int_{b'}^{\infty} \cos\left(t^2 - \frac{n^2}{4m}\right) dt + \\ &+ \frac{1}{\sqrt{m}} \int_{b''}^{\infty} \cos\left(t^2 - \frac{n^2}{4m}\right) dt \end{aligned} \tag{13}$$

where

$$b' = \sqrt{m}b + \frac{n}{2\sqrt{m}}, \quad b'' = \sqrt{m}b - \frac{n}{2\sqrt{m}} \tag{14}$$

taking into account the convergence of the Fresnel integrals and the fact for fixed n, the set $\{n/2\sqrt{m},\ m \geq 1\}$ is bounded. Analogously, for $\varepsilon > 0$ and fixed n, we can find $A(\varepsilon) < 0$ such that

$$\left|\int_{-\infty}^{a} \cos mt^2 \cos nt\,dt\right| < \varepsilon \tag{15}$$

for every $a \leq A(\varepsilon)$ and for every $m \geq 1$. From the inequality (11), we then deduce that, for every fixed n, the inequality

$$\left|\int_a^b \left(\sum_{m\geq 1} k'_m \cos mt^2\right)\cos nt\,dt - \sum_{m\geq 1} k'_m \cos\frac{\alpha}{\sqrt{m}} a_{n,m}\right| \leq 2\varepsilon \sum_{m\geq 1} |k'_m| \tag{16}$$

holds for $b \geq B(\varepsilon)$ and $a \leq A(\varepsilon)$, which implies

$$\begin{aligned} \int_{-\infty}^{\infty} \left(\sum_{m\geq 1} k'_m \cos mt^2\right)\cos nt\,dt &= \sum_{m\geq 1} k'_m \int_{-\infty}^{\infty} \cos mt^2 \cos nt\,dt = \\ &= \sum_{m\geq 1} k'_m \frac{\alpha}{\sqrt{m}} a_{n,m} \end{aligned} \tag{17}$$

true for every $n \geq 1$. In a similar way, we can obtain

$$\begin{aligned} \int_{-\infty}^{\infty} \left(\sum_{m\geq 1} k''_m \sin mt^2\right)\cos nt\,dt &= \sum_{m\geq 1} k''_m \int_{-\infty}^{\infty} \sin mt^2 \cos nt\,dt = \\ &= \sum_{m\geq 1} k''_m \frac{\alpha}{\sqrt{m}} b_{n,m} \end{aligned} \tag{18}$$

also true for every $n \geq 1$, where $b_{n,m}$ is given by (7). Consequently, the formula (6) is proved.

Remark 1. From (2),(6) and (7), it follows that there exists a constant $C > 0$, given by

$$C = \sqrt{\pi} \sum_{m \geq 1} \frac{1}{\sqrt{m}} (|k'_m| + |k''_m|) \tag{19}$$

such that

$$\left| \int_{-\infty}^{\infty} k(t) \cos nt dt \right| \leq C, \quad n = 1, 2, 3, \ldots \tag{20}$$

Remark 2. Under the condition (2), we can prove that the integral

$$\int_{-\infty}^{\infty} k(t) \sin nt dt, \quad n = 1, 2, 3, \ldots \tag{21}$$

is convergent. Since $k = k(t)$ is an even function, its value is obviously zero.

Lemma 2. *Under the condition (2) and (4), we have*

$$\int_{-\infty}^{\infty} \left(\sum_{m \geq 1} y'_n \cos nt \right) k(t) dt = \sum_{m \geq 1} y'_n \int_{-\infty}^{\infty} k(t) \cos nt dt \tag{22}$$

Proof. From (4) and (20), we deduce that the series involved in the right hand side of (22) is absolutely convergent. We prove that there exists a positive constant M, such that we have for every a and b (finite or infinite)

$$\left| \int_a^b k(t) \cos nt dt \right| \leq M, \quad n = 1, 2, 3, \ldots \tag{23}$$

This fact easily follows from the inequality

$$\begin{aligned} \left| \int_a^b k(t) \cos nt dt \right| &\leq \sum_{m \geq 1} |k'_m| \cdot \left| \int_a^b \cos mt^2 \cos nt dt \right| + \\ &+ \sum_{m \geq 1} |k''_m| \cdot \left| \int_a^b \sin mt^2 \cos nt dt \right| \end{aligned} \tag{24}$$

in which a and b are supposed to be finite, writting the first integral in the right hand side under the form

$$\begin{aligned} 2 \int_a^b \cos mt^2 \cos nt dt &= \int_a^b \cos \left[m \left(t + \frac{n}{2m} \right)^2 - \frac{n^2}{4m} \right] dt + \\ &+ \int_a^b \cos \left[m \left(t - \frac{n}{2m} \right)^2 - \frac{n^2}{4m} \right] dt \end{aligned} \tag{25}$$

and remarking that for every $a', b' \in R$ and for every $m \geq 1$, the absolute values of the integrals

$$\int_{a'}^{b'} \cos mt^2 dt \quad \textit{and} \quad \int_{a'}^{b'} \sin mt^2 dt \tag{26}$$

are upper bounded by some positive constant. A similar argument may be made relatively to the second integral in the right hand side of the inequality (24) and thus the bounding of (23) holds. Let now a and b be finite, $a < 0 < b$. We have

$$\begin{aligned} &\left| \int_a^b \left(\sum_{n \geq 1} y'_n \cos nt \right) k(t) dt - \sum_{n \geq 1} y'_n \int_{-\infty}^{\infty} k(t) \cos nt dt \right| \leq \\ &\leq \sum_{n \geq 1} |y'_n| \cdot \left| \int_b^{\infty} k(t) \cos nt dt \right| + \sum_{n \geq 1} |y'_n| \cdot \left| \int_{-\infty}^{a} k(t) \cos nt dt \right| \end{aligned} \tag{27}$$

Taking $\varepsilon > 0$ and fixing N =a natural number, sufficiently large such that

$$\sum_{n > N} |y'_n| < \varepsilon, \tag{28}$$

we can find $B(\varepsilon) > 0$ such that for every $b \geq B(\varepsilon)$, we have

$$\left| \int_b^{\infty} k(t) \cos nt dt \right| < \varepsilon, \quad n = 1, 2, \ldots, N. \tag{29}$$

Consequently, we have

$$\sum_{n \geq 1} |y'_n| \cdot \left| \int_b^{\infty} k(t) \cos nt dt \right| < \varepsilon \left(\sum_{n \geq 1} |y'_n| + M \right) \tag{30}$$

for $b \geq B(\varepsilon)$ and in a similar way, we find $A(\varepsilon) < 0$ such that

$$\sum_{n\geq 1} |y_n'| \cdot \left|\int_{-\infty}^{a} k(t)\cos ntdt\right| < \varepsilon \left(\sum_{n\geq 1} |y_n'| + M\right) \tag{31}$$

for every $a \leq A(\varepsilon)$. This proves the formula (22).

Remark 3. In a similar way, we can prove that under the hypotheses (2) and (4), it follows

$$\int_{-\infty}^{\infty} \left(\sum_{n\geq 1} y_n'' \sin nt\right) k(t)dt = \sum_{n\geq 1} y_n'' \int_{-\infty}^{\infty} k(t)\sin ntdt = 0 \tag{32}$$

which shows the convergence of the involved here integrals and the possibility to interchange the symbols $\int$ and Σ.

In what follows, we use the notation

$$\sigma_n = \int_{-\infty}^{\infty} k(t)\cos ntdt = \sum_{m\geq 1} \frac{\alpha}{\sqrt{m}} (k_m' a_{n,m} + k_m'' b_{n,m}),\ n \geq 1. \tag{33}$$

Proposition 1. *Under the conditions (2) and (3), the equation (1) has a unique solution $y = y(x)$ of the form (4), if*

$$|\lambda| < C^{-1} \tag{34}$$

where C is the constant defined by (19). The coefficients y_n' and y_n'' are given by

$$y_n' = \frac{f_n'}{1+\lambda\sigma_n}, \quad y_n'' = \frac{f_n''}{1+\lambda\sigma_n}, \quad n = 1,2,3,\ldots \tag{35}$$

Proof. The equation (1), in which $y = y(x)$ is assumed to be of the form (4), may be written as

$$\begin{aligned} &\sum_{n\geq 1} (y_n' \cos nx + y_n'' \sin nx) + \\ &+ \lambda \int_{-\infty}^{\infty} \left[\sum_{n\geq 1} (y_n' \cos n(x-t) + y_n'' \sin n(x-t))\right] k(t)dt = \\ &= \sum_{n\geq 1} (f_n' \cos nx + f_n'' \sin nx) \end{aligned} \tag{36}$$

Taking into account the formulas (22),(32),(6) and (33), we obtain

$$y_n'(1+\lambda\sigma_n) = f_n', \quad y_n''(1+\lambda\sigma_n) = f_n'', \tag{37}$$

for $n \geq 1$ and from the inequalities

$$|y_n'| \leq (1-|\lambda|C)^{-1} |f_n'|, \quad |y_n''| \leq (1-|\lambda|C)^{-1} |f_n''|, \tag{38}$$

true for $n \geq 1$, it follows that the second condition (4) is also satisfied. The unicity, in the class defined by condition (4), of the solution $y = y(x)$ given by (37), follows from the fact that the function $f = f(x)$ defined by (3) uniquely determines the coefficients f_n' and f_n'' for $n \geq 1$.

Consider now the singular Fredholm integral equation of the first kind

$$\int_{-\infty}^{\infty} k(x-t)y(t)dt = f(x), \quad x \in R \tag{39}$$

where $k = k(x)$, $f = f(x)$ are given and $y = y(x)$ is the unknown function. We can state

Proposition 2. *Assume that for equation (39), the conditions (2) and (3) are fulfilled. Assume also that*

(i) if $\sigma_n \neq 0$, y_n' and y_n'' are defined by

$$y_n' = f_n'/\sigma_n, \quad y_n'' = f_n''/\sigma_n; \tag{40}$$

(ii) if $\sigma_n = 0$ for some n, then $f_n' = f_n'' = 0$ and y_n', y_n'' are arbitrarily taken;

(iii) the sequences (y_n') and (y_n'') defined by (i) and (ii) satisfy the second condition (4).

Then, the function

$$y(x) = \sum_{n\geq 1} (y_n' \cos nx + y_n'' \sin nx), \quad x \in R \tag{41}$$

is a solution of the equation (39).

For the proof, we write the integral contained in equation (39) under the form (5) and we continue the argument as in the preceding case. We remark that if there exists $\sigma_n = 0$ (for some n), we have an infinity of solutions of the form (41), because the corresponding y_n' and y_n'' may be taken arbitrarily.

Example. If the kernel $k = k(x)$ satisfies the condition (2), it suffices to have

$$|f'_n|,\ |f''_n| \le c_n|\sigma_n|,\quad for \quad n \ge 1, \quad \sum_{n\ge 1} c_n < \infty \tag{42}$$

such that the conclusion of Proposition 2 holds. In other words, given $k = k(x)$, there exists an infinity of functions $f = f(x)$ of the form (3), such that the equation (39) has solution of the form (4).

Corollary 2.1. *Under the conditions (2) and (3), the equation (39) has a unique solution $y = y(x)$ of the form (4), if $\sigma_n \neq 0$ for all $n \ge 1$ and if the sequences*

$$y'_n = f'_n/\sigma_n, \quad y''_n = f''_n/\sigma_n, \quad n \ge 1 \tag{43}$$

satisfy the second condition (4).

Remark 4. The same method may be used to solve singular integral equations

$$y(x) + \lambda \int_{-\infty}^{\infty} k(x+t)y(t)dt = f(x), \quad x \in R \tag{44}$$

and

$$\int_{-\infty}^{\infty} k(x+t)y(t)dt = f(x), \quad x \in R. \tag{45}$$

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Department of Mathematics
Technical University "Gh.Asachi"
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ROMANIA

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