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AUTOTOPISM CHARACTERIZATION OF G-LOOPS

BY

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1.Introduction. The concept of $G-$*loops* has been studied, by B e l o u s o v [2] and B a s a r a b [1]. G o o d a i r e and R o b i n s o n [7] have studied the class of conjugacy closed loops and proved that they are $G-$*loops*. They have given a characterization of conjugacy closed loops, and established some algebraic properties of such loops. Recently, S h a r m a and S o l a r i n [11] proved that Bol loops of order 3p, where p is a prime number greater than 3 are $G-$ *loops*. S o l a r i n and C h i b o k a [12] have proved that extra loops and central loops are conjugacy closed and hence they are $G-$ *loops*. This they achieved by considering the algebras satisfying extra and central identities respectively. B r u c k [3] had raised the question as to the necessary and sufficient conditions for a loop to be a $G-$*loop*. He noted that associativity is sufficient, but not necessary since the multiplicative loop of any alternative division ring has this property. We shall give a characterization for $G-$*loops* in general. The autotopism characterization for conjugacy closed loops will be seen to be a special case of our general characterization. We prove that while every subloop of a $G-$ *loop* is a $G-$ *loop*, the homomorphic image of a $G-$ *loop* is also a $G-$ *loop*. It is also proved that certain diassociative Hamiltonian loops are $G-$ *loops*.

Definition 1.1. If $(G,\cdot)$ is a loop and $a\in G$, the permutations $R(a)$ and $L(a)$, called the right and left multiplications, respectively, are defined by $xR(a)=xa$, and $xL(a)=ax$, for all $x\in G$.

Definition 1.2. A loop $(G,.)$ satisfies the right inverse property if and only if $xy\cdot y^{\rho}=x$ for all $x,y\in G$. If $y^{\lambda}\cdot yx=x$ for all $x,y\in G$, then $(G,\cdot)$ is said to satisfy the left inverse property. A loop which satisfies both inverse properties is called an inverse property loop.

Definition 1.3. A loop $(G,\cdot)$ is said to be conjugacy closed if and only if $R(x)^{-1}R(y)R(x) \in \Pi_R$, and $L(x)^{-1}L(y)L(x) \in \Pi_L$ for all $x, y \in G$, where $\Pi_R = \{R(a) \mid a \in G\}$ and $\Pi_L = \{L(a) \mid a \in G\}$. Π_R is called the right regular representation of G, and Π_L is called the left representation of G.

Definition 1.4. Let $(G, \cdot)$ and $(K, \circ)$ be any two loops. Then $(K, \circ)$ is said to be an isotope of $(G, \cdot)$ if and only if there exist three one-to-one mappings α, β, γ of G onto K such that

$$x\alpha \circ y\beta = (x \cdot y)\gamma \tag{1}$$

for all $x, y \in G$. If K consists of the same elements as G and γ is the identity mapping of G, then $(G, \circ)$ is called a principal isotope of $(G, \cdot)$.

B r u c k [3] has shown that every isotope of a loop is isomorphic to a principal isotope of the loop, and every principal isotope $(G, \circ)$ of a loop $(G, \cdot)$ is defined by

$$x \circ y = xR(b)^{-1} \cdot yL(a)^{-1} \tag{2}$$

for all $x, y \in G$, and some $a, b \in G$.

Definition 1.5. Let $(G, \cdot)$ be a loop, the triple (α, β, γ) of permutations of G is called an autotopism of G if and only if

$$x\alpha \cdot y\beta = (x \cdot y)\gamma \tag{3}$$

for all $x, y \in G$.

B r u c k [3] noted that if $(G, \circ)$ is an isotope of the loop $(G, \cdot)$ and τ is a homomorphism of $(G, \cdot)$ onto $(H, \cdot)$, then θ induces a homomorphism θ_0 of $(G, \cdot)$ onto a loop $(H, \circ)$ defined by

$$x\theta_0 \circ y\theta_0 = x\theta R(b\theta)^{-1} \cdot y\theta L(a\theta)^{-1} \tag{4}$$

for all $x, y \in G$.

Definition 1.6. A loop $(G, \cdot)$ is a $G-loop$ if and only if it is isomorphic to all its principal isotopes.

Definiton 1.7. If $(G, \cdot)$ is a loop, F e n y v e s [6] defines an extra loop and a central loop by $(xy \cdot z)x = x(y \cdot zx)$ and $(xy \cdot y)z = x(y \cdot yz)$ for all $x, y, z \in G$.

Definition 1.8. A diassociative loop $(G, \cdot)$ of order 16 with three generators a_1, a_2, a_3 such that

$$a_i^4 = 1; \quad a_i^2 = a_j^2 \neq 1; \quad a_ia_j = a_j^3a_i; \quad i \neq j.,$$

$$a_ia_j \cdot a_k = a_i^3 \cdot a_ja_k. \quad i, j, k \text{ all distinct},$$

is called a *Cayley loop*. The basis elements of the Cayley - Dickson algebra (5, p.561) form a *Cayley loop*.

This definition is due to N o r t o n [8].

Definition 1.9. A loop $(G, \cdot)$ is called an *M-loop* if it satifies the identical relation

$$xy \cdot zx^\alpha = (x \cdot yz)x^\alpha \tag{6}$$

for all $x, y, z \in G$, where x^α is the image of x under some single- valued mapping of G into itself. If $\alpha : x \to x^k$, k a positive integer, the law (6) is called an M_k -law. If $(G, \cdot)$ is an *M-loop*, but not an M_k -loop for any $k > 1$, then $(G, \cdot)$ is called a strictly *Moufang loop*.

P f l u g f e l d e r [9] and C h e i n and P f l u g f e l d e r [4] have studied $M-$ *loops* extensively. C h e i n and P f l u g f e l d e r [4] constructed some $M-$ *loops* in which $x : x \to x^k$, k a positive integer. They also constructed some strictly Moufang loops which are $G-$ *loops*.

2.Result. Lemma 2.1. *A loop $(G, \cdot)$ is a $G-$ loop if and only if there exsits a permutation θ of G such that $< \theta R(b)^{-1},\ \theta L(a)^{-1}, \theta >$ is an autotopism of G for all $a, b \in G$.*

Proof. Let $(G, \cdot)$ be a $G-$ *loop* and $(G, \circ)$ the principal isotope of $(G, \cdot)$ defined by $x \circ y = xR(b)^{-1} \cdot yL(a)^{-1}$ for all $x, y \in G$. Let θ be a permutation of G such that $(x \cdot y)\theta = x\theta \circ y\theta$ (since $(G, \cdot)$ is isomorphic to its principal isotopes). Then $(x \cdot y)\theta = x\theta \circ y\theta = x\theta R(b)^{-1} \cdot y\theta L(a)^{-1}$ for all $x, y \in G$, so $< \theta R(b)^{-1},\ \theta L(a)^{-1}, \theta >$ is autotopism of G for all $a, b \in G$.

Suppose there exists a mapping θ of G such that $< \theta R(b)^{-1},\ \theta L(a)^{-1}, \theta >$ is an autotopism of G for all $a, b \in G$. Since each of the mappings in the triple is a permutation, it follows that θ is a permutation of G. Now $x\theta R(b)^{-1} \cdot y\theta L(a)^{-1} = (x \cdot y)\theta$ for all $x, y \in G$. Let $(G, \circ)$ be the principal isotope of $(G, \cdot)$ defined by $x \circ y = xR(b)^{-1} \cdot yL(a)^{-1}$ for all $x, y \in G$, then $x\theta \circ y\theta = x\theta R(b)^{-1} \cdot y\theta L(a)^{-1} = (x \cdot y)\theta$, so θ is a homomorphism from $(G, \cdot)$ to $(G, \circ)$, and hence an isomorphism since it is 1-1 and onto. Thus $(G, \cdot)$ is isomorphic to all its principal isotopes, and hence a $G-$ *loop*.

Corollary 2.1. *A loop $(G, \cdot)$ is a $G-$ loop if and only if there exist permutations θ and ψ of G such that the triples $< \theta R(a)^{-1}, \theta, \theta >$ and $< \psi, \psi L(a)^{-1}, \psi >$ are autotopims of G for all $a \in G$.*

Proof. In Lemma 2.1, if we put $b = e$ (the identity element of $(G, \cdot)$), we obtain the autotopism $< \theta, \theta L(a)^{-1}, \theta >$ for all $a \in G$. For $a = e$, we obtain the autotopism $< \theta R(b)^{-1}, \theta, \theta >$ for all $b \in G$. We note that the permutation θ is not the same for both autotopisms.

Remark 2.1. In Corollary 2.1, if we take $\theta = L(a)$ and $\psi = R(a)$, the loop reduces to a conjugacy closed loop, in accordance with the result of Goodaire and Robinson [7].

Lemma 2.2. *Let $(H,\cdot)$ be a subloop of the G-loop $(G,\cdot)$, then $(H,\cdot)$ is also a G-loop.*

Proof. If $(G,\cdot)$ is a $G-loop$, then there exists a permutation θ of G such that $< \theta R(b)^{-1}, \theta L(a)^{-1}, \theta >$ is an autotopism of G for all $a, b \in G$, that is,

$$x\theta R(b)^{-1} \cdot y\theta L(a)^{-1} = (x \cdot y)\theta \tag{7}$$

for all $x, y \in G$. Suppose $(H,\cdot)$ is a subloop of $(G,\cdot)$, then $(H\theta, \circ)$ is a subloop of $(G,\circ) = (G\theta,\cdot)$. Now,

$$x\theta \circ y\theta = x\theta R(b)^{-1} \cdot y\theta L(a)^{-1} = (x \cdot y)\theta \tag{8}$$

where

$$x \circ y = xR(b)^{-1} \cdot yL(a)^{-1} \tag{9}$$

for all $x, y \in G$. Since equations (7), (8) and (9) hold for all $x, y \in G$, they also hold for $x, y \in G$ for which $a, b \in H$. Hence, for such $a, b \in H$, θ is an isomorphism from $(H,\cdot)$ to $(H,\circ)$.

Theorem 2.1. *Let $(G,\cdot)$ be a G-loop. If H is a normal subloop of G, then G/H is a G-loop.*

Proof. Since $(G,\cdot)$ is a $G-loop$, there exists a permutation θ, of $(G,\cdot)$ such that $< \theta R(b)^{-1}, \theta L(a)^{-1}, \theta >$ is an autotopism of $(G,\cdot)$ for every principal isotope $(G,\circ)$ of $(G,\cdot)$ given by $xH \circ yH = xHR(bH)^{-1} \cdot yHL(aH)^{-1}$, where $R(bH)^{-1}$ and $L(aH)^{-1}$ are given by $uHR(bH) = uH \cdot bH$, and $uHL(aH) = aH \cdot uH$ for all $u \in G$, and hence all $uH \in G/H$. Now,

$$\begin{aligned} uH &= (uH \cdot bH)R(bH)^{-1} = (ub)HR(bH)^{-1} = u(bH)R(bH)^{-1} = \\ &= u(Hb)R(bH)^{-1} = (uH)bR(bH)^{-1} = (uH)R(b)R(bH)^{-1}. \end{aligned}$$

So $R(b)R(bH)^{-1} = I$ implies that $R(bH)^{-1} = R(b)^{-1}$. Similarly, $uHL(aH) = aH \cdot uH$ implies that

$$\begin{aligned} uH &= (aH \cdot uH)L(aH)^{-1} = (au)HL(aH)^{-1} = a(uH)L(aH)^{-1} = \\ &= (uH)L(a)L(aH)^{-1}, \end{aligned}$$

So, $L(a)L(aH)^{-1} = I$ gives $L(aH)^{-1} = L(a)^{-1}$.

Now,

$$\begin{aligned} xH \circ yH &= xHR(bH)^{-1} \cdot yHL(aH)^{-1} = \\ &= xHR(b)^{-1} \cdot yHL(a)^{-1} = \\ &= (Hx)R(b)^{-1} \cdot (Hy)L(a)^{-1} = \\ &= H(xR(b)^{-1}) \cdot H(yL(a)^{-1}) = \\ &= xR(b)^{-1}H \cdot yL(a)^{-1}H. \end{aligned}$$

Define $\psi : (G/H,\cdot) \to (G/H,\circ)$ by $(xH)\psi = (x\theta)H$.
Then,

$$\begin{aligned} (xH \cdot yH)\psi &= ((xy)H)\psi = ((xy)\theta)H = (x\theta R(b)^{-1} \cdot y\theta L(a)^{-1}H = \\ &= x\theta R(b)^{-1}H \cdot y\theta L(a)^{-1}H, \end{aligned}$$

and $(xH)\psi \circ (yH)\psi = (x\theta)H \circ (y\theta)H = x\theta R(b)^{-1}H \cdot y\theta L(a)^{-1}H$, so $(xH)\psi \circ (yH)\psi = (xH \cdot yH)\psi$.
Hence ψ is a homomorphism from $(G/H,\cdot)$ to $(G/H,\circ)$.

By definition, $(xH)\psi \in G/H$, so

$$(xH)\psi \circ (yH)\psi = (xH)\psi R(bH)^{-1} \cdot (yH)\psi L(aH)^{-1}.$$

But, $(xH \cdot yH)\psi = (xH)\psi \circ (yH)\psi$. So,

$$(xH \cdot yH)\psi = (xH)\psi R(bH)^{-1} \cdot (yH)\psi L(aH)^{-1}.$$

Thus, the triple $< \psi R(bH)^{-1}, \psi L(aH)^{-1}, \psi >$ is an autotopism of G/H for all $a, b \in G$, and hence all $aH, bH \in G/H$. By Lemma 2.2, G/H is a $G-loop$.

Theorem 2.2. *Let $(L,\cdot)$ be a loop and α a mapping of $(L,\cdot)$ into itself such that $R_xR_yR_x\alpha \in \Pi_R$ and $L_xL_yL_x\alpha$ or $L_x\alpha L_yL_x \in \Pi_L$; then $(L,\cdot)$ is an inverse property loop.*

Proof. $R_xR_yR_x \in \Pi_R$ implies that $(zx \cdot y)x^\alpha = za$ for all $x, y, z \in L$ and some $a \in L$. In particular, $z = 1$ gives $xy \cdot x^\alpha$, so $(zx \cdot y)x^\alpha = z(xy \cdot x^\alpha)$ for all $x, y, z \in L$. Substituting $y = x^\rho$, we obtain $(zx \cdot x^\rho)x^\alpha = zx^\alpha$, and $zx \cdot x^\rho = z$ for all $x, z \in L$. $L_xL_yL_x\alpha \in \Pi_L$ implies that $x^\alpha(y \cdot xz) = bz$ for all $x, y, z \in L$ and some $b \in L$. $z = 1$ gives $x^\alpha \cdot yx = b$ and $x^\alpha(y \cdot xz) = (x^\alpha \cdot yx)z$ for all $x, y, z \in L$. Setting $y = x^\lambda$ gives $x^\alpha(x^\lambda \cdot xz) = (x^\alpha \cdot x^\lambda x)z = x^\alpha z$, so $x^\lambda \cdot xz = z$ for all $x, z \in L$. Now, $L_x\alpha L_yL_x \in L$ implies that $x(y \cdot x^\alpha z) = bz$ for all $x, y, z \in L$ and some $b \in L$. $z = 1$ gives $x \cdot yx^\alpha = b$ and $x(y \cdot x^\alpha z) = (x \cdot x^\alpha)z$ for all $x, y, z \in L$. But $y = (x^\alpha)^\lambda$ gives $x[(x^\alpha)^\lambda \cdot x^\alpha z] = (x \cdot (x^\alpha)^\lambda x^\alpha)z = xz$. So $(x^\alpha)^\lambda \cdot x^\alpha z = z$ and $(L,\cdot)$ is an inverse property loop.

The following Lemma is due to S h a r m a [10].

Lemma 2.3. *If the loop $(G,\cdot)$ satisfies any one of the following (M-loop) identities, then $(G,\cdot)$ satisfies all three*

$$xy\cdot zx^\alpha = (x\cdot yz)x^\alpha;$$
$$(yx\cdot z)x^\alpha = y(x\cdot zx^\alpha);$$
$$x^{j\alpha j}(y\cdot xz) = (x^{j\alpha j}y\cdot x)z;$$

where x^α is the image of x under some mapping of the loop $(G,\cdot)$ into itself and j is the inverse permutation of $(G,\cdot)$ defined by $xj = x^{-1}$.

Theorem 2.3. *Let $(G,\cdot)$ be an M-loop with α the inverse mapping, then $(G,\cdot)$ is conjugacy closed, hence a G-loop.*

Proof. From Lemma 2.3, with $x = x^{-1}$, we have

$$(yx\cdot z)x^{-1} = y(x\cdot zx^{-1}) \tag{10}$$

and

$$((x^j)^{-1})^j[y\cdot xz] = [((x^j)^{-1})^j y\cdot x]z \tag{11}$$

Equation (10) implies that $yR(x)R(z)R(x)^{-1} = yR(x\cdot zx^{-1})$, which gives $R(x)R(z)R(x)^{-1}\in\Pi$ for all $x,z\in G$.
Equation (11) implies that $zL(x)L(y)L(x)^{-1} = zL(x^{-1}y\cdot x)$ which gives $L(x)L(y)L(x)^{-1}\in\Pi_L$ for all $x,y\in G$. Hence $(G,\cdot)$ is conjugacy closed.

Theorem 2.4. *The following statements are equivalent for a conjugacy closed loop.*

(i) $(G,\cdot)$ *is diassociative*
(ii) $(G,\cdot)$ *has the inverse property*
(iii) $(G,\cdot)$ *is an M-loop*

Proof. Since $(G,\cdot)$ is conjugacy closed, we have

$$A = < L(x)R(x)^{-1}, L(x), L(x) >$$

and

$$B = < R(x), R(x)L(x)^{-1}, R(x) >$$

are autotopism of G for all $x\in G$. If $(G,\cdot)$ is diassociative, then $xy\cdot x = x\cdot yx$ for all $x,y\in G$, so $yL(x)R(x) = yR(x)L(x)$ implies that $L(x)R(x) = R(x)L(x)$ for all $x\in G$,. Now,

$$BA = < L(x), R(x), R(x)L(x) > = < L(x), R(x), L(x)R(x) >$$

is an autotopism of G for all $x\in G$, hence $(G,\cdot)$ is a *Moufang loop*. Thus (i) implies (iii), hence (i) implies (ii).

If $(G,\cdot)$ is an inverse property loop, then $R(x)^{-1} = R(x^{-1})$ and $L(x)^{-1} = L(x^{-1})$ for all $x\in G$, so $zR(x)R(y)R(x)^{-1} = zR(x\cdot yx^{-1})$, implies that $(zx\cdot y)x^{-1} = z(x\cdot yx^{-1})$ for all $x,y,z\in G$. Using Sharma's result [10] with $x^\alpha = x^{-1}$, $(G,\cdot)$ is *M-loop*. Thus (ii) implies (iii) and (i). Since every M-loop is diassociative and has the inverse property, the proof is complete.

By considering the loop formed by the units of a central algebra, S o l a r i n and C h i b o k a [12] proved that every central loop is a *G – loop*. We now show that a *Cayley loop* is a central loop, hence a *G – loop*.

Lemma 2.4. *Let $(L,\cdot)$ be a Cayley loop, then $(L,\cdot)$ is a central loop, and hence a $G-$ loop.*

Proof. Let a_1, a_2, a_3 be the generators of L. Then

$$(a_ia_j\cdot a_j)a_k = a_ia_j^2\cdot a_k = a_ia_i^2a = a_i^3a_k = a_ka_i,$$

$$a_i(a_j\cdot a_ja_k) = a_i(a_j^2a_k) = a_i\cdot a_k^2a_k = a_ia_k^3 = a_ka_i.$$

Therefore, $(L,\cdot)$ is a central loop, and hence a $G-$ *loop*.

In his investigation of Hamiltonian loops, N o r t o n [8] established the following result:

Lemma 2.5. *A Hamiltonian diassociative loop satisfying the condition:*

If three distinct elements associate in some order, then the subloop generated by the three elements is a group, is (i) an Abelian group, or (ii) a Hamiltonian group, or (iii) the direct product of a Cayley loop, an Abelian goup with exponent 2, and an Abelian group with elements of odd order.

W i l s o n [13] has proved that the direct product of $G-$ *loops* is $G-$ *loop*, hence we have

Theorem 2.5. *If $(G,\cdot)$ is a Hamiltonian diassociative loop satisfying the condition of Lemma 2.5, then $(G,\cdot)$ is a $G-$ loop.*

Proof. The result follows from Lemma 2.4, Lemma 2.5, and Wilson's result [13].

Corollary 2.2. *A Moufang loop which is Hamiltonian is a $G-$ loop.*

Proof. Since every *Moufang loop* is diassociative and satisfies the condition of Theorem 2.5 the result follows.

Theorem 2.6. *A commutative conjugacy closed loop is a group.*

Proof. Let $(G,\cdot)$ be a conjugacy loop, then $xy\cdot f = (xf)\cdot(yf)L(f)^{-1}$ for all $x,y,f\in G$. Since G is comutative, we have $f\cdot xy = xf\cdot(fy)L(f)^{-1}$ which implies that $f\cdot xy = fx\cdot yL(f)L(f)^{-1} = fx\cdot y$ for all $x,y,f\in G$, hence $(G,\cdot)$ is associative. Similarly, $g \cdot xy = (gx)R(g)^{-1} \cdot gy$ implies that $xy \cdot g = x \cdot yg$ for all $x, y, g \in G$.

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ON CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

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Introduction. Let A_n denote the class of all functions of the form $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + ...$ that are regular in the unit disk $E = \{z : |z| < 1\}$. A function $f(z)$ in A_n is said to be starlike if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \text{ for } z \in E,$$

and is said to be convex if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \text{ for } z \in E.$$

Let

$$D^\alpha f(z) = \frac{z}{(1-z)^{\alpha+1}} * f(z), \ \alpha > -1 =$$
$$= z + \sum_{m=n}^{\infty} \frac{(\alpha+1)(\alpha+2)...(\alpha+m)}{m!} a_{m+1} z^{m+1}$$

and the operation $(*)$ denotes the Hadamard product of the power series. R u s c h e w e y h introduced the symbol D^α in [3].

In this paper we obtain certain properties of functions $f(z)$ in A_n that satisfy $\mathrm{Re}(D^\alpha f(z)/z) > 0$ for $z \in E$. Some of the results of M a c G r e g o r [1,2] are obtained as particular cases.

Theorem 1. *Suppose that $f(z) = z + a_{n+1}z^{n+1} + ...$ is regular and satisfies $\mathrm{Re}(D^\alpha f(z)/z) > 0$ for $z \in E$. Then*

$$\mathrm{Re}\frac{D^{\alpha+1} f(z)}{D^\alpha f(z)} > \frac{\alpha}{\alpha+1} \text{ in } |z| < \left((n^2+1)^{1/2} - n\right)^{1/n}$$